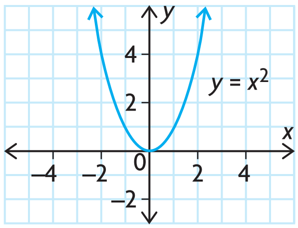
Section 4.1—Increasing and Decreasing Functions

The graph of the quadratic function $f(x) = x^2$ is a parabola. If we imagine a particle moving along this parabola from left to right, we can see that, while the x-coordinates of the ordered pairs steadily increase, the y-coordinates of the ordered pairs along the particle's path first decrease and then increase. Determining the intervals in which a function increases and decreases is extremely useful for understanding the behaviour of the function. The following statements give a clear picture:



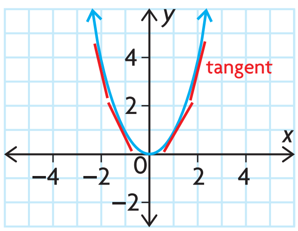
Intervals of Increase and Decrease

We say that a function f *is decreasing on an interval* if, for any value of $x_1 < x_2$ on the interval, $f(x_1) > f(x_2)$.

Similarly, we say that a function f *is increasing on an interval* if, for any value of $x_1 < x_2$ on the interval, $f(x_1) < f(x_2)$.

For the parabola with the equation $y = x^2$, the change from decreasing y-values to increasing y-values occurs at $(0, 0)$, the vertex of the parabola. The function $f(x) = x^2$ is decreasing on the interval $x < 0$ and is increasing on the interval $x > 0$.

If we examine tangents to the parabola anywhere on the interval where the y-values are decreasing (that is, on $x < 0$), we see that all of these tangents have negative slopes. Similarly, the slopes of tangents to the graph on the interval where the y-values are increasing are all positive.



For functions that are both continuous and differentiable, we can determine intervals of increasing and decreasing y-values using the derivative of the function. In the case of $y = x^2$, $\frac{dy}{dx} = 2x$. For $x < 0$, $\frac{dy}{dx} < 0$, and the slopes of the tangents are negative. The interval $x < 0$ corresponds to the decreasing portion of the graph of the parabola. For $x > 0$, $\frac{dy}{dx} > 0$, and the slopes of the tangents are positive on the interval where the graph is increasing.

We summarize this as follows: For a continuous and differentiable function, f, the function values (y-values) are increasing for all x-values where $f'(x) > 0$, and the function values (y-values) are decreasing for all x-values where $f'(x) < 0$.

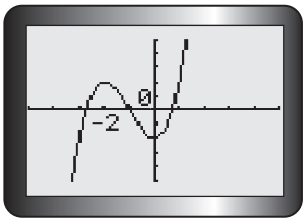
EXAMPLE 1 Using the derivative to reason about intervals of increase and decrease

Use your calculator to graph the following functions. Use the graph to estimate the values of x for which the function values (y-values) are increasing, and the values of x for which the y-values are decreasing. Verify your estimates with an algebraic solution.

a. $y = x^3 + 3x^2 - 2$　　　　b. $y = \frac{x}{x^2 + 1}$

Solution

a. Using a calculator, we obtain the graph of $y = x^3 + 3x^2 - 2$. Using the TRACE key on the calculator, we estimate that the function values are increasing on $x < -2$, decreasing on $-2 < x < 0$, and increasing again on $x > 0$. To verify these estimates with an algebraic solution, we consider the slopes of the tangents.



The slope of a general tangent to the graph of $y = x^3 + 3x^2 - 2$ is given by $\frac{dy}{dx} = 3x^2 + 6x$. We first determine the values of x for which $\frac{dy}{dx} = 0$. These values tell us where the function has a **local maximum** or **local minimum** value. These are greatest and least values respectively of a function in relation to its neighbouring values.

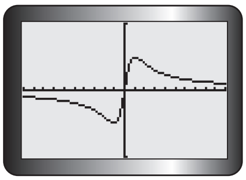
Setting $\frac{dy}{dx} = 0$, we obtain
$$3x^2 + 6x = 0$$
$$3x(x + 2) = 0$$
$$x = 0, x = -2$$

These values of x locate points on the graph where the slope of the tangent is zero (that is, where the tangent is horizontal).

Since this is a polynomial function it is continuous so $\frac{dy}{dx}$ is defined for all values of x. Because $\frac{dy}{dx} = 0$ only at $x = -2$ and $x = 0$, the derivative must be either positive or negative for all other values of x. We consider the intervals $x < -2$, $-2 < x < 0$, and $x > 0$.

Value of *x*	$x < -2$	$-2 < x < 0$	$x > 0$
Sign of $\frac{dy}{dx} = 3x(x + 2)$	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} > 0$
Slope of Tangents	positive	negative	positive
Values of *y* Increasing or Decreasing	increasing	decreasing	increasing

So $y = x^3 + 3x^2 - 2$ is increasing on the intervals $x < -2$ and $x > 0$ and is decreasing on the interval $-2 < x < 0$.



b. Using a calculator, we obtain the graph of $y = \frac{x}{x^2 + 1}$. Using the TRACE key on the calculator, we estimate that the function values (y-values) are decreasing on $x < -1$, increasing on $-1 < x < 1$, and decreasing again on $x > 1$.

We analyze the intervals of increasing/decreasing y-values for the function by determining where $\frac{dy}{dx}$ is positive and where it is negative.

$$y = \frac{x}{x^2 + 1} \qquad \text{(Express as a product)}$$

$$= x(x^2 + 1)^{-1}$$

$$\frac{dy}{dx} = 1(x^2 + 1)^{-1} + x(-1)(x^2 + 1)^{-2}(2x) \qquad \text{(Product and chain rules)}$$

$$= \frac{1}{x^2 + 1} - \frac{2x^2}{(x^2 + 1)^2}$$

$$= \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} \qquad \text{(Simplify)}$$

$$= \frac{-x^2 + 1}{(x^2 + 1)^2}$$

Setting $\frac{dy}{dx} = 0$, we obtain $\frac{-x^2 + 1}{(x^2 + 1)^2} = 0$ (Solve)

$$-x^2 + 1 = 0$$

$$x^2 = 1$$

$$x = 1 \text{ or } x = -1$$

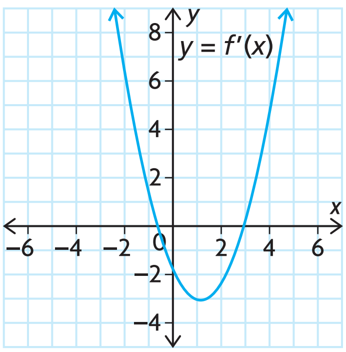
These values of x locate the points on the graph where the slope of the tangent is 0. Since the denominator of this rational function can never be 0, this function is continuous so $\frac{dy}{dx}$ is defined for all values of x. Because $\frac{dy}{dx} = 0$ at $x = -1$ and $x = 1$, we consider the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

Value of x	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Sign of $\frac{dy}{dx} = \frac{-x^2 + 1}{(x^2 + 1)^2}$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$
Slope of Tangents	negative	positive	negative
Values of y Increasing or Decreasing	decreasing	increasing	decreasing

Then $y = \frac{x}{x^2 + 1}$ is increasing on the interval $(-1, 1)$ and is decreasing on the intervals $(-\infty, -1)$ and $(1, \infty)$.

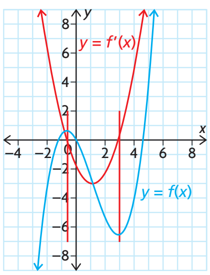
EXAMPLE 2 **Graphing a function given the graph of the derivative**

Consider the graph of $f'(x)$. Graph $f(x)$.



Solution

When the derivative, $f'(x)$, is positive, the graph of $f(x)$ is rising. When the derivative is negative, the graph is falling. In this example, the derivative changes sign from positive to negative at $x \doteq -0.6$. This indicates that the graph of $f(x)$ changes from increasing to decreasing, resulting in a local maximum for this value of x. The derivative changes sign from negative to positive at $x = 2.9$, indicating the graph of $f(x)$ changes from decreasing to increasing resulting in a local minimum for this value of x.

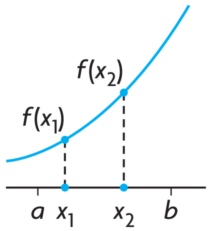


One possible graph of $f(x)$ is shown.

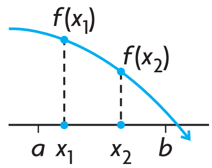
IN SUMMARY

Key Ideas

- A function f is **increasing** on an interval if, for any value of $x_1 < x_2$ in the interval, $f(x_1) < f(x_2)$.



- A function f is **decreasing** on an interval if, for any value of $x_1 < x_2$ in the interval, $f(x_1) > f(x_2)$.



- For a function f that is continuous and differentiable on an interval I
  - $f(x)$ is **increasing** on I if $f'(x) > 0$ for all values of x in I
  - $f(x)$ is **decreasing** on I if $f'(x) < 0$ for all values of x in I

Need to Know

- A function increases on an interval if the graph rises from left to right.
- A function decreases on an interval if the graph falls from left to right.
- The slope of the tangent at a point on a section of a curve that is increasing is always positive.
- The slope of the tangent at a point on a section of a curve that is decreasing is always negative.

Exercise 4.1

PART A

K 1. Determine the points at which $f'(x) = 0$ for each of the following functions:

a. $f(x) = x^3 + 6x^2 + 1$

b. $f(x) = \sqrt{x^2 + 4}$

c. $f(x) = (2x - 1)^2(x^2 - 9)$

d. $f(x) = \dfrac{5x}{x^2 + 1}$

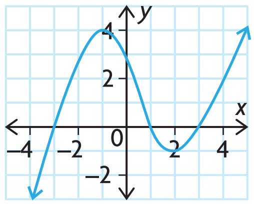
C 2. Explain how you would determine when a function is increasing or decreasing.

3. For each of the following graphs, state

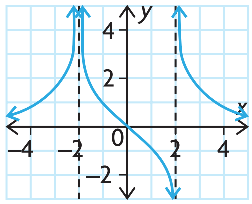
i. the intervals where the function is increasing

ii. the intervals where the function is decreasing

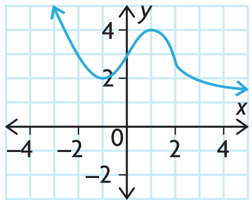
iii. the points where the tangent to the function is horizontal a.



c.



b.



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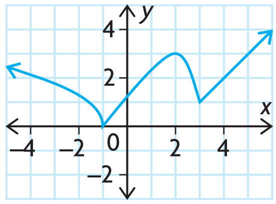


4. Use a calculator to graph each of the following functions. Inspect the graph to estimate where the function is increasing and where it is decreasing. Verify your estimates with algebraic solutions.

a. $f(x) = x^3 + 3x^2 + 1$

b. $f(x) = x^5 - 5x^4 + 100$

c. $f(x) = x + \frac{1}{x}$

d. $f(x) = \frac{x - 1}{x^2 + 3}$

e. $f(x) = 3x^4 + 4x^3 - 12x^2$

f. $f(x) = x^4 + x^2 - 1$

PART B

5. Suppose that f is a differentiable function with the derivative $f'(x) = (x - 1)(x + 2)(x + 3)$. Determine the values of x for which the function f is increasing and the values of x for which the function is decreasing.

A 6. Sketch a graph of a function that is differentiable on the interval $-2 \le x \le 5$ and that satisfies the following conditions:

- The graph of f passes through the points $(-1, 0)$ and $(2, 5)$.
- The function f is decreasing on $-2 < x < -1$, increasing on $-1 < x < 2$, and decreasing again on $2 < x < 5$.

7. Find constants a, b, and c such that the graph of $f(x) = x^3 + ax^2 + bx + c$ will increase to the point $(-3, 18)$, decrease to the point $(1, -14)$, and then continue increasing.

8. Sketch a graph of a function f that is differentiable and that satisfies the following conditions:

- $f'(x) > 0$, when $x < -5$
- $f'(x) < 0$, when $-5 < x < 1$ and when $x > 1$
- $f'(-5) = 0$ and $f'(1) = 0$
- $f(-5) = 6$ and $f(1) = 2$

9. Each of the following graphs represents the derivative function $f'(x)$ of a function $f(x)$. Determine

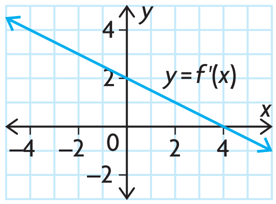
   i. the intervals where $f(x)$ is increasing

   ii. the intervals where $f(x)$ is decreasing

   iii. the x-coordinate for all local extrema of $f(x)$

   Assuming that $f(0) = 2$, make a rough sketch of the graph of each function.

a.



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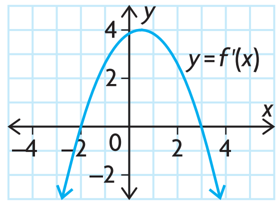
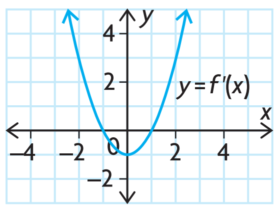
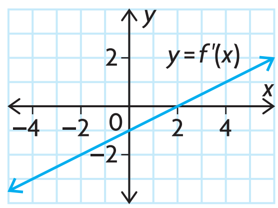


b.



d.



10. Use the derivative to show that the graph of the quadratic function $f(x) = ax^2 + bx + c$, $a > 0$, is decreasing on the interval $x < -\frac{b}{2a}$ and increasing on the interval $x > -\frac{b}{2a}$.

11. For $f(x) = x^4 - 32x + 4$, find where $f'(x) = 0$, the intervals on which the function increases and decreases, and all the local extrema. Use graphing technology to verify your results.

12. Sketch a graph of the function g that is differentiable on the interval $-2 \leq x \leq 5$, decreases on $0 < x < 3$, and increases elsewhere on the domain. The absolute maximum of g is 7, and the absolute minimum is -3. The graph of g has local extrema at $(0, 4)$ and $(3, -1)$.

PART C

T 13. Let f and g be continuous and differentiable functions on the interval $a \leq x \leq b$. If f and g are both increasing on $a \leq x \leq b$, and if $f(x) > 0$ and $g(x) > 0$ on $a \leq x \leq b$, show that the product fg is also increasing on $a \leq x \leq b$.

14. Let f and g be continuous and differentiable functions on the interval $a \leq x \leq b$. If f and g are both increasing on $a \leq x \leq b$, and if $f(x) < 0$ and $g(x) < 0$ on $a \leq x \leq b$, is the product fg increasing on $a \leq x \leq b$, decreasing, or neither?