Section 3.4—Optimization Problems in Economics and Science

In the world of business, it is extremely important to manage costs effectively. Good control will allow for minimization of costs and maximization of profit. At the same time, there are human considerations. If your company is able to maximize profit but antagonizes customers or employees in the process, there may be problems in the future. For this reason, it may be important that, in addition to any mathematical constraints, you consider other more practical constraints on the domain when you construct a workable function.

The following examples will illustrate economic situations and domain constraints you may encounter.

EXAMPLE 1 **Solving a problem to maximize revenue**

A commuter train carries 2000 passengers daily from a suburb into a large city. The cost to ride the train is \$7.00 per person. Market research shows that 40 fewer people would ride the train for each \$0.10 increase in the fare, and 40 more people would ride the train for each \$0.10 decrease. If the capacity of the train is 2600 passengers, and carrying fewer than 1600 passengers means costs exceed revenue, what fare should the railway charge to get the largest possible revenue?

Solution

To maximize revenue, we require a revenue function. We know that revenue $=$ (number of passengers) $\times$ (fare per passenger).

To form a revenue function, the most straightforward choice for the independent variable comes from noticing that both the number of passengers and the fare per passenger change with each \$0.10 increase or decrease in the fare. If we let x represent the number of \$0.10 increases in the fare (for example, $x = 3$ represents a \$0.30 increase in the fare, whereas $x = -1$ represents a \$0.10 decrease in the fare), then we can write expressions for both the number of passengers and the fare per passenger in terms of x, as follows:

- the fare per passenger is $7 + 0.10x$
- the number of passengers is $2000 - 40x$

Since the number of passengers must be at least 1600, $2000 - 40x \geq 1600$, and $x \leq 10$. Since the number of passengers cannot exceed 2600, $2000 - 40x \leq 2600$, and $x \geq -15$.

The domain is $-15 \leq x \leq 10$.

The revenue function is

$$R(x) = (7 + 0.10x)(2000 - 40x)$$
$$= -4x^2 - 80x + 14\,000$$

From a practical point of view, we also require x to be an integer, so that the fare only varies by increments of \$0.10. We do not wish to consider fares that are not multiples of 10 cents.

Therefore, we need to find the absolute maximum value of the revenue function $R(x) = -4x^2 - 80x + 14\,000$ on the interval $-15 \leq x \leq 10$, where x must be an integer.

$$R'(x) = -8x - 80$$

$$R'(x) = 0 \text{ when } -8x - 80 = 0\ x = -10$$

$R'(x)$ is never undefined. Notice that $x = -10$, is in the domain. To determine the maximum revenue, we evaluate

$$R(-15) = -4(-15)^2 - 80(-15) + 14\,000$$
$$= 14\,300$$
$$R(-10) = -4(-10)^2 - 80(-10) + 14\,000$$
$$= 14\,400$$
$$R(10) = -4(10)^2 - 80(10) + 14\,000$$
$$= 12\,800$$

Therefore, the maximum revenue occurs when there are -10 fare increases of \$0.10 each, or a fare decrease of $10(0.10) = \$1.00$. At a fare of \$6.00, the daily revenue is \$14 400, and the number of passengers is $2000 - 40(-10) = 2400$.

EXAMPLE 2 Solving a problem to minimize cost

A cylindrical chemical storage tank with a capacity of 1000 m^3 is going to be constructed in a warehouse that is 12 m by 15 m, with a height of 11 m. The specifications call for the base to be made of sheet steel that costs \$100/m^2, the top to be made of sheet steel that costs \$50/m^2, and the wall to be made of sheet steel that costs \$80/m^2.

a. Determine whether it is possible for a tank of this capacity to fit in the warehouse. If it *is* possible, state the restrictions on the radius.

b. If fitting the tank in the warehouse is possible, determine the proportions that meet the conditions and that minimize the cost of the steel for construction.

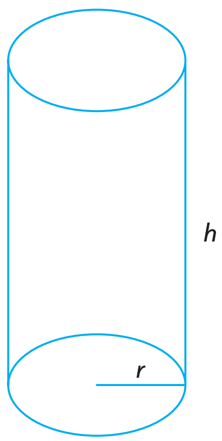
All calculations should be accurate to two decimal places.

Solution

a. The radius of the tank cannot exceed 6 m, and the maximum height is 11 m.
The volume, using $r = 6$ and $h = 11$, is $V = \pi r^2 h \doteq 1244 \text{ m}^3$.
It is possible to build a tank with a volume of 1000 m^3.
There are limits on the radius and the height. Clearly, $0 < r \leq 6$. Also, if $h = 11$, then $\pi r^2(11) \geq 1000$, so $r \geq 5.38$.
The tank can be constructed to fit in the warehouse. Its radius must be $5.38 \leq r \leq 6$.

b. If the height is h metres and the radius is r metres, then

- the cost of the base is $\$100(\pi r^2)$
- the cost of the top is $\$50(\pi r^2)$
- the cost of the wall is $\$80(2\pi rh)$



The cost of the tank is $C = 150\pi r^2 + 160\pi rh$.
Here we have two variable quantities, r and h.

However, since $V = \pi r^2 h = 1000$, $h = \frac{1000}{\pi r^2}$.
Substituting for h, we have a cost function in terms of r.

$$C(r) = 150\pi r^2 + 160\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$\text{or } C(r) = 150\pi r^2 + \frac{160\,000}{r}$$

From part a., we know that the domain is $5.38 \leq r \leq 6$.
To find points where extreme values could occur, set $C'(r) = 0$.

$$300\pi r - \frac{160\,000}{r^2} = 0$$

$$300\pi r = \frac{160\,000}{r^2}$$

$$r^3 = \frac{1600}{3\pi}$$

$$r \doteq 5.54$$

This value is within the given domain, so we use the algorithm for finding maximum and minimum values.

$$C(5.38) = 150\pi(5.38)^2 + \frac{160\,000}{5.38} \doteq 43\,380$$

$$C(5.54) = 150\pi(5.54)^2 + \frac{160\,000}{5.54} \doteq 43\,344$$

$$C(6) = 150\pi(6)^2 + \frac{160\,000}{6} \doteq 43\,631$$

The minimal cost is approximately \$43 344, with a tank of radius 5.54 m and a height of $\frac{1000}{\pi(5.54)^2} = 10.37$ m.

When solving real-life optimization problems, there are often many factors that can affect the required functions and their domains. Such factors may not be obvious from the statement of the problem. We must do research and ask many questions to address all the factors. Solving an entire problem is a series of many steps, and optimization using calculus techniques is only one step in determining a solution.

IN SUMMARY

Key Ideas

- Profit, cost, and revenue are quantities whose rates of change are measured in terms of the number of units produced or sold.
- Economic situations usually involve minimizing costs or maximizing profits.

Need to Know

- To maximize revenue, we can use the revenue function.

  revenue = total revenue from the sale of x units = (price per unit) $\times$ x.
- Practical constraints, as well as mathematical constraints, must always be considered when constructing a model.
- Once the constraints on the model have been determined—that is the domain of the function—apply the extreme value algorithm to the function over the appropriately defined domain to determine the absolute extrema.

Exercise 3.4

PART A

K 1. The cost, in dollars, to produce x litres of maple syrup for the Elmira Maple Syrup Festival is $C(x) = 75(\sqrt{x} - 10)$, where $x \geq 400$.

a. What is the average cost of producing 625 L?

b. The marginal cost is $C'(x)$, and the marginal revenue is $R'(x)$. Marginal cost at x litres is the expected change in cost if we were to produce one additional litre of syrup. Similarly for marginal revenue. What is the marginal cost at 1225 L?

c. How much production is needed to achieve a marginal cost of \$0.50/L?

2. A sociologist determines that a foreign-language student has learned $N(t) = 20t - t^2$ vocabulary terms after t hours of uninterrupted study.

   a. How many terms are learned between times $t = 2$ and $t = 3$?

   b. What is the rate, in terms per hour, at which the student is learning at time $t = 2$?

   c. What is the maximum rate, in terms per hour, at which the student is learning?

3. A researcher found that the level of antacid in a person's stomach, t minutes after a certain brand of antacid tablet is taken, is $L(t) = \frac{6t}{t^2 + 2t + 1}$.

   a. Determine the value of t for which $L'(t) = 0$.

   b. Determine $L(t)$ for the value you found in part a.

   c. Using your graphing calculator, graph $L(t)$.

   d. From the graph, what can you predict about the level of antacid in a person's stomach after 1 min?

   e. What is happening to the level of antacid in a person's stomach from $2 \le t \le 8$?

PART B

A

4. The operating cost, C, in dollars per hour, for an airplane cruising at a height of h metres and an air speed of 200 km/h is given by $C = 4000 + \frac{h}{15} + \frac{15\,000\,000}{h}$ for the domain $1000 \le h \le 20\,000$. Determine the height at which the operating cost is at a minimum, and find the operating cost per hour at this height.

5. A rectangular piece of land is to be fenced using two kinds of fencing. Two opposite sides will be fenced using standard fencing that costs \$6/m, while the other two sides will require heavy-duty fencing that costs \$9/m. What are the dimensions of the rectangular lot of greatest area that can be fenced for a cost of \$9000?

6. A real estate office manages 50 apartments in a downtown building. When the rent is \$900 per month, all the units are occupied. For every \$25 increase in rent, one unit becomes vacant. On average, all units require \$75 in maintenance and repairs each month. How much rent should the real estate office charge to maximize profits?

7. A bus service carries 10 000 people daily between Ajax and Union Station, and the company has space to serve up to 15 000 people per day. The cost to ride the bus is \$20. Market research shows that if the fare increases by \$0.50, 200 fewer people will ride the bus. What fare should be charged to get the maximum revenue, given that the bus company must have at least \$130 000 in fares a day to cover operating costs?

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8. The fuel cost per hour for running a ship is approximately one half the cube of the speed (measured in knots) plus additional fixed costs of $216 per hour. Find the most economical speed to run the ship for a 500 M (nautical mile) trip. *Note:* Assume that there are no major disturbances, such as heavy tides or stormy seas.

9. A 20 000 m^3 rectangular cistern is to be made from reinforced concrete such that the interior length will be twice the height. If the cost is $40/$m^2$ for the base, $100/$m^2$ for the side walls, and $200/$m^2$ for the roof, find the interior dimensions (to one decimal place) that will keep the cost to a minimum. To protect the water table, the building code specifies that no excavation can be more than 22 m deep. It also specifies that all cisterns must be at least 1 m deep.

C

10. The cost of producing an ordinary cylindrical tin can is determined by the materials used for the wall and the end pieces. If the end pieces are twice as expensive per square centimetre as the wall, find the dimensions (to the nearest millimetre) to make a 1000 cm^3 can at minimal cost.

11. Your neighbours operate a successful bake shop. One of their specialties is a very rich whipped-cream-covered cake. They buy the cakes from a supplier who charges $6.00 per cake, and they sell 200 cakes weekly at $10.00 each. Research shows that profit from the cake sales can be increased by increasing the price. Unfortunately, for every increase of $0.50 cents, sales will drop by seven cakes.

    a. What is the optimal retail price for a cake to obtain a maximum weekly profit?

    b. The supplier, unhappy with reduced sales, informs the owners that if they purchase fewer than 165 cakes weekly, the cost per cake will increase to $7.50. Now what is the optimal retail price per cake, and what is the bake shop's total weekly profit?

    c. Situations like this occur regularly in retail trade. Discuss the implications of reduced sales with increased total profit versus greater sales with smaller profits. For example, a drop in the number of customers could mean fewer sales of associated products.

12. Sandy is making a closed rectangular jewellery box with a square base from two different woods. The wood for the top and bottom costs $20/$m^2$. The wood for the sides costs $30/$m^2$. Find the dimensions that will minimize the cost of the wood for a volume of 4000 cm^3.

13. An electronics store is selling personal CD players. The regular price for each CD player is $90. During a typical two weeks, the store sells 50 units. Past sales indicate that for every $1 decrease in price, the store sells five more units during two weeks. Calculate the price that will maximize revenue.

14. A professional basketball team plays in an arena that holds 20 000 spectators. Average attendance at each game has been 14 000. The average ticket price is \$75. Market research shows that for each \$5 reduction in the ticket price, attendance increases by 800. Find the price that will maximize revenue.

15. Through market research, a computer manufacturer found that x thousand units of its new laptop will sell at a price of $2000 - 5x$ dollars per unit. The cost, C, in dollars to produce this many units is $C(x) = 15\,000\,000 + 1\,800\,000x + 75x^2$. Determine the level of sales that will maximize profit.

PART C

16. If the cost of producing x items is given by the function $C(x)$, and the total revenue when x items are sold is $R(x)$, then the profit function is $P(x) = R(x) - C(x)$. Show that the instantaneous rate of change in profit is 0 when the marginal revenue equals the marginal cost.

17. A fuel tank is being designed to contain 200 m^3 of gasoline, but the maximum length of a tank (measured from the tips of each hemisphere) that can be safely transported to clients is 16 m long. The design of the tank calls for a cylindrical part in the middle, with hemispheres at each end. If the hemispheres are twice as expensive per unit area as the cylindrical part, find the radius and height of the cylindrical part so the cost of manufacturing the tank will be minimal. Give your answers correct to the nearest centimetre.

18. A truck crossing the prairies at a constant speed of 110 kilometres per hour gets gas mileage of 8 kilometre per litre. Gas costs \$1.15 per litre. The truck loses 0.10 kilometres per litre in fuel efficiency for each kilometre per hour increase in speed. The driver is paid \$35 per hour in wages and benefits. Fixed costs for running the truck are \$15.50 per hour. If a trip of 450 kilometres is planned, what speed will minimize operating expenses?

19. During a cough, the diameter of the trachea decreases. The velocity, v, of air in the trachea during a cough may be modelled by the formula $v(r) = Ar^2(r_0 - r)$, where A is a constant, r is the radius of the trachea during the cough, and r_0 is the radius of the trachea in a relaxed state. Find the radius of the trachea when the velocity is the greatest, and find the associated maximum velocity of air. Note that the domain for the problem is $0 \leq r \leq r_0$.