Cumulative Review of Calculus

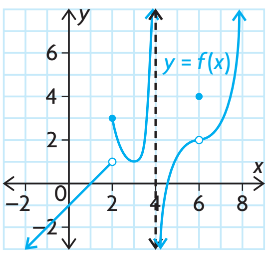
1. Using the limit definition of the slope of a tangent, determine the slope of the tangent to each curve at the given point.
   a. $f(x) = 3x^2 + 4x - 5, (2, 15)$
   b. $f(x) = \dfrac{2}{x - 1}, (2, 2)$
   c. $f(x) = \sqrt{x + 3}, (6, 3)$
   d. $f(x) = 2^{5x}, (1, 32)$

2. The position, in metres, of an object is given by $s(t) = 2t^2 + 3t + 1$, where t is the time in seconds.
   a. Determine the average velocity from $t = 1$ to $t = 4$.
   b. Determine the instantaneous velocity at $t = 3$.

3. If $\lim\limits_{h\to 0} \dfrac{(4 + h)^3 - 64}{h}$ represents the slope of the tangent to $y = f(x)$ at $x = 4$, what is the equation of $f(x)$?

4. An object is dropped from the observation deck of the Skylon Tower in Niagara Falls, Ontario. The distance, in metres, from the deck at t seconds is given by $d(t) = 4.9t^2$.
   a. Determine the average rate of change in distance with respect to time from $t = 1$ to $t = 3$.
   b. Determine the instantaneous rate of change in distance with respect to time at 2 s.
   c. The height of the observation deck is 146.9 m. How fast is the object moving when it hits the ground?

5. The model $P(t) = 2t^2 + 3t + 1$ estimates the population of fish in a reservoir, where P represents the population, in thousands, and t is the number of years since 2000.
   a. Determine the average rate of population change between 2000 and 2008.
   b. Estimate the rate at which the population was changing at the start of 2005.



6. a. Given the graph of $f(x)$ at the left, determine the following:
      i. $f(2)$
      ii. $\lim\limits_{x\to 2^-} f(x)$
      iii. $\lim\limits_{x\to 2^+} f(x)$
      iv. $\lim\limits_{x\to 6} f(x)$
   b. Does $\lim\limits_{x\to 4} f(x)$ exist? Justify your answer.

7. Consider the following function:

$$f(x) = \begin{cases} x^2 + 1, \text{ if } x < 2 \\ 2x + 1, \text{ if } x = 2 \\ -x + 5, \text{ if } x > 2 \end{cases}$$

Determine where $f(x)$ is discontinuous, and justify your answer.

8. Use algebraic methods to evaluate each limit (if it exists).

a. $\lim_{x \to 0} \frac{2x^2 + 1}{x - 5}$

b. $\lim_{x \to 3} \frac{x - 3}{\sqrt{x + 6} - 3}$

c. $\lim_{x \to -3} \frac{\frac{1}{x} + \frac{1}{3}}{x + 3}$

d. $\lim_{x \to 2} \frac{x^2 - 4}{x^2 - x - 2}$

e. $\lim_{x \to 2} \frac{x - 2}{x^3 - 8}$

f. $\lim_{x \to 0} \frac{\sqrt{x + 4} - \sqrt{4 - x}}{x}$

9. Determine the derivative of each function from first principles.

a. $f(x) = 3x^2 + x + 1$

b. $f(x) = \frac{1}{x}$

10. Determine the derivative of each function.

a. $y = x^3 - 4x^2 + 5x + 2$

b. $y = \sqrt{2x^3 + 1}$

c. $y = \frac{2x}{x + 3}$

d. $y = (x^2 + 3)^2(4x^5 + 5x + 1)$

e. $y = \frac{(4x^2 + 1)^5}{(3x - 2)^3}$

f. $y = [x^2 + (2x + 1)^3]^5$

11. Determine the equation of the tangent to $y = \frac{18}{(x + 2)^2}$ at the point $(1, 2)$.

12. Determine the slope of the tangent to $y = x^2 + 9x + 9$ at the point where the curve intersects the line $y = 3x$.

13. In 1980, the population of Littletown, Ontario, was 1100. After a time t, in years, the population was given by $p(t) = 2t^2 + 6t + 1100$.

a. Determine $p'(t)$, the function that describes the rate of change of the population at time t.

b. Determine the rate of change of the population at the start of 1990.

c. At the beginning of what year was the rate of change of the population 110 people per year?

14. Determine f' and f'' for each function.

a. $f(x) = x^5 - 5x^3 + x + 12$

b. $f(x) = \frac{-2}{x^2}$

c. $f(x) = \frac{4}{\sqrt{x}}$

d. $f(x) = x^4 - \frac{1}{x^4}$

15. Determine the extreme values of each function on the given interval.

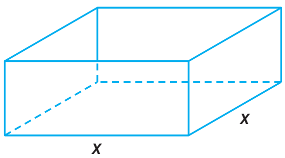
a. $f(x) = 1 + (x + 3)^2, -2 \le x \le 6$

b. $f(x) = x + \frac{1}{\sqrt{x}}, 1 \le x \le 9$

c. $f(x) = \frac{e^x}{1 + e^x}, x \in [0, 4]$

d. $f(x) = 2 \sin 4x + 3, x \in [0, \pi]$

16. The position, at time t, in seconds, of an object moving along a line is given by $s(t) = 3t^3 - 40.5t^2 + 162t$ for $0 \leq t \leq 8$.
    a. Determine the velocity and the acceleration at any time t.
    b. When is the object stationary? When is it advancing? When is it retreating?
    c. At what time, t, is the velocity not changing?
    d. At what time, t, is the velocity decreasing?
    e. At what time, t, is the velocity increasing?
17. A farmer has 750 m of fencing. The farmer wants to enclose a rectangular area on all four sides, and then divide it into four pens of equal size with the fencing parallel to one side of the rectangle. What is the largest possible area of each of the four pens?
18. A cylindrical metal can is made to hold 500 mL of soup. Determine the dimensions of the can that will minimize the amount of metal required. (Assume that the top and sides of the can are made from metal of the same thickness.)
19. A cylindrical container, with a volume of 4000 cm^3, is being constructed to hold candies. The cost of the base and lid is $0.005/$cm^2$, and the cost of the side walls is $0.0025/$cm^2$. Determine the dimensions of the cheapest possible container.
20. An open rectangular box has a square base, with each side measuring x centimetres.



    a. If the length, width, and depth have a sum of 140 cm, find the depth in terms of x.
    b. Determine the maximum possible volume you could have when constructing a box with these specifications. Then determine the dimensions that produce this maximum volume.
21. The price of x MP3 players is $p(x) = 50 - x^2$, where $x \in \mathbf{N}$. If the total revenue, $R(x)$, is given by $R(x) = xp(x)$, determine the value of x that corresponds to the maximum possible total revenue.
22. An express railroad train between two cities carries 10 000 passengers per year for a one-way fare of $50. If the fare goes up, ridership will decrease because more people will drive. It is estimated that each $10 increase in the fare will result in 1000 fewer passengers per year. What fare will maximize revenue?
23. A travel agent currently has 80 people signed up for a tour. The price of a ticket is $5000 per person. The agency has chartered a plane seating 150 people at a cost of $250 000. Additional costs to the agency are incidental fees of $300 per person. For each $30 that the price is lowered, one new person will sign up. How much should the price per person be lowered to maximize the profit for the agency?

24. For each function, determine the derivative, all the critical numbers, and the intervals of increase and decrease.

a. $y = -5x^2 + 20x + 2$

b. $y = 6x^2 + 16x - 40$

c. $y = 2x^3 - 24x$

d. $y = \dfrac{x}{x - 2}$

25. For each of the following, determine the equations of any horizontal, vertical, or oblique asymptotes and all local extrema:

a. $y = \dfrac{8}{x^2 - 9}$

b. $y = \dfrac{4x^3}{x^2 - 1}$

26. Use the algorithm for curve sketching to sketch the graph of each function.

a. $f(x) = 4x^3 + 6x^2 - 24x - 2$

b. $y = \dfrac{3x}{x^2 - 4}$

27. Determine the derivative of each function.

a. $f(x) = (-4)e^{5x+1}$

b. $f(x) = xe^{3x}$

c. $y = 6^{3x-8}$

d. $y = e^{\sin x}$

28. Determine the equation of the tangent to the curve $y = e^{2x-1}$ at $x = 1$.

29. In a research laboratory, a dish of bacteria is infected with a particular disease. The equation $N(d) = (15d)e^{-\frac{d}{5}}$ models the number of bacteria, N, that will be infected after d days.

a. How many days will pass before the maximum number of bacteria will be infected?

b. Determine the maximum number of bacteria that will be infected.

30. Determine the derivative of each function.

a. $y = 2 \sin x - 3 \cos 5x$

b. $y = (\sin 2x + 1)^4$

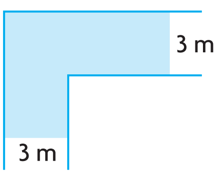
c. $y = \sqrt{x^2 + \sin 3x}$

d. $y = \dfrac{\sin x}{\cos x + 2}$

e. $y = \tan x^2 - \tan^2 x$

f. $y = \sin(\cos x^2)$

31. A tool shed, 250 cm high and 100 cm deep, is built against a wall. Calculate the shortest ladder that can reach from the ground, over the shed, to the wall behind.



32. A corridor that is 3 m wide makes a right-angle turn, as shown on the left. Find the longest rod that can be carried horizontally around this corner. Round your answer to the nearest tenth of a metre.