Section 4.5—An Algorithm for Curve Sketching

You now have the necessary skills to sketch the graphs of most elementary functions. However, you might be wondering why you should spend time developing techniques for sketching graphs when you have a graphing calculator. The answer is that, in doing so, you develop an understanding of the qualitative features of the functions you are analyzing. Also, for certain functions, maximum/minimum/inflection points are not obvious if the window setting is not optimal. In this section, you will combine the skills you have developed. Some of them use the calculus properties. Others were learned earlier. Putting all the skills together will allow you to develop an approach that leads to simple, yet accurate, sketches of the graphs of functions.

An Algorithm for Sketching the Graph of $y = f(x)$

Note: As each piece of information is obtained, use it to build the sketch.

1: Determine any discontinuities or limitations in the domain. For discontinuities, investigate the function's values on either side of the discontinuity.

2: Determine any vertical asymptotes.

3: Determine any intercepts.

4: Determine any critical numbers by finding where $\frac{dy}{dx} = 0$ or where $\frac{dy}{dx}$ is undefined.

5: Determine the intervals of increase/decrease, and then test critical points to see whether they are local maxima, local minima, or neither.

6: Determine the behaviour of the function for large positive and large negative values of x. This will identify horizontal asymptotes, if they exist. Identify if the functions values approach the horizontal asymptote from above or below.

7: Determine $\frac{d^2y}{dx^2}$ and test for points of inflection using the intervals of concavity.

8: Determine any oblique asymptotes. Identify if the functions values approach the obliques asymptote from above or below.

9: Complete the sketch using the above information.

When using this algorithm, keep two things in mind:

1. You will not use all the steps in every situation. Use only the steps that are essential.

2. You are familiar with the basic shapes of many functions. Use this knowledge when possible.

INVESTIGATION

Use the algorithm for curve sketching to sketch the graph of each of the following functions. After completing your sketch, use graphing technology to verify your results.

a. $y = x^4 - 3x^2 + 2x$

b. $y = \frac{x}{x^2 - 1}$

EXAMPLE 1 Sketching an accurate graph of a polynomial function

Use the algorithm for curve sketching to sketch the graph of $f(x) = -3x^3 - 2x^2 + 5x$.

Solution

This is a polynomial function, so there are no discontinuities and no asymptotes. The domain is $\{x \in \mathbf{R}\}$. **Analyze $f(x)$.** Determine any intercepts.

x-intercept, $y = 0$

$$-3x^3 - 2x^2 + 5x = 0$$
$$-x(3x^2 + 2x - 5) = 0$$
$$-x(3x + 5)(x - 1) = 0$$
$$x = 0, x = -\frac{5}{3}, x = 1$$
$$(0, 0), \left(-\frac{5}{3}, 0\right), (1, 0)$$

y-intercept, $x = 0$

$$y = 0$$
$$(0, 0)$$

Now determine the critical points.

Analyze $f'(x)$.

$f'(x) = -9x^2 - 4x + 5$

Setting $f'(x) = 0$, we obtain

$$-9x^2 - 4x + 5 = 0$$
$$-(9x^2 + 4x - 5) = 0$$
$$-(9x - 5)(x + 1) = 0$$
$$x = \frac{5}{9} \text{ or } x = -1$$

When we sketch the function, we can use approximate values $x = 0.6$ and $y = 1.6$ for $x = \frac{5}{9}$ and $f\left(\frac{5}{9}\right)$.

Analyze $f''(x)$.

$f''(x) = -18x - 4$

At $x = \frac{5}{9}$,

$$f''\left(\frac{5}{9}\right) = -18\left(\frac{5}{9}\right) - 4 = -10 - 4 = -14 < 0$$

At $x = -1$,

$$f''(-1) = -18(-1) - 4 = 18 - 4 = 14 > 0$$

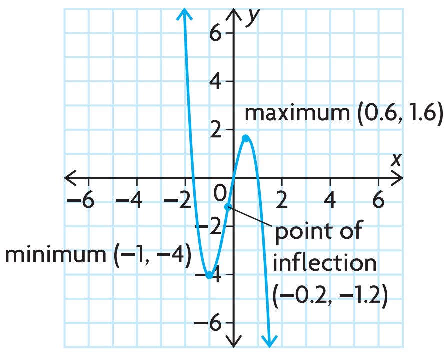
Therefore, by the second derivative test $x = \frac{5}{9}$ gives a local maximum and $x = -1$ gives a local minimum. Since this is a polynomial function, $f(x)$ must be decreasing when $x < -1$, increasing when $-1 < x < \frac{5}{9}$ and decreasing when $x > \frac{5}{9}$. For a point of inflection, $f''(x) = 0$ and changes sign.

$-18x - 4 = 0$ or $x = -\frac{2}{9}$

Now we determine the sign of $f''(x)$ in, the intervals determined by $x = -\frac{2}{9}$. A point of inflection occurs at about $(-0.2, -1.2)$.

We can now draw our sketch.

Interval	$x < -\frac{2}{9}$	$x = -\frac{2}{9}$	$x > -\frac{2}{9}$
$f''(x)$	> 0	0	< 0
Graph of $f(x)$	concave up	point of inflection	concave down



EXAMPLE 2 Sketching an accurate graph of a rational function

Sketch the graph of $f(x) = \frac{x - 4}{x^2 - x - 2}$.

Solution

Analyze $f(x)$.

$f(x)$ is a rational function.

Determine any intercepts.

x-intercept, $y = 0$	y-intercept, $x = 0$
$\frac{x - 4}{x^2 - x - 2} = 0$	$y = \frac{0 - 4}{0 - 0 - 2}$
$x - 4 = 0$	$y = \frac{-4}{-2}$
$x = 4$	$y = 2$
$(4, 0)$	$(0, 2)$

Determine any asymptotes.

The function is not defined if

$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$
$$x = 2 \text{ and } x = -1$$

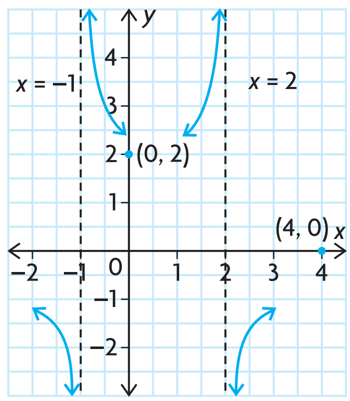
The domain is $\{x \in \mathbf{R} \mid x \neq 2 \text{ and } x \neq -1\}$.

There are vertical asymptotes at $x = 2$ and $x = -1$.

Using $f(x) = \frac{x - 4}{x^2 - x - 2}$, we examine function values near the asymptotes.

$$\lim_{x \to -1^-} f(x) = -\infty \qquad \lim_{x \to -1^+} f(x) = +\infty$$
$$\lim_{x \to 2^-} f(x) = +\infty \qquad \lim_{x \to 2^+} f(x) = -\infty$$

Sketch the information you have so far, as shown.



Analyze $f'(x)$. Now determine the critical points.

$$f(x) = (x - 4)(x^2 - x - 2)^{-1}$$
$$f'(x) = (1)(x^2 - x - 2)^{-1} + (x - 4)(-1)(x^2 - x - 2)^{-2}(2x - 1)$$
$$= \frac{1}{x^2 - x - 2} - \frac{(x - 4)(2x - 1)}{(x^2 - x - 2)^2}$$
$$= \frac{(x^2 - x - 2)}{(x^2 - x - 2)^2} - \frac{(2x^2 - 9x + 4)}{(x^2 - x - 2)^2}$$
$$= \frac{-x^2 + 8x - 6}{(x^2 - x - 2)^2}$$
$$f'(x) = 0 \text{ if } -x^2 + 8x - 6 = 0$$
$$x = \frac{8 \pm 2\sqrt{10}}{2}$$
$$x = 4 \pm \sqrt{10}$$

Since we are sketching, approximate values 7.2 and 0.8 are acceptable. These values give the approximate points (7.2, 0.1) and (0.8, 1.5).

Interval	$(-\infty, -1)$	$(-1, 0.8)$	$(0.8, 2)$	$(2, 7.2)$	$(7.2, \infty)$
$-x^2 + 8x - 6$	$-$	$-$	$+$	$+$	$-$
$(x^2 - x - 2)^2$	$+$	$+$	$+$	$+$	$+$
$f'(x)$	< 0	< 0	> 0	> 0	< 0
$f(x)$	decreasing	decreasing	increasing	increasing	decreasing

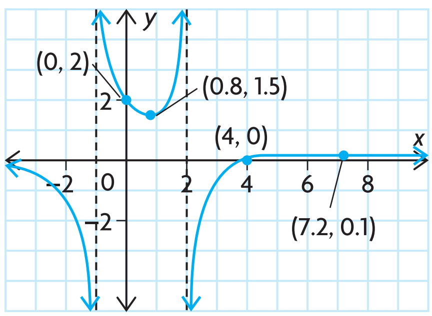
From the information obtained, we can see that $(7.2, 0.1)$ is likely a local maximum and $(0.8, 1.5)$ is likely a local minimum. To verify this using the second derivative test is a difficult computational task. Instead, verify using the first derivative test, as follows.
$x \doteq 0.8$ gives the local minimum. $x \doteq 7.2$ gives the local maximum.
Now check the end behaviour of the function.

$\lim_{x \to +\infty} f(x) = 0$ but $y > 0$ always.
$\lim_{x \to -\infty} f(x) = 0$ but $y < 0$ always.

Therefore, $y = 0$ is a horizontal asymptote. The curve approaches from above on the right and below on the left.

There is a point of inflection beyond $x = 7.2$, since the curve opens down at that point but changes as x becomes larger. The amount of work necessary to

determine the point is greater than the information we gain, so we leave it undone. (If you wish to check it, it occurs for $x \doteq 10.4$.) The finished sketch is given below and, because it is a sketch, it is not to scale.



IN SUMMARY

Key Idea

- The first and second derivatives of a function give information about the shape of the graph of the function.

Need to Know

Sketching the Graph of a Polynomial or Rational Function

1. Use the function to
   - determine the domain and any discontinuities
   - determine the intercepts
   - find any asymptotes, and determine function behaviour relative to these asymptotes
2. Use the first derivative to
   - find the critical numbers
   - determine where the function is increasing and where it is decreasing
   - identify any local maxima or minima
3. Use the second derivative to
   - determine where the graph is concave up and where it is concave down
   - find any points of inflection

The second derivative can also be used to identify local maxima and minima.

4. Calculate the values of y that correspond to critical points and points of inflection. Use the information above to sketch the graph.

Remember that you will not use all the steps in every situation! Use only the steps that are necessary to give you a good idea of what the graph will look like.

Exercise 4.5

PART A

1. If a polynomial function of degree three has a local minimum, explain how the function's values behave as $x \to +\infty$ and as $x \to -\infty$. Consider all cases.

C 2. How many local maximum and local minimum values are possible for a polynomial function of degree three, four, or n? Explain.

3. Determine whether each function has vertical asymptotes. If it does, state the equations of the asymptotes.

   a. $y = \dfrac{x}{x^2 + 4x + 3}$ b. $y = \dfrac{5x - 4}{x^2 - 6x + 12}$ c. $y = \dfrac{3x + 2}{x^2 - 6x + 9}$

PART B

K 4. Use the algorithm for curve sketching to sketch the following:

a. $y = x^3 - 9x^2 + 15x + 30$

b. $f(x) = -4x^3 + 18x^2 + 3$

c. $y = 3 + \frac{1}{(x + 2)^2}$

d. $f(x) = x^4 - 4x^3 - 8x^2 + 48x$

e. $y = \frac{2x}{x^2 - 25}$

f. $f(x) = \frac{1}{x^2 - 4x}$

g. $y = \frac{6x^2 - 2}{x^3}$

h. $f(x) = \frac{x + 3}{x^2 - 4}$

i. $y = \frac{x^2 - 3x + 6}{x - 1}$

j. $f(x) = (x - 4)^{\frac{2}{3}}$

5. Verify your results for question 4 using graphing technology.

A 6. Determine the constants a, b, c, and d so that the curve defined by $y = ax^3 + bx^2 + cx + d$ has a local maximum at the point $(2, 4)$ and a point of inflection at the origin. Sketch the curve.

7. Given the following results of the analysis of a function, sketch a possible graph for the function:

a. $f(0) = 0$, the horizontal asymptote is $y = 2$, the vertical asymptote is $x = 3$, and $f'(x) < 0$ and $f''(x) < 0$ for $x < 3$; $f'(x) < 0$ and $f''(x) > 0$ for $x > 3$.

b. $f(0) = 6, f(-2) = 0$ the horizontal asymptote is $y = 7$, the vertical asymptote is $x = -4$, and $f'(x) > 0$ and $f''(x) > 0$ for $x < -4$; $f'(x) > 0$ and $f''(x) < 0$ for $x > -4$.

PART C

8. Sketch the graph of $f(x) = \frac{k - x}{k^2 + x^2}$, where k is any positive constant.

9. Sketch the curve defined by $g(x) = x^{\frac{1}{3}}(x + 3)^{\frac{2}{3}}$.

10. Find the horizontal asymptotes for each of the following:

a. $f(x) = \frac{x}{\sqrt{x^2 + 1}}$

b. $g(t) = \sqrt{t^2 + 4t} - \sqrt{t^2 + t}$

T 11. Show that, for any cubic function of the form $y = ax^3 + bx^2 + cx + d$, there is a single point of inflection, and the slope of the curve at that point is $c - \frac{b^2}{3a}$.

CAREER LINK WRAP-UP Investigate and Apply

CHAPTER 4: PREDICTING STOCK VALUES

In the Career Link earlier in the chapter, you investigated a graphical model used to predict stock values for a new stock. A brand new stock is also called an initial public offering, or IPO. Remember that, in this model, the period immediately after the stock is issued offers excess returns on the stock—that is, the stock is selling for more than it is really worth.

One such model for a class of Internet IPOs predicts the percent overvaluation of a stock as a function of time as $R(t) = 250\left(\frac{t^2}{(2.718)^{3t}}\right)$, where $R(t)$ is the overvaluation in percent and t is the time in months after the initial issue.

a. Use the information provided by the first derivative, second derivative, and asymptotes to prepare advice for clients as to when they should expect a signal to prepare to buy or sell (inflection point), the exact time when they should buy or sell (local maximum/minimum), and any false signals prior to a horizontal asymptote. Explain your reasoning.

b. Make a sketch of the function without using a graphing calculator.