Section 3.3—Optimization Problems

We frequently encounter situations in which we are asked to do the best we can. Such a request is vague unless we are given some conditions. Asking us to minimize the cost of making tables and chairs is not clear. Asking us to make the maximum number of tables and chairs possible, with a given amount of material, so that the costs of production are minimized allows us to construct a function that describes the situation. We can then determine the minimum (or maximum) of the function.

Such a procedure is called **optimization**. To optimize a situation is to realize the best possible outcome, subject to a set of restrictions. Because of these restrictions, the domain of the function is usually restricted. As you have seen earlier, in such situations, the maximum or minimum can be identified through the use of calculus, but might also occur at the ends of the restricted domain.

EXAMPLE 1 Solving a problem involving optimal area

A farmer has 800 m of fencing and wishes to enclose a rectangular field. One side of the field is against a country road that is already fenced, so the farmer needs to fence only the remaining three sides of the field. The farmer wants to enclose the maximum possible area and to use all the fencing. How does the farmer determine the dimensions to achieve this goal?

Solution

The farmer can achieve this goal by determining a function that describes the area, subject to the condition that the amount of fencing used is to be exactly 800 m, and by finding the maximum of the function. To do so, the farmer proceeds as follows:

Let the width of the enclosed area be x metres.

Then the length of the rectangular field is $(800 - 2x)$ m. The area of the field can be represented by the function $A(x)$, where

$$\begin{aligned} A(x) &= x(800 - 2x) \\ &= 800x - 2x^2 \end{aligned}$$

The domain of the function is $0 < x < 400$, since the amount of fencing is 800 m. To find the minimum and maximum values, determine $A'(x)$: $A'(x) = 800 - 4x$. Setting $A'(x) = 0$, we obtain $800 - 4x = 0$, so $x = 200$.

The minimum and maximum values can occur at $x = 200$ or at the ends of the domain, $x = 0$ and $x = 400$. Evaluating the area function at each of these gives

$$\begin{aligned} A(0) &= 0 \\ A(200) &= 200(800 - 400) \\ &= 80\,000 \\ A(400) &= 400(800 - 800) \\ &= 0 \end{aligned}$$

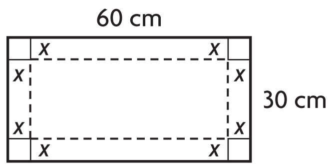
Sometimes, the ends of the domain produce results that are either not possible or unrealistic. In this case, $x = 200$ produces the maximum. The ends of the domain do not result in possible dimensions of a rectangle.

The maximum area that the farmer can enclose is 80 000 m^2, within a field 200 m by 400 m.

EXAMPLE 2

Solving a problem involving optimal volume

A piece of sheet metal, 60 cm by 30 cm, is to be used to make a rectangular box with an open top. Determine the dimensions that will give the box with the largest volume.

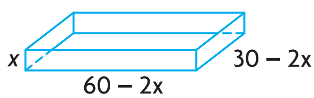


Solution

From the diagram, making the box requires the four corner squares to be cut out and discarded. Folding up the sides creates the box. Let each side of the squares be x centimetres.

$$\begin{aligned} \text{Therefore, height} &= x \\ \text{length} &= 60 - 2x \\ \text{width} &= 30 - 2x \end{aligned}$$

Since all dimensions are positive, $0 < x < 15$.



The volume of the box is the product of its dimensions and is given by the function $V(x)$, where

$$\begin{aligned} V(x) &= x(60 - 2x)(30 - 2x) \\ &= 4x^3 - 180x^2 + 1800x \end{aligned}$$

For extreme values, set $V'(x) = 0$.

$$\begin{aligned} V'(x) &= 12x^2 - 360x + 1800 \\ &= 12(x^2 - 30x + 150) \end{aligned}$$

Setting $V'(x) = 0$, we obtain $x^2 - 30x + 150 = 0$. Solving for x using the quadratic formula results in

$$\begin{aligned} x &= \frac{30 \pm \sqrt{300}}{2} \\ &= 15 \pm 5\sqrt{3} \end{aligned}$$

$x \doteq 23.7$ or $x \doteq 6.3$

Since $0 < x < 15$, $x = 15 - 5\sqrt{3} \doteq 6.3$. This is the only place within the interval where the derivative is 0.

To find the largest volume, substitute $x = 6.3$ in $V(x) = 4x^3 - 180x^2 + 1800x$.

$$\begin{aligned} V(6.3) &= 4(6.3)^3 - 180(6.3)^2 + 1800(6.3) \\ &= 5196 \end{aligned}$$

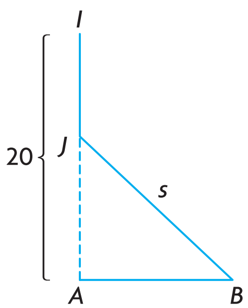
Notice that the endpoints of the domain did not have to be tested since it is impossible to make a box using the values $x = 0$ or $x = 15$.

The maximum volume is obtained by cutting out corner squares of side length 6.3 cm. The length of the box is $60 - 2 \times 6.3 = 47.4$ cm, the width is about $30 - 2 \times 6.3 = 17.4$ cm, and the height is about 6.3 cm.

EXAMPLE 3 Solving a problem that minimizes distance

Ian and Ada are both training for a marathon. Ian's house is located 20 km north of Ada's house. At 9:00 a.m. one Saturday, Ian leaves his house and jogs south at 8 km/h. At the same time, Ada leaves her house and jogs east at 6 km/h. When are Ian and Ada closest together, given that they both run for 2.5 h?

Solution



If Ian starts at point I, he reaches point J after time t hours. Then $IJ = 8t$ km, and $JA = (20 - 8t)$ km.

If Ada starts at point A, she reaches point B after t hours, and $AB = 6t$ km. Now the distance they are apart is $s = JB$, and s can be expressed as a function of t by

$$\begin{aligned} s(t) &= \sqrt{JA^2 + AB^2} \\ &= \sqrt{(20 - 8t)^2 + (6t)^2} \\ &= \sqrt{100t^2 - 320t + 400} \\ &= (100t^2 - 320t + 400)^{\frac{1}{2}} \end{aligned}$$

The domain for t is $0 \leq t \leq 2.5$.

$$s'(t) = \frac{1}{2}(100t^2 - 320t + 400)^{-\frac{1}{2}}(200t - 320)$$

$$= \frac{100t - 160}{\sqrt{100t^2 - 320t + 400}}$$

To obtain a minimum or maximum value, let $s'(t) = 0$.

$$\frac{100t - 160}{\sqrt{100t^2 - 320t + 400}} = 0$$

$$100t - 160 = 0$$

$$t = 1.6$$

Using the algorithm for finding extreme values,

$$s(0) = \sqrt{400} = 20$$

$$s(1.6) = \sqrt{100(1.6)^2 - 320(1.6) + 400} = 12$$

$$s(2.5) = \sqrt{225} = 15$$

Therefore, the minimum value of $s(t)$ is 12 km, which occurs at time 10:36 a.m.

IN SUMMARY

Key Ideas

- In an optimization problem, you must determine the maximum or minimum value of a quantity.
- An optimization problem can be solved using a mathematical model that is developed using information given in the problem. The numerical solution represents the extreme value of the model.

Need to Know

- **Algorithm for Solving Optimization Problems:**
  1. Understand the problem, and identify quantities that can vary. Determine a function in one variable that represents the quantity to be optimized.
  2. Whenever possible, draw a diagram, labelling the given and required quantities.
  3. Determine the domain of the function to be optimized, using the information given in the problem.
  4. Use the algorithm for extreme values to find the absolute maximum or minimum value in the domain.
  5. Use your result for step 4 to answer the original problem.

Exercise 3.3

PART A

1. A piece of wire, 100 cm long, needs to be bent to form a rectangle. Determine the dimensions of a rectangle with the maximum area.

C 2. Discuss the result of maximizing the area of a rectangle, given a fixed perimeter.

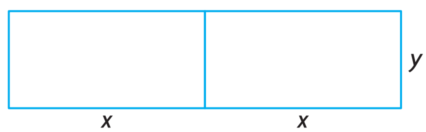
3. A farmer has 600 m of fence and wants to enclose a rectangular field beside a river. Determine the dimensions of the fenced field in which the maximum area is enclosed. (Fencing is required on only three sides: those that aren't next to the river.)

4. A rectangular piece of cardboard, 100 cm by 40 cm, is going to be used to make a rectangular box with an open top by cutting congruent squares from the corners. Calculate the dimensions (to one decimal place) for a box with the largest volume.

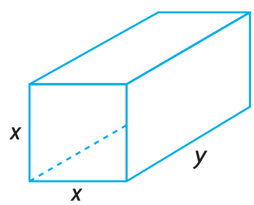
5. A rectangle has a perimeter of 440 cm. What dimensions will maximize the area of the rectangle?

6. What are the dimensions of a rectangle with an area of 64 m^2 and the smallest possible perimeter?

7. A rancher has 1000 m of fencing to enclose two rectangular corrals. The corrals have the same dimensions and one side in common. What dimensions will maximize the enclosed area?



8. A net enclosure for practising golf shots is open at one end, as shown. Find the dimensions that will minimize the amount of netting needed and give a volume of 144 m^2. (Netting is required only on the sides, the top, and the far end.)



PART B

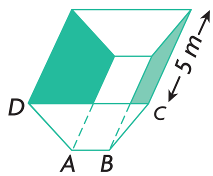
9. The volume of a square-based rectangular cardboard box needs to be 1000 cm^3. Determine the dimensions that require the minimum amount of material to manufacture all six faces. Assume that there will be no waste material. The machinery available cannot fabricate material smaller than 2 cm in length.

10. Determine the area of the largest rectangle that can be inscribed inside a semicircle with a radius of 10 units. Place the length of the rectangle along the diameter.

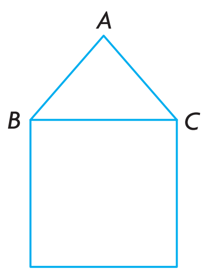
A 11. A cylindrical-shaped tin can must have a capacity of 1000 cm^3.
   a. Determine the dimensions that require the minimum amount of tin for the can. (Assume no waste material.) According to the marketing department, the smallest can that the market will accept has a diameter of 6 cm and a height of 4 cm.
   b. Express your answer for part a. as a ratio of height to diameter. Does this ratio meet the requirements outlined by the marketing department?

12. a. Determine the area of the largest rectangle that can be inscribed in a right triangle if the legs adjacent to the right angle are 5 cm and 12 cm long. The two sides of the rectangle lie along the legs.
   b. Repeat part a. for a right triangle that has sides 8 cm and 15 cm.
   c. Hypothesize a conclusion for any right triangle.

T 13. a. An isosceles trapezoidal drainage gutter is to be made so that the angles at A and B in the cross-section $ABCD$ are each 120°. If the 5 m long sheet of metal that has to be bent to form the open-topped gutter and the width of the sheet of metal is 60 cm, then determine the dimensions so that the cross-sectional area will be a maximum.



   b. Calculate the maximum volume of water that can be held by this gutter.



14. The 6 segments of the window frame shown in the diagram are to be constructed from a piece of window framing material 6m in length. A carpenter wants to build a frame for a rural gothic style window, where $\triangle ABC$ is equilateral. The window must fit inside a space that is 1 m wide and 3 m high.

a. Determine the dimensions that should be used for the six pieces so that the maximum amount of light will be admitted. Assume no waste material for corner cuts and so on.

b. Would the carpenter get more light if the window was built in the shape of an equilateral triangle only? Explain.

15. A train leaves the station at 10:00 a.m. and travels due south at a speed of 60 km/h. Another train has been heading due west at 45 km/h and reaches the same station at 11:00 a.m. At what time were the two trains closest together?

16. A north–south highway intersects an east–west highway at point P. A vehicle crosses P at 1:00 p.m., travelling east at a constant speed of 60 km/h. At the same instant, another vehicle is 5 km north of P, travelling south at 80 km/h. Find the time when the two vehicles are closest to each other and the distance between them at this time.

PART C

17. In question 12, part c., you looked at two specific right triangles and observed that a rectangle with the maximum area that can be inscribed inside the triangle had dimensions equal to half the lengths of the sides adjacent to the rectangle. Prove that this is true for any right triangle.

18. Prove that any cylindrical can of volume k cubic units that is to be made using a minimum amount of material must have the height equal to the diameter.

19. A piece of wire, 100 cm long, is cut into two pieces. One piece is bent to form a square, and the other piece is bent to form a circle. Determine how the wire should be cut so that the total area enclosed is

    a. a maximum

    b. a minimum

20. Determine the minimal distance from point $(-3, 3)$ to the curve given by $y = (x - 3)^2$.

21. A chord joins any two points A and B on the parabola whose equation is $y^2 = 4x$. If C is the midpoint of AB, and CD is drawn parallel to the x-axis to meet the parabola at D, prove that the tangent at D is parallel to chord AB.

22. A rectangle lies in the first quadrant, with one vertex at the origin and two of the sides along the coordinate axes. If the fourth vertex lies on the line defined by $x + 2y - 10 = 0$, find the rectangle with the maximum area.

23. The base of a rectangle lies along the x-axis, and the upper two vertices are on the curve defined by $y = k^2 - x^2$. Determine the dimensions of the rectangle with the maximum area.