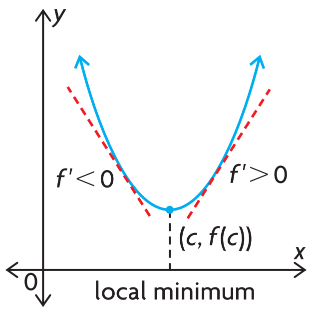
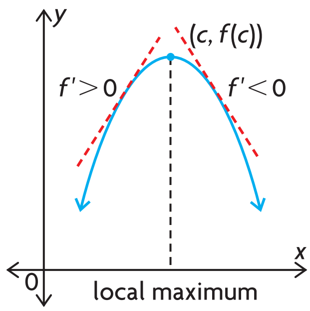
# *Section 4.2*—Critical Points, Local Maxima, and Local Minima

In Chapter 3, we learned that a maximum or minimum function value might occur at a point $(c, f(c))$ if $f'(c) = 0$. It is also possible that a maximum or minimum function value might occur at a point $(c, f(c))$ if $f'(c)$ is undefined. Since these points help to define the shape of the function's graph, they are called **critical points** and the values of $c$ are called **critical numbers**. Combining this with the properties of increasing and decreasing functions, we have a **first derivative test** for local extrema.

### The First Derivative Test

Test for local minimum and local maximum points. Let $f'(c) = 0$.




When moving left to right through $x$-values:

- if $f'(x)$ changes sign from negative to positive at $x = c$, then $f(x)$ has a local minimum at this point.
- if $f'(x)$ changes sign from positive to negative at $x = c$, then $f(x)$ has a local maximum at this point.

$f'(c) = 0$ may imply something other than the existence of a maximum or a minimum at $x = c$. There are also simple functions for which the derivative does not exist at certain points. In Chapter 2, we demonstrated three different ways that this could happen. For example, extrema could occur at points that correspond to cusps and corners on a function's graph and in these cases the derivative is undefined.

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**EXAMPLE 1** **Connecting the first derivative test to local extrema of a polynomial function**

For the function $y = x^4 - 8x^3 + 18x^2$, determine all the critical numbers. Determine whether each of these values of $x$ gives a local maximum, a local minimum, or neither for the function.

## Solution

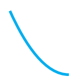
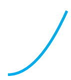
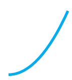
First determine $\frac{dy}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= 4x^3 - 24x^2 + 36x\\ &= 4x(x^2 - 6x + 9)\\ &= 4x(x - 3)^2\end{aligned}$$

For critical numbers, let $\frac{dy}{dx} = 0$.

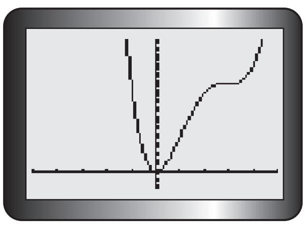
$$\begin{aligned}4x(x - 3)^2 &= 0\\ x &= 0 \text{ or } x = 3\end{aligned}$$

Both values of $x$ are in the domain of the function. There is a horizontal tangent at each of these values of $x$. To determine which of these values of $x$ yield local maximum or minimum values of the function, we use a table to analyze the behaviour of $\frac{dy}{dx}$ and $y = x^4 - 8x^3 + 18x^2$.

| **Interval** | $x < 0$ | $0 < x < 3$ | $x > 3$ |
|---|---|---|---|
| **$4x$** | $-$ | $+$ | $+$ |
| **$(x - 3)^2$** | $+$ | $+$ | $+$ |
| **$4x(x - 3)^2$** | $(-)(+) = -$ | $(+)(+) = +$ | $(+)(+) = +$ |
| **$\frac{dy}{dx}$** | $< 0$ | $> 0$ | $> 0$ |
| **$y = x^4 - 8x^3 + 18x^2$** | decreasing | increasing | increasing |
| **Shape of the Curve** |  |  |  |

Using the information from the table, we see that there is a local minimum value of the function at $x = 0$, since the function values are decreasing before $x = 0$ and increasing after $x = 0$. We can also tell that there is neither a local maximum nor minimum value at $x = 3$, since the function values increase toward this point and increase away from it.

A calculator gives the following graph for $y = x^4 - 8x^3 + 18x^2$, which verifies our solution:



## EXAMPLE 2 Reasoning about the significance of horizontal tangents

Determine whether or not the function $f(x) = x^3$ has a maximum or minimum at $(c, f(c))$, where $f'(c) = 0$.

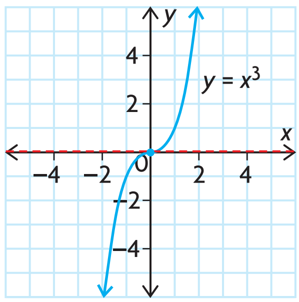
### Solution

The derivative is $f'(x) = 3x^2$.

Setting $f'(x) = 0$ gives

$$3x^2 = 0$$
$$x = 0$$

$f(x)$ has a horizontal tangent at $(0, 0)$.




From the graph, it is clear that $(0, 0)$ is neither a maximum nor a minimum value since the values of this function are always increasing. Note that $f'(x) > 0$ for all values of $x$ other than 0.

From this example, we can see that it is possible for a horizontal tangent to exist when $f'(c) = 0$, but that $(c, f(c))$ is neither a maximum nor a minimum. In the next example you will see that it is possible for a maximum or minimum to occur at a point at which the derivative does not exist.

## EXAMPLE 3 Reasoning about the significance of a cusp

For the function $f(x) = (x + 2)^{\frac{2}{3}}$, determine the critical numbers. Use your calculator to sketch a graph of the function.

### Solution

First determine $f'(x)$.

$$f'(x) = \frac{2}{3}(x + 2)^{-\frac{1}{3}}$$
$$= \frac{2}{3(x + 2)^{\frac{1}{3}}}$$

Note that there is no value of $x$ for which $f'(x) = 0$ since the numerator is always positive. However, $f'(x)$ is undefined for $x = -2$.

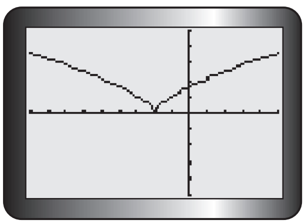
Since $f(-2) = (-2 + 2)^{\frac{2}{3}} = 0$, $x = -2$ is in the domain of $f(x) = (x + 2)^{\frac{2}{3}}$. We determine the slopes of tangents for $x$-values close to $-2$.

| $x$ | $f'(x) = \dfrac{2}{3(x+2)^{\frac{1}{3}}}$ | $x$ | $f'(x) = \dfrac{2}{3(x+2)^{\frac{1}{3}}}$ |
|---|---|---|---|
| −2.1 | −1.436 29 | −1.9 | 1.436 29 |
| −2.01 | −3.094 39 | −1.99 | 3.094 39 |
| −2.001 | −6.666 67 | −1.999 | 6.666 67 |
| −2.000 01 | −30.943 9 | −1.999 99 | 30.943 9 |

The slope of the tangent is undefined at the point $(-2, 0)$. Therefore, the function has one critical point, when $x = -2$.

In this example, the slopes of the tangents to the left of $x = -2$ are approaching $-\infty$, while the slopes to the right of $x = -2$ are approaching $+\infty$. Since the slopes on opposite sides of $x = -2$ are not approaching the same value, there is no tangent at $x = -2$ even though there is a point on the graph.

A calculator gives the following graph of $f(x) = (x + 2)^{\frac{2}{3}}$. There is a cusp at $(-2, 0)$.



If a value $c$ is in the domain of a function $f(x)$, and if this value is such that $f'(c) = 0$ or $f'(c)$ is undefined, then $(c, f(c))$ is a critical point of the function $f$ and $c$ is called a critical number for $f''$.

In summary, critical points that occur when $\frac{dy}{dx} = 0$ give the locations of horizontal tangents on the graph of a function. Critical points that occur when $\frac{dy}{dx}$ is undefined give the locations of either vertical tangents or cusps (where we say that no tangent exists). Besides giving the location of interesting tangents (or lack thereof), critical points also determine other interesting features of the graph of a function.

**Critical Numbers and Local Extrema**

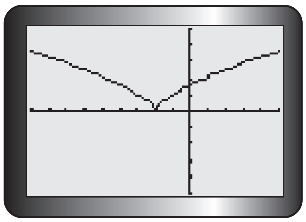
The critical number $c$ determines the location of a local minimum value for a function $f$ if $f(c) < f(x)$ for all values of $x$ near $c$.

Similarly, the critical number $c$ determines the location of a local maximum value for a function $f$ if $f(c) > f(x)$ for all values of $x$ near $c$.

Together, local maximum and minimum values of a function are called local extrema.

As mentioned earlier, a local minimum value of a function does not have to be the smallest value in the entire domain, just the smallest value in its neighbourhood. Similarly, a local maximum value of a function does not have to be the largest value in the entire domain, just the largest value in its neighbourhood. Local extrema occur graphically as peaks or valleys. The peaks and valleys can be either smooth or sharp.

To apply this reasoning, let's reconsider the graph of $f(x) = (x + 2)^{\frac{2}{3}}$.



The function $f(x) = (x + 2)^{\frac{2}{3}}$ has a local minimum value at $x = -2$, which also happens to be a critical value of the function.

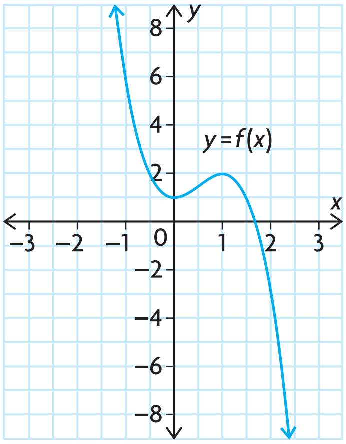
Every local maximum or minimum value of a function occurs at a critical point of the function.

In simple terms, peaks or valleys occur on the graph of a function at places where the tangent to the graph is horizontal, vertical, or does not exist.

How do we determine whether a critical point yields a local maximum or minimum value of a function without examining the graph of the function? We use the first derivative test to analyze whether the function is increasing or decreasing on either side of the critical point.

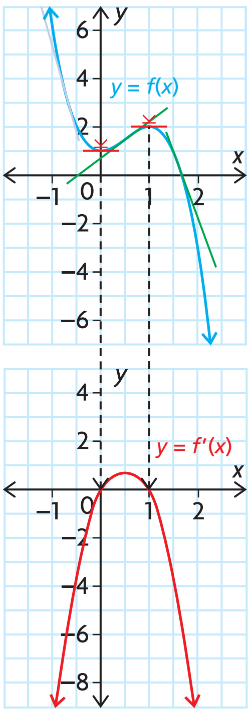
## EXAMPLE 4 Graphing the derivative given the graph of a polynomial function

Given the graph of a polynomial function $y = f(x)$, graph $y = f'(x)$.




### Solution

A polynomial function $f$ is continuous for all values of $x$ in the domain of $f$. The derivative of $f$, $f'$, is also continuous for all values of $x$ in the domain of $f$.




To graph $y = f'(x)$ using the graph of $y = f(x)$, first determine the slopes of the tangent lines, $f'(x_i)$, at certain $x$-values, $x_i$. These $x$-values include zeros, critical numbers, and numbers in each interval where $f$ is increasing or decreasing. Then plot the corresponding ordered pairs on a graph. Draw a smooth curve through these points to complete the graph.

The given graph has a local minimum at (0, 1) and a local maximum at (1, 2). At these points, the tangents are horizontal. Therefore, $f'(0) = 0$ and $f'(1) = 0$.

At $x = \frac{1}{2}$, which is halfway between $x = 0$ and $x = 1$, the slope of the tangent is about $\frac{2}{3}$. So $f'\left(\frac{1}{2}\right) \doteq \frac{2}{3}$

The function $f(x)$ is decreasing when $f'(x) < 0$. The tangent lines show that $f'(x) < 0$ when $x < 0$ and when $x > 1$. Similarly, $f(x)$ is increasing when $f'(x) > 0$. The tangent lines show that $f'(x) > 0$ when $0 < x < 1$.

The shape of the graph of $f(x)$ suggests that $f(x)$ is a cubic polynomial with a negative leading coefficient. Assume that this is true. The derivative, $f'(x)$, may be a quadratic function with a negative leading coefficient. If it is, the graph of $f'(x)$ is a parabola that opens down.

Plot (0, 0), (1, 0), and $\left(\frac{1}{2}, \frac{2}{3}\right)$ on the graph of $f'(x)$. The graph of $f'(x)$ is a parabola that opens down and passes through these points.

## IN SUMMARY

### Key Idea

For a function $f(x)$, a **critical number** is a number, $c$, in the domain of $f(x)$ such that $f'(x) = 0$ or $f'(x)$ is undefined. As a result $(c, f(c))$ is called a critical point and usually corresponds to local or absolute extrema.

### Need to Know

**First Derivative Test**

Let $c$ be a critical number of a function $f$.

When moving through $x$-values from left to right:

- if $f'(x)$ changes from negative to positive at $c$, then $(c, f(c))$ is a **local minimum** of $f$.
- if $f'(x)$ changes from positive to negative at $c$, then $(c, f(c))$ is a **local maximum** of $f$.
- if $f'(x)$ does not change its sign at $c$, then $(c, f(c))$ is neither a local minimum or a local maximum.

**Algorithm for Finding Local Maximum and Minimum Values of a Function *f***

1. Find critical numbers of the function (that is, determine where $f'(x) = 0$ and where $f'(x)$ is undefined) for all $x$-values in the domain of $f$.
2. Use the first derivative to analyze whether $f$ is increasing or decreasing on either side of each critical number.
3. Based upon your findings in step 2., conclude whether each critical number locates a local maximum value of the function $f$, a local minimum value, or neither.

# Exercise 4.2

## PART A

C

1. Explain what it means to determine the critical points of the graph of a given function.

2. a. For the function $y = x^3 - 6x^2$, explain how you would find the critical points.

   b. Determine the critical points for $y = x^3 - 6x^2$, and then sketch the graph.

3. Find the critical points for each function. Use the first derivative test to determine whether the critical point is a local maximum, local minimum, or neither.

   a. $y = x^4 - 8x^2$  b. $f(x) = \dfrac{2x}{x^2 + 9}$  c. $y = x^3 + 3x^2 + 1$

4. Find the $x$- and $y$-intercepts of each function in question 3, and then sketch the curve.

5. Determine the critical points for each function. Determine whether the critical point is a local maximum or minimum, and whether or not the tangent is parallel to the horizontal axis.

   a. $h(x) = -6x^3 + 18x^2 + 3$

   b. $g(t) = t^5 + t^3$

   c. $y = (x - 5)^{\frac{1}{3}}$

   d. $f(x) = (x^2 - 1)^{\frac{1}{3}}$

6. Use graphing technology to graph the functions in question 5 and verify your results.

## PART B

K 7. Determine the critical points for each of the following functions, and determine whether the function has a local maximum value, a local minimum value, or neither at the critical points. Sketch the graph of each function.

   a. $f(x) = -2x^2 + 8x + 13$

   b. $f(x) = \dfrac{1}{3}x^3 - 9x + 2$

   c. $f(x) = 2x^3 + 9x^2 + 12x$

   d. $f(x) = -3x^3 - 5x$

   e. $f(x) = \sqrt{x^2 - 2x + 2}$

   f. $f(x) = 3x^4 - 4x^3$

A 8. Suppose that $f$ is a differentiable function with the derivative $f'(x) = (x + 1)(x - 2)(x + 6)$. Find all the critical numbers of $f$, and determine whether each corresponds to a local maximum, a local minimum, or neither.

9. Sketch a graph of a function $f$ that is differentiable on the interval $-3 \le x \le 4$ and that satisfies the following conditions:

   - The function $f$ is decreasing on $-1 < x < 3$ and increasing elsewhere on $-3 \le x \le 4$.
   - The largest value of $f$ is 6, and the smallest value is 0.
   - The graph of $f$ has local extrema at $(-1, 6)$ and $(3, 1)$.

10. Determine values of $a$, $b$, and $c$ such that the graph of $y = ax^2 + bx + c$ has a relative maximum at $(3, 12)$ and crosses the $y$-axis at $(0, 1)$.

11. For $f(x) = x^2 + px + q$, find the values of $p$ and $q$ such that $f(1) = 5$ is an extremum of $f$ on the interval $0 \le x \le 2$. Is this extremum a maximum value or a minimum value? Explain.

12. For $f(x) = x^3 - kx$, where $k \in \mathbf{R}$, find the values of $k$ such that $f$ has

    a. no critical numbers

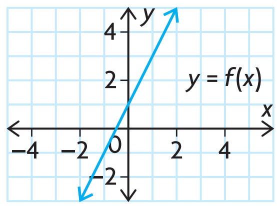
    b. one critical number

    c. two critical numbers

T 13. Find values of $a$, $b$, $c$, and $d$ such that $g(x) = ax^3 + bx^2 + cx + d$ has a local maximum at $(2, 4)$ and a local minimum at $(0, 0)$.

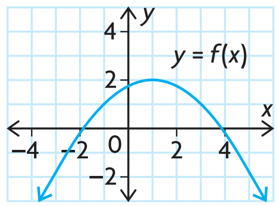
14. For each of the following graphs of the function $y = f(x)$, make a rough sketch of the derivative function $f'(x)$. By comparing the graphs of $f(x)$ and $f'(x)$, show that the intervals for which $f(x)$ is increasing correspond to the intervals where $f'(x)$ is positive. Also show that the intervals where $f(x)$ is decreasing correspond to the intervals for which $f'(x)$ is negative.

a.



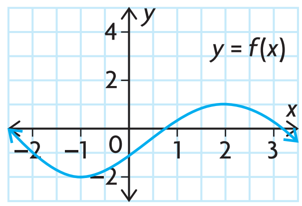

$f(x)$ is a linear function.

b.



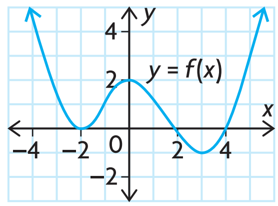

$f(x)$ is a quadratic function.

c.




$f(x)$ is a cubic function.

d.




$f(x)$ is a quartic function.

15. Consider the function $f(x) = 3x^4 + ax^3 + bx^2 + cx + d$.

    a. Find constants $a$, $b$, $c$, and $d$ such that the graph of $f$ will have horizontal tangents at $(-2, -73)$ and $(0, -9)$.

    b. There is a third point that has a horizontal tangent. Find this point.

    c. For all three points, determine whether each corresponds to a local maximum, a local minimum, or neither.

## PART C

16. For each of the following polynomials, find the local extrema and the direction that the curve is opening for $x = 100$. Use this information to make a quick sketch of the curve.

    a. $y = 4 - 3x^2 - x^4$

    b. $y = 3x^5 - 5x^3 - 30x$

17. Suppose that $f(x)$ and $g(x)$ are positive functions (functions where $f(x) > 0$ and $g(x) > 0$) such that $f(x)$ has a local maximum and $g(x)$ has a local minimum at $x = c$. Show that the function $h(x) = \frac{f(x)}{g(x)}$ has a local maximum at $x = c$.