Section 5.1—Derivatives of Exponential Functions, $y = e^x$

Many mathematical relations in the world are nonlinear. We have already studied various polynomial and rational functions and their rates of change. Another type of nonlinear model is the exponential function. Exponential functions are often used to model rapid change. Compound interest, population growth, the intensity of an earthquake, and radioactive decay are just a few examples of exponential change.

In this section, we will study the exponential function $y = e^x$ and its derivative. The number e is a special irrational number, like the number π. It is called the natural number, or Euler's number in honour of the Swiss mathematician Leonhard Euler (pronounced "oiler"), who lived from 1707 to 1783. We use a rational approximation for e of about 2.718. The rules developed thus far have been applied to polynomial functions and rational functions. We are now going to show how the derivative of an exponential function can be found.

INVESTIGATION

Tech *Support*

To evaluate powers of e, such as e^{-2}, press 2ND LN (−) 2 ENTER.

In this investigation, you will

- graph the exponential function $f(x) = e^x$ and its derivative
- determine the relationship between the exponential function and its derivative

A. Consider the function $f(x) = e^x$. Create a table similar to the one shown below. Complete the $f(x)$ column by using a graphing calculator to calculate the values of e^x for the values of x provided. Round all values to three decimal places.

x	$f(x)$	$f'(x)$
−2	0.135	
−1		
0		
1		
2		
3		

B. Graph the function $f(x) = e^x$.

C. Use a graphing calculator to calculate the value of the derivative $f'(x)$ at each of the given points.

To calculate $f'(x)$, press MATH and scroll down to **8:nDeriv(** under the MATH menu. Press ENTER and the display on the screen will be **nDeriv(**. To find the derivative, key in the expression e^x, the variable x, and the x-value at which you want the derivative, for example, to determine $\frac{d}{dx}(e^x)$ at $x = -2$, the display will be **nDeriv(** e^x, X, -2). Press ENTER, and the approximate value of $f'(-2)$ will be returned.

D. What do you notice about the values of $f(x)$ and $f'(x)$?

E. Draw the graph of the derivative function $f'(x)$ on the same set of axes as $f(x)$. How do the two graphs compare?

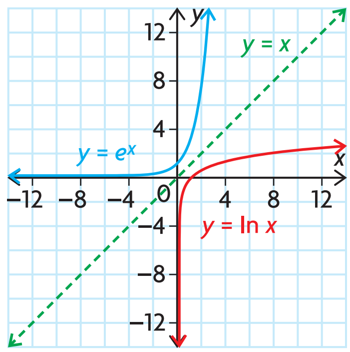
F. Try a few other values of x to see if the pattern continues.

G. What conclusion can you make about the function $f(x) = e^x$ and its derivative?

Properties of $y = e^x$

Since $y = e^x$ is an exponential function, it has the same properties as other exponential functions you have studied.

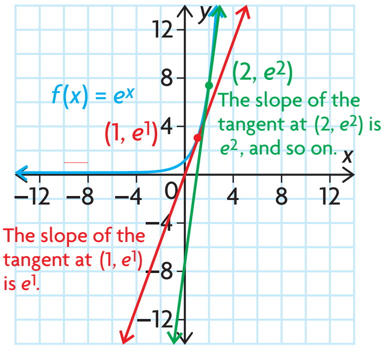
Recall that the logarithm function is the inverse of the exponential function. For example, $y = \log_2 x$ is the inverse of $y = 2^x$. The function $y = e^x$ also has an inverse, $y = \log_e x$. Their graphs are reflections in the line $y = x$. The function $y = \log_e x$ can be written as $y = \ln x$ and is called the **natural logarithm function.**



All the properties of exponential functions and logarithmic functions you are familiar with also apply to $y = e^x$ and $y = \ln x$.

$y = e^x$	$y = \ln x$
• The domain is $\{x \in \mathbf{R}\}$.	• The domain is $\{x \in \mathbf{R} \mid x > 0\}$.
• The range is $\{y \in \mathbf{R} \mid y > 0\}$.	• The range is $\{y \in \mathbf{R}\}$.
• The function passes through (0, 1).	• The function passes through (1, 0).
• $e^{\ln x} = x, x > 0$.	• $\ln e^x = x, x \in \mathbf{R}$.
• The line $y = 0$ is the horizontal asymptote.	• The line $x = 0$ is the vertical asymptote.

From the investigation, you should have noticed that all the values of the derivative $f'(x)$ were exactly the same as those of the original function $f(x) = e^x$. This is a very significant result, since this function is its own derivative—that is, $f(x) = f'(x)$. Since the derivative also represents the slope of the tangent at any given point, the function $f(x) = e^x$ has the special property that the slope of the tangent at a point is the value of the function at this point.



Derivative of $f(x) = e^x$
For the function $f(x) = e^x$, $f'(x) = e^x$.

EXAMPLE 1 Selecting a strategy to differentiate a composite function involving e^x

Determine the derivative of $f(x) = e^{3x}$.

Solution

To find the derivative, use the chain rule.

$$\begin{aligned}\frac{df(x)}{dx} &= \frac{d(e^{3x})}{d(3x)}\frac{d(3x)}{dx}\\ &= e^{3x} \times 3\\ &= 3e^{3x}\end{aligned}$$

Derivative of a Composite Function Involving e^x
In general, if $f(x) = e^{g(x)}$, then $f'(x) = e^{g(x)}\,g'(x)$ by the chain rule.

EXAMPLE 2 Derivatives of exponential functions involving e^x

Determine the derivative of each function.

a. $g(x) = e^{x^2 - x}$

b. $f(x) = x^2 e^x$

Solution

a. To find the derivative of $g(x) = e^{x^2 - x}$, we use the chain rule.

$$\begin{aligned}\frac{dg(x)}{dx} &= \frac{d(e^{x^2-x})}{dx} \\ &= \frac{d(e^{x^2-x})}{d(x^2-x)} \times \frac{d(x^2-x)}{dx} && \text{(Chain rule)} \\ &= e^{x^2-x}(2x-1)\end{aligned}$$

b. Using the product rule,

$$\begin{aligned}f'(x) &= \frac{d(x^2)}{dx} \times e^x + x^2 \times \frac{de^x}{dx} && \text{(Product rule)} \\ &= 2xe^x + x^2e^x && \text{(Factor)} \\ &= e^x(2x + x^2)\end{aligned}$$

EXAMPLE 3 Selecting a strategy to determine the value of the derivative

Given $f(x) = 3e^{x^2}$, determine $f'(-1)$.

Solution

First, find an expression for the derivative of $f'(x)$.

$$\begin{aligned}f'(x) &= \frac{d(3e^{x^2})}{d(x^2)}\frac{dx^2}{dx} && \text{(Chain rule)} \\ &= 3e^{x^2}(2x) \\ &= 6xe^{x^2}\end{aligned}$$

Then $f'(-1) = -6e$.

Answers are usually left as exact values in this form. If desired, numeric approximations can be obtained from a calculator. Here, using the value of e provided by the calculator, we obtain the answer -16.3097, rounded to four decimal places.

EXAMPLE 4 Connecting the derivative of an exponential function to the slope of a tangent

Determine the equation of the line tangent to $y = \frac{e^x}{x^2}$, where $x = 2$.

Solution

Use the derivative to determine the slope of the required tangent.

$$y = \frac{e^x}{x^2} \qquad \text{(Rewrite as a product)}$$

$$= x^{-2}e^x$$

$$\frac{dy}{dx} = (-2x^{-3})e^x + x^{-2}e^x \qquad \text{(Product rule)}$$

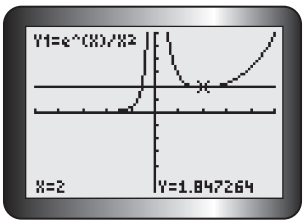
$$= \frac{-2e^x}{x^3} + \frac{e^x}{x^2} \qquad \text{(Determine a common denominator)}$$

$$= \frac{-2e^x}{x^3} + \frac{xe^x}{x^3} \qquad \text{(Simplify)}$$

$$= \frac{-2e^x + xe^x}{x^3} \qquad \text{(Factor)}$$

$$= \frac{(-2 + x)e^x}{x^3}$$

When $x = 2$, $y = \frac{e^2}{4}$. When $x = 2$, $\frac{dy}{dx} = 0$ and the tangent is horizontal. Therefore, the equation of the required tangent is $y = \frac{e^2}{4}$. A calculator yields the following graph for $y = \frac{e^x}{x^2}$, and we see the horizontal tangent at $x = 2$. The number Y = 1.847264 in the display is an approximation to the exact number $\frac{e^2}{4}$.



How does the derivative of the general exponential function $g(x) = b^x$ compare with the derivative of $f(x) = e^x$? We will answer this question in Section 5.2.

IN SUMMARY

Key Ideas

- For $f(x) = e^x$, $f'(x) = e^x$.

  In Leibniz notation, $\frac{d}{dx}(e^x) = e^x$.
- For $f(x) = e^{g(x)}$, $f'(x) = e^{g(x)} \times g'(x)$.

  In Leibniz notation, $\frac{d(e^{g(x)})}{dx} = \frac{d(e^{g(x)})}{d(g(x))}\frac{d(g(x))}{dx}$.
- The slope of the tangent at a point on the graph of $y = e^x$ equals the value of the function at this point.

Need to Know

- The rules for differentiating functions, such as the product, quotient, and chain rules, also apply to combinations involving exponential functions of the form $f(x) = e^{g(x)}$.
- e is called Euler's number or the natural number, where $e \doteq 2.718$.

Exercise 5.1

PART A

1. Why can you not use the power rule for derivatives to differentiate $y = e^x$?

2. Differentiate each of the following:

   a. $y = e^{3x}$
   b. $s = e^{3t-5}$
   c. $y = 2e^{10t}$
   d. $y = e^{-3x}$
   e. $y = e^{5-6x+x^2}$
   f. $y = e^{\sqrt{x}}$

K 3. Determine the derivative of each of the following:

   a. $y = 2e^{x^3}$
   b. $y = xe^{3x}$
   c. $f(x) = \frac{e^{-x^3}}{x}$
   d. $f(x) = \sqrt{x}e^x$
   e. $h(t) = et^2 + 3e^{-t}$
   f. $g(t) = \frac{e^{2t}}{1 + e^{2t}}$

4. a. If $f(x) = \frac{1}{3}(e^{3x} + e^{-3x})$, calculate $f'(1)$.
   b. If $f(x) = e^{-\left(\frac{1}{x+1}\right)}$, calculate $f'(0)$.
   c. If $h(z) = z^2(1 + e^{-z})$, calculate $h'(-1)$.

5. a. Determine the equation of the tangent to the curve defined by $y = \frac{2e^x}{1 + e^x}$ at the point $(0, 1)$.
   b. Use graphing technology to graph the function in part a., and draw the tangent at $(0, 1)$.
   c. Compare the equation in part a. with the equation generated by graphing technology. Do they agree?

PART B

6. Determine the equation of the tangent to the curve $y = e^{-x}$ at the point where $x = -1$. Graph the original curve and the tangent.

7. Determine the equation of the tangent to the curve defined by $y = xe^{-x}$ at the point $A(1, e^{-1})$.

8. Determine the coordinates of all points at which the tangent to the curve defined by $y = x^2e^{-x}$ is horizontal.

9. If $y = \frac{5}{2}(e^{\frac{x}{5}} + e^{-\frac{x}{5}})$, prove that $y'' = \frac{y}{25}$.

10. a. For the function $y = e^{-3x}$, determine $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and $\frac{d^3y}{dx^3}$.

    b. From the pattern in part a., state the value of $\frac{d^ny}{dx^n}$.

11. Determine the first and second derivatives of each function.

    a. $y = -3e^x$ b. $y = xe^{2x}$ c. $y = e^x(4 - x)$

A 12. The number, N, of bacteria in a culture at time t, in hours, is $N(t) = 1000\left[30 + e^{-\frac{t}{30}}\right]$

    a. What is the initial number of bacteria in the culture?

    b. Determine the rate of change in the number of bacteria at time t.

    c. How fast is the number of bacteria changing when $t = 20$?

    d. Determine the largest number of bacteria in the culture during the interval $0 \leq t \leq 50$.

    e. What is happening to the number of bacteria in the culture as time passes?

13. The distance s, in metres, fallen by a skydiver t seconds after jumping (and before the parachute opens) is $s = 160\left(\frac{1}{4}t - 1 + e^{-\frac{t}{4}}\right)$.

    a. Determine the velocity, v, at time t.

    b. Show that acceleration is given by $a = 10 - \frac{1}{4}v$.

    c. Determine $v_T = \lim_{t\to\infty} v$. This is the "terminal" velocity, the constant velocity attained when the air resistance balances the force of gravity.

    d. At what time is the velocity 95% of the terminal velocity? How far has the skydiver fallen at that time?

C 14. a. Use a table of values and successive approximation to evaluate each of the following:

    i. $\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)$ ii. $\lim_{x\to 0}(1 + x)^{\frac{1}{x}}$

    b. Discuss your results.

PART C

T 15. Use the definition of the derivative to evaluate each limit.

a. $\lim_{h \to 0} \frac{e^h - 1}{h}$

b. $\lim_{h \to 0} \frac{e^{2+h} - e^2}{h}$

16. For what values of m does the function $y = Ae^{mt}$ satisfy the following equation?

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

17. The hyperbolic functions are defined as $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

a. Prove $\frac{d(\sinh x)}{dx} = \cosh x$.

b. Prove $\frac{d(\cosh x)}{dx} = \sinh x$.

c. Prove $\frac{d(\tanh x)}{dx} = \frac{1}{(\cosh x)^2}$ if $\tanh x = \frac{\sinh x}{\cosh x}$.

Extension: Graphing the Hyperbolic Function

1. Use graphing technology to graph $y = \cosh x$ by using the definition $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

CATALOG

2. Press 2ND 0 for the list of CATALOG items, and select **cosh(** to investigate if cosh is a built-in function.
3. In the same window as problem 1, graph $y = 1.25x^2 + 1$ and $y = 1.05x^2 + 1$. Investigate changes in the coefficient a in the equation $y = ax^2 + 1$ to see if you can create a parabola that will approximate the hyperbolic cosine function.

18. a. Another expression for e is $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$. Evaluate this expression using four, five, six, and seven consecutive terms of this expression. (*Note:* 2! is read "two factorial"; $2! = 2 \times 1$ and $5! = 5 \times 4 \times 3 \times 2 \times 1$.)

b. Explain why the expression for e in part a. is a special case of $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$. What is the value of x?