Chapter 4

CURVE SKETCHING

If you are having trouble figuring out a mathematical relationship, what do you do? Many people find that visualizing mathematical problems is the best way to understand them and to communicate them more meaningfully. Graphing calculators and computers are powerful tools for producing visual information about functions. Similarly, since the derivative of a function at a point is the slope of the tangent to the function at this point, the derivative is also a powerful tool for providing information about the graph of a function. It should come as no surprise, then, that the Cartesian coordinate system in which we graph functions and the calculus that we use to analyze functions were invented in close succession in the seventeenth century. In this chapter, you will see how to draw the graph of a function using the methods of calculus, including the first and second derivatives of the function.

CHAPTER EXPECTATIONS

In this chapter, you will

- determine properties of the graphs of polynomial and rational functions, **Sections 4.1, 4.3, 4.5**
- describe key features of a given graph of a function, **Sections 4.1, 4.2, 4.4**
- determine intercepts and positions of the asymptotes of a graph, **Section 4.3**
- determine the values of a function near its asymptotes, **Section 4.3**
- determine key features of the graph of a function, **Section 4.5, Career Link**
- sketch, by hand, the graph of the derivative of a given graph, **Section 4.2**
- determine, from the equation of a simple combination of polynomial or rational functions $\left(\text{such as } f(x) = x^2 + \frac{1}{x}\right)$, the key features of the graph of the function, using the techniques of differential calculus, and sketch the graph by hand, **Section 4.4**

Review of Prerequisite Skills

There are many features that we can analyze to help us sketch the graph of a function. For example, we can try to determine the x- and y-intercepts of the graph, we can test for horizontal and vertical asymptotes using limits, and we can use our knowledge of certain kinds of functions to help us determine domains, ranges, and possible symmetries.

In this chapter, we will use the derivatives of functions, in conjunction with the features mentioned above, to analyze functions and their graphs. Before you begin, you should

- be able to solve simple equations and inequalities
- know how to sketch graphs of parent functions and simple transformations of these graphs (including quadratic, cubic, and root functions)
- understand the intuitive concept of a limit of a function and be able to evaluate simple limits
- be able to determine the derivatives of functions using all known rules

Exercise

1. Solve each equation.

a. $2y^2 + y - 3 = 0$

b. $x^2 - 5x + 3 = 17$

c. $4x^2 + 20x + 25 = 0$

d. $y^3 + 4y^2 + y - 6 = 0$

2. Solve each inequality.

a. $3x + 9 < 2$

b. $5(3 - x) \geq 3x - 1$

c. $t^2 - 2t < 3$

d. $x^2 + 3x - 4 > 0$

3. Sketch the graph of each function.

a. $f(x) = (x + 1)^2 - 3$

b. $f(x) = x^2 - 5x - 6$

c. $f(x) = \dfrac{2x - 4}{x + 2}$

d. $f(x) = \sqrt{x - 2}$

4. Evaluate each limit.

a. $\lim\limits_{x \to 2^-} (x^2 - 4)$

b. $\lim\limits_{x \to 2} \dfrac{x^2 + 3x - 10}{x - 2}$

c. $\lim\limits_{x \to 3} \dfrac{x^3 - 27}{x - 3}$

d. $\lim\limits_{x \to 4^+} \sqrt{2x + 1}$

5. Determine the derivative of each function.

a. $f(x) = \frac{1}{4}x^4 + 2x^3 - \frac{1}{x}$

b. $f(x) = \frac{x + 1}{x^2 - 3}$

c. $f(x) = (3x^2 - 6x)^2$

d. $f(t) = \frac{2t}{\sqrt{t - 4}}$

6. Divide, and then write your answer in the form $ax + b + \frac{r}{q(x)}$. For example, $(x^2 + 4x - 5) \div (x - 2) = x + 6 + \frac{7}{x - 2}$.

a. $(x^2 - 5x + 4) \div (x + 3)$

b. $(x^2 + 6x - 9) \div (x - 1)$

7. Determine the points where the tangent is horizontal to $f(x) = x^3 + 0.5x^2 - 2x + 3$.

8. State each differentiation rule in your own words.

a. power rule

b. constant rule

c. product rule

d. quotient rule

e. chain rule

f. power of a function rule

9. Describe the end behaviour of each function as $x \to \infty$ and $x \to -\infty$.

a. $f(x) = 2x^2 - 3x + 4$

b. $f(x) = -2x^3 + 4x - 1$

c. $f(x) = -5x^4 + 2x^3 - 6x^2 + 7x - 1$

d. $f(x) = 6x^5 - 4x - 7$

10. For each function, determine the reciprocal, $y = \frac{1}{f(x)}$, and the equations of the vertical asymptotes of $y = \frac{1}{f(x)}$. Verify your results using graphing technology.

a. $f(x) = 2x$

b. $f(x) = -x + 3$

c. $f(x) = (x + 4)^2 + 1$

d. $f(x) = (x + 3)^2$

11. State the equation of the horizontal asymptote of each function.

a. $y = \frac{5}{x + 1}$

b. $y = \frac{4x}{x - 2}$

c. $y = \frac{3x - 5}{6x - 3}$

d. $y = \frac{10x - 4}{5x}$

12. For each function in question 11, determine the following:

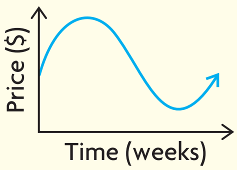
a. the x- and y-intercepts

b. the domain and range

CHAPTER 4: PREDICTING STOCK VALUES

Stock-market analysts collect and interpret vast amounts of information and then predict trends in stock values. Stock analysts are classified into two main groups: the fundamentalists who predict stock values based on analysis of the companies' economic situations, and the technical analysts who predict stock values based on trends and patterns in the market. Technical analysts spend a significant amount of their time constructing and interpreting graphical models to find undervalued stocks that will give returns in excess of what the market predicts. In this chapter, your skills in producing and analyzing graphical models will be extended through the use of differential calculus.

Case Study: Technical Stock Analyst



To raise money for expansion, many privately owned companies give the public a chance to own part of their company through purchasing stock. Those who buy ownership expect to obtain a share in the future profits of the company. Some technical analysts believe that the greatest profits to be had in the stock market are through buying brand new stocks and selling them quickly. A technical analyst predicts that a stock's price over its first several weeks on the market will follow the pattern shown on the graph. The technical analyst is advising a person who purchased the stock the day it went on sale.

DISCUSSION QUESTIONS

Make a rough sketch of the graph, and answer the following questions:

1. When would you recommend that the owner sell her shares? Label this point S on your graph. What do you notice about the slope, or instantaneous rate of change, of the graph at this point?
2. When would you recommend that the owner get back into the company and buy shares again? Label this point B on your graph. What do you notice about the slope, or instantaneous rate of change, of the graph at this point?
3. A concave-down section of a graph opens in a downward direction, and a concave-up section opens upward. On your graph, find the point where the concavity changes from concave down to concave up, and label this point C. Another analyst says that a change in concavity from concave down to concave up is a signal that a selling opportunity will soon occur. Do you agree with the analyst? Explain.

At the end of this chapter, you will have an opportunity to apply the tools of curve sketching to create, evaluate, and apply a model that could be used to advise clients on when to buy, sell, and hold new stocks.