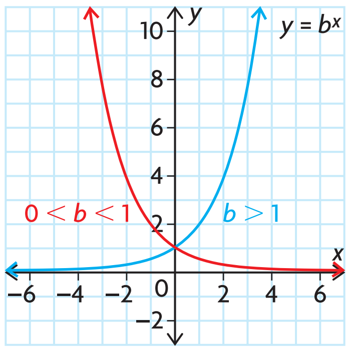
Section 5.2—The Derivative of the General Exponential Function, $y = b^x$

In the previous section, we investigated the exponential function $y = e^x$ and its derivative. The exponential function has a special property—it is its own derivative. The graph of the derivative function is the same as the graph of $y = e^x$. In this section, we will look at the general exponential function $y = b^x$ and its derivative.



INVESTIGATION

In this investigation, you will

- graph and compare the general exponential function and its derivative using the slopes of the tangents at various points and with different bases
- determine the relationship between the general exponential function and its derivative by means of a special ratio

A. Consider the function $f(x) = 2^x$. Create a table with the headings shown below. Use the equation of the function to complete the $f(x)$ column.

x	$f(x)$	$f'(x)$	$\frac{f'(x)}{f(x)}$
-2			
-1			
0			
1			
2			
3			

B. Graph the function $f(x) = 2^x$.

C. Calculate the value of the derivative $f'(x)$ at each of the given points to three decimal places. To calculate $f'(x)$, use the **nDeriv(** function. (See the investigation in Section 5.1 for detailed instructions.)

D. Draw the graph of the derivative function on the same set of axes as $f(x)$ using the given x values and the corresponding values of $f'(x)$.

E. Compare the graph of the derivative with the graph of $f(x)$.

F. i. Calculate the ratio $\frac{f'(x)}{f(x)}$, and record these values in the last column of your table.
   ii. What do you notice about this ratio for the different values of x?
   iii. Is the ratio greater or less than 1?

G. Repeat parts A to F for the function $f(x) = 3^x$.

H. Compare the ratio $\frac{f'(x)}{f(x)}$ for the functions $f(x) = 2^x$ and $f(x) = 3^x$.

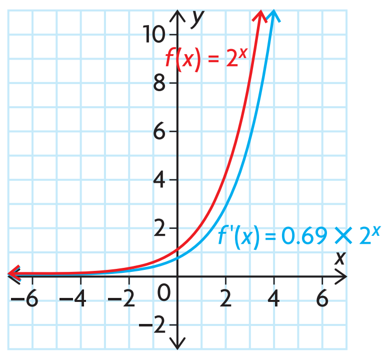
I. Repeat parts A to F for the function $f(x) = b^x$ using different values of b. Does the pattern you found for $f(x) = 2^x$ and $f(x) = 3^x$ continue?

J. What conclusions can you make about the general exponential function and its derivative?

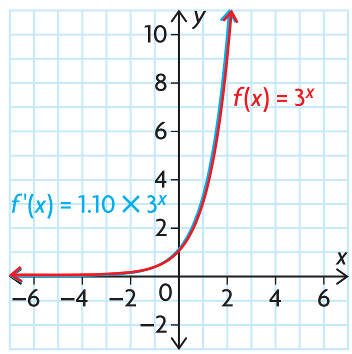
Properties of $y = b^x$

In this investigation, you worked with the functions $f(x) = 2^x$ and $f(x) = 3^x$, and their derivatives. You should have made the following observations:

- For the function $f(x) = 2^x$, the ratio $\frac{f'(x)}{f(x)}$ is approximately equal to 0.69.
- The derivative of $f(x) = 2^x$ is approximately equal to 0.69×2^x.
- For the function $f(x) = 3^x$, the ratio $\frac{f'(x)}{f(x)}$ is approximately equal to 1.10.
- The derivative of $f(x) = 3^x$ is approximately equal to 1.10×3^x.



The derivative of $f(x) = 2^x$ is an exponential function.
The graph of $f'(x)$ is a vertical compression of the graph of $f(x)$.



The derivative of $f(x) = 3^x$ is an exponential function.
The graph of $f'(x)$ is a vertical stretch of the graph of $f(x)$.

In general, for the exponential function $f(x) = b^x$, we can conclude that

- $f(x)$ and $f'(x)$ are both exponential functions
- the slope of the tangent at a point on the curve is proportional to the value of the function at this point
- $f'(x)$ is a vertical stretch or compression of $f(x)$, dependent on the value of b
- the ratio $\frac{f'(x)}{f(x)}$ is a constant and is equivalent to the stretch/compression factor

We can use the definition of the derivative to determine the derivative of the exponential function $f(x) = b^x$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} && \text{(Substitution)} \\ &= \lim_{h \to 0} \frac{b^x \times b^h - b^x}{h} && \text{(Properties of the exponential function)} \\ &= \lim_{h \to 0} \frac{b^x(b^h - 1)}{h} && \text{(Common factor)} \end{aligned}$$

The factor b^x is constant as $h \to 0$ and does not depend on h. Therefore, $f'(x) = b^x \lim_{h \to 0} \frac{b^h - 1}{h}$.

Consider the functions from our investigation:

- For $f(x) = 2^x$, we determined that $f'(x) \doteq 0.69 \times 2^x$ and so $\lim_{h \to 0} \frac{2^h - 1}{h} \doteq 0.69$.
- For $f(x) = 3^x$, we determined that $f'(x) \doteq 1.10 \times 3^x$ and so $\lim_{h \to 0} \frac{3^h - 1}{h} \doteq 1.10$.

In the previous section, for $f(x) = e^x$, we determined that $f'(x) = e^x$ and so $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

Can we find a way to determine this constant of proportionality without using a table of values?

The derivative of $f(x) = e^x$ might give us a hint at the answer to this question. From the previous section, we know that $f'(x) = 1 \times e^x$.

We also know that $\log_e e = 1$, or $\ln e = 1$.
Now consider $\ln 2$ and $\ln 3$.

$\ln 2 \doteq 0.693\ 147$ and $\ln 3 \doteq 1.098\ 612$

These match the constants $\frac{f'(x)}{f(x)}$ that we determined in our investigation.

This leads to the following conclusion:

Derivative of $f(x) = b^x$

$\lim\limits_{h \to 0} \frac{b^h - 1}{h} = \ln b$ and if $f(x) = b^x$, then $f'(x) = (\ln b) \times b^x$

EXAMPLE 1 Selecting a strategy to determine derivatives involving b^x

Determine the derivative of

a. $f(x) = 5^x$

b. $f(x) = 5^{3x-2}$

Solution

a. $f(x) = 5^x$

Use the derivative of $f(x) = b^x$.

$f'(x) = (\ln 5) \times 5^x$

b. To differentiate $f(x) = 5^{3x-2}$, use the chain rule and the derivative of $f(x) = b^x$.

$f(x) = 5^{3x-2}$

We have $f(x) = 5^{g(x)}$ with $g(x) = 3x - 2$.

Then $g'(x) = 3$

Now, $f'(x) = 5^{3x-2} \times (\ln 5) \times 3$

$= 3(5^{3x-2})\ln 5$

Derivative of $f(x) = b^{g(x)}$

For $f(x) = b^{g(x)}$, $f'(x) = b^{g(x)}(\ln b)(g'(x))$

EXAMPLE 2 Solving a problem involving an exponential model

On January 1, 1850, the population of Goldrushtown was 50 000. The size of the population since then can be modelled by the function $P(t) = 50\ 000(0.98)^t$, where t is the number of years since January 1, 1850.

a. What was the population of Goldrushtown on January 1, 1900?

b. At what rate was the population of Goldrushtown changing on January 1, 1900? Was it increasing or decreasing at that time?

Solution

a. January 1, 1900, is exactly 50 years after January 1, 1850, so we let $t = 50$.

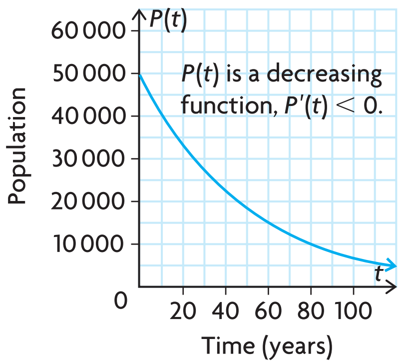
$$P(50) = 50\,000(0.98)^{50}$$
$$= 18\,208.484$$

The population on January 1, 1900, was approximately 18 208.

b. To determine the rate of change in the population, we require the derivative of P.

$$P'(t) = 50\,000(0.98)^t \ln(0.98)$$
$$P'(50) = 50\,000(0.98)^{50} \ln(0.98)$$
$$\doteq -367.861$$

Hence, after 50 years, the population was decreasing at a rate of approximately 368 people per year. (We expected the rate of change to be negative, because the original population function was a decaying exponential function since the base was less than 1.)



IN SUMMARY

Key Ideas

- If $f(x) = b^x$, then $f'(x) = b^x \times \ln b$.

  In Leibniz notation, $\frac{d}{dx}(b^x) = b^x \times \ln b$.

- If $f(x) = b^{g(x)}$, then $f'(x) = b^{g(x)} \times \ln b \times g'(x)$.

  In Leibniz notation, $\frac{d}{dx}(b^{g(x)}) = \frac{d(b^{g(x)})}{d(g(x))} \frac{d(g(x))}{dx}$.

Need to Know

- $\lim_{h \to 0} \frac{b^h - 1}{h} = \ln b$
- When you are differentiating a function that involves an exponential function, use the rules given above, along with the sum, difference, product, quotient, and chain rules as required.

Exercise 5.2

PART A

K 1. Differentiate each of the following functions:

a. $y = 2^{3x}$

b. $y = 3.1^x + x^3$

c. $s = 10^{3t-5}$

d. $w = 10^{(5-6n+n^2)}$

e. $y = 3^{x^2+2}$

f. $y = 400(2)^{x+3}$

2. Determine the derivative of each function.

a. $y = x^5 \times (5)^x$

b. $y = x(3)^{x^2}$

c. $v = \dfrac{2^t}{t}$

d. $f(x) = \dfrac{\sqrt{3^x}}{x^2}$

3. If $f(t) = 10^{3t-5} \times e^{2t^2}$, determine the values of t so that $f'(t) = 0$.

PART B

4. Determine the equation of the tangent to $y = 3(2^x)$ at $x = 3$.

5. Determine the equation of the tangent to $y = 10^x$ at $(1, 10)$.

A 6. A certain radioactive material decays exponentially. The percent, P, of the material left after t years is given by $P(t) = 100(1.2)^{-t}$.

a. Determine the half-life of the substance.

b. How fast is the substance decaying at the point where the half-life is reached?

T 7. Historical data show that the amount of money sent out of Canada for interest and dividend payments during the period from 1967 to 1979 can be approximated by the model $P = (5 \times 10^8)e^{0.200\,15t}$, where t is measured in years $(t = 0$ in $1967)$ and P is the total payment in Canadian dollars.

a. Determine and compare the rates of increase for the years 1968 and 1978.

b. Assuming this trend continues, compare the rate of increase for 1988 with the rate of increase for 1998.

c. Check the Statistics Canada website to see if the rates of increase predicted by this model were accurate for 1988 and 1998.

8. Determine the equation of the tangent to the curve $y = 2^{-x^2}$ at the point on the curve where $x = 0$. Graph the curve and the tangent at this point.

PART C

C 9. The velocity of a car is given by $v(t) = 120(1 - 0.85^t)$. Graph the function. Describe the acceleration of the car.