Chapter 5

DERIVATIVES OF EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

The world's population experiences exponential growth—the rate of growth becomes more rapid as the size of the population increases. Can this be explained in the language of calculus? Well, the rate of growth of the population is described by an exponential function, and the derivative of the population with respect to time is a constant multiple of the population at any time *t*. There are also many situations that can be modelled by trigonometric functions, whose derivative also provides a model for instantaneous rate of change at any time *t*. By combining the techniques in this chapter with the derivative rules seen earlier, we can find the derivative of an exponential or trigonometric function that is combined with other functions. Logarithmic functions and exponential functions are inverses of each other, and, in this chapter, you will also see how their graphs and properties are related to each other.

CHAPTER EXPECTATIONS

In this chapter, you will

- define e and the derivative of $y = e^x$, **Section 5.1**
- determine the derivative of the general exponential function $y = b^x$, **Section 5.2**
- compare the graph of an exponential function with the graph of its derivative, **Sections 5.1, 5.2**
- solve optimization problems using exponential functions, **Section 5.3**
- investigate and determine the derivatives of sinusoidal functions, **Section 5.4**
- determine the derivative of the tangent function, **Section 5.5**
- solve rate of change problems involving exponential and trigonometric function models using their derivatives, **Sections 5.1 to 5.5**

Review of Prerequisite Skills

In Chapter 5, you will be studying the derivatives of two classes of functions that occur frequently in calculus problems: exponential functions and trigonometric functions. To begin, we will review some of the properties of exponential and trigonometric functions.

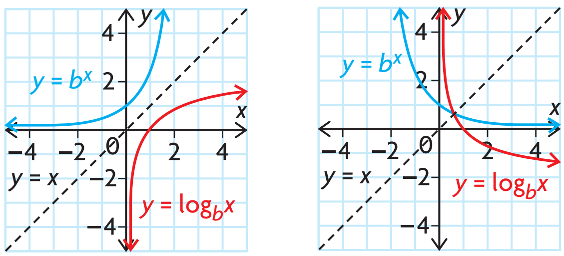
Properties of Exponents

- $b^m b^n = b^{m+n}$
- $\dfrac{b^m}{b^n} = b^{m-n}, b^n \neq 0$
- $(b^m)^n = b^{mn}$
- $b^{\log_b m} = m$
- $\log_b b^m = m$

Properties of the Exponential Function, $y = b^x$

- The base b is positive and $b \neq 1$.
- The y-intercept is 1.
- The x-axis is a horizontal asymptote.
- The domain is the set of real numbers, **R**.
- The range is the set of positive real numbers.
- The exponential function is always increasing if $b > 1$.
- The exponential function is always decreasing if $0 < b < 1$.
- The inverse of $y = b^x$ is $x = b^y$.
- The inverse is called the logarithmic function and is written as $\log_b x = y$.

Graphs of $y = \log_b x$ and $y = b^x$



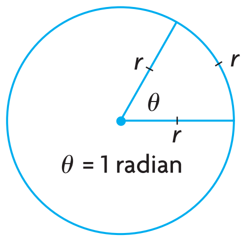
for $b > 1$ for $0 < b < 1$

- If $b^m = n$ for $b > 0$, then $\log_b n = m$.

Radian Measure

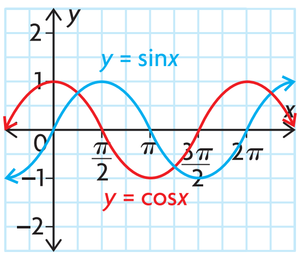
A radian is the measure of an angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.

π radians $= 180°$



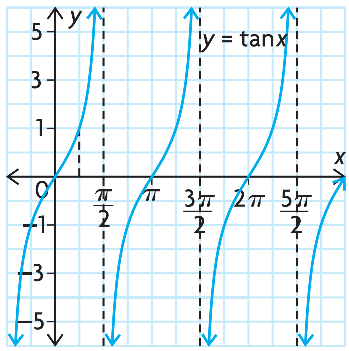
Sine and Cosine Functions
$f(x) = \sin x$ and $f(x) = \cos x$

Domain	$x \in \mathbf{R}$
Range	$-1 \le \sin x \le 1$
	$-1 \le \cos x \le 1$
Periodicity	$\sin(x + 2\pi) = \sin x$
	$\cos(x + 2\pi) = \cos x$



Tangent Function
$f(x) = \tan x$

Domain	$\left\{x \in \mathbf{R} \mid x \ne \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\right\}$
Range	$\{y \in \mathbf{R}\}$
Periodicity	$\tan(x + \pi) = \tan x$



Transformations of Sinusoidal Functions

For $y = a \sin k(x - d) + c$ and $y = a \cos k(x - d) + c$,

- the amplitude is $|a|$
- the period is $\frac{2\pi}{|k|}$
- the horizontal shift is d, and
- the vertical translation is c

Trigonometric Identities

Reciprocal Identities

$\csc\theta = \dfrac{1}{\sin\theta}$

$\sec\theta = \dfrac{1}{\cos\theta}$

$\cot\theta = \dfrac{1}{\tan\theta}$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Quotient Identities

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Reflection Identities

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

Cofunction Identities

$\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$

$\sin\left(\dfrac{\pi}{2} - x\right) = \cos x$

Exercise

1. Evaluate each of the following:

a. 3^{-2} b. $32^{\frac{2}{5}}$ c. $27^{-\frac{2}{3}}$ d. $\left(\dfrac{2}{3}\right)^{-2}$

2. Express each of the following in the equivalent logarithmic form:

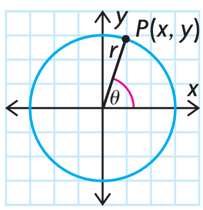
a. $5^4 = 625$ c. $x^3 = 3$ e. $3^8 = z$

b. $4^{-2} = \dfrac{1}{16}$ d. $10^w = 450$ f. $a^b = T$

3. Sketch the graph of each function, and state its x-intercept.

a. $y = \log_{10}(x + 2)$ b. $y = 5^{x+3}$

4. Refer to the following figure. State the value of each trigonometric ratio below.



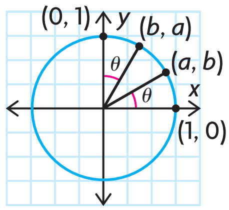
a. $\sin\theta$ b. $\cos\theta$ c. $\tan\theta$

5. Convert the following angles to radian measure:

a. $360°$ c. $-90°$ e. $270°$ g. $225°$

b. $45°$ d. $30°$ f. $-120°$ h. $330°$

6. Refer to the following figure. State the value of each trigonometric ratio below.



a. $\sin \theta$ c. $\cos \theta$ e. $\cos\left(\frac{\pi}{2} - \theta\right)$

b. $\tan \theta$ d. $\sin\left(\frac{\pi}{2} - \theta\right)$ f. $\sin(-\theta)$

7. The value of $\sin \theta$, $\cos \theta$, or $\tan \theta$ is given. Determine the values of the other two functions if θ lies in the given interval.

a. $\sin \theta = \frac{5}{13}, \frac{\pi}{2} \leq \theta \leq \pi$ c. $\tan \theta = -2, \frac{3\pi}{2} \leq \theta \leq 2\pi$

b. $\cos \theta = -\frac{2}{3}, \pi \leq \theta \leq \frac{3\pi}{2}$ d. $\sin \theta = 1, 0 \leq \theta \leq \pi$

8. State the period and amplitude of each of the following:

a. $y = \cos 2x$ d. $y = \frac{2}{7}\cos(12x)$

b. $y = 2 \sin \frac{x}{2}$ e. $y = 5 \sin\left(\theta - \frac{\pi}{6}\right)$

c. $y = -3 \sin(\pi x) + 1$ f. $y = |3 \sin x|$

9. Sketch the graph of each function over two complete periods.

a. $y = \sin 2x + 1$ b. $y = 3 \cos\left(x + \frac{\pi}{2}\right)$

10. Prove the following identities:

a. $\tan x + \cot x = \sec x \csc x$ b. $\frac{\sin x}{1 - \sin^2 x} = \tan x \sec x$

11. Solve the following equations, where $x \in [0, 2\pi]$.

a. $3 \sin x = \sin x + 1$ b. $\cos x - 1 = -\cos x$

CHAPTER 5: RATE-OF-CHANGE MODELS IN MICROBIOLOGY

While many real-life situations can be modelled fairly well by polynomial functions, there are some situations that are best modelled by other types of functions, including exponential, logarithmic, and trigonometric functions. Because determining the derivative of a polynomial function is simple, finding the rate of change for models described by polynomial functions is also simple. Often the rate of change at various times is more important to the person studying the scenario than the value of the function is. In this chapter, you will learn how to differentiate exponential and trigonometric functions, increasing the number of function types you can use to model real-life situations and, in turn, analyze using rates of change.

Case Study—Microbiologist

Microbiologists contribute their expertise to many fields, including medicine, environmental science, and biotechnology. Enumerating, the process of counting bacteria, allows microbiologists to build mathematical models that predict populations after a given amount of time has elapsed. Once they can predict a population accurately, the model can be used in medicine, for example, to predict the dose of medication required to kill a certain bacterial infection. The data in the table shown was used by a microbiologist to produce a polynomial-based mathematical model to predict population $p(t)$ as a function of time t, in hours, for the growth of a certain strain of bacteria:

$$p(t) = 1000\left(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5\right)$$

Time (h)	Population
0	1000
0.5	1649
1.0	2718
1.5	4482
2.0	7389

DISCUSSION QUESTIONS

1. How well does the function fit the data? Use the data, the equation, a graph, and/or a graphing calculator to comment on the "goodness of fit."

2. Use $p(t)$ and $p'(t)$ to determine the following:

a) the population after 0.5 h and the rate at which the population is growing at this time.

b) the population after 1.0 h and the rate at which the population is growing at this time.

3. What pattern did you notice in your calculations? Explain this pattern by examining the terms of the equation to find the reason why.

The polynomial function in this case study is an approximation of a special function in mathematics, natural science, and economics, $f(x) = e^x$, where e has a value of 2.718 28…. At the end of this chapter, you will complete a task on rates of change of exponential growth in a biotechnology case study.