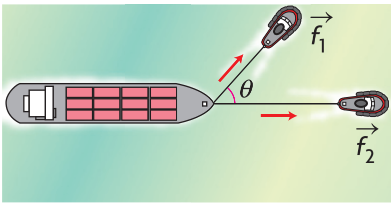
Section 6.2—Vector Addition

In this section, we will examine ways that vectors can be used in different physical situations. We will consider a variety of contexts and use them to help develop rules for the application of vectors.

Examining Vector Addition

Suppose that a cargo ship has a mechanical problem and must be towed into port by two tugboats. This situation is represented in the following diagram.



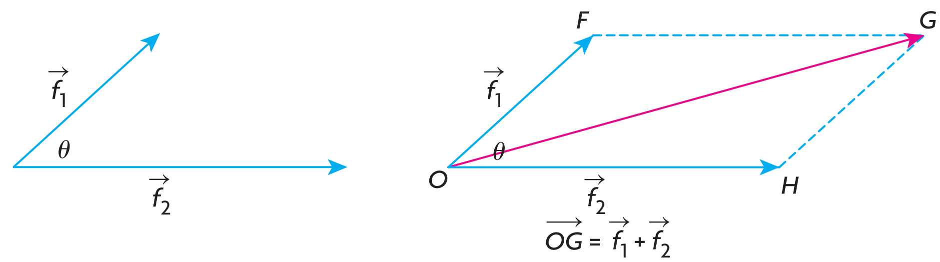
The force exerted by the first tugboat is denoted by $\vec{f_1}$ and that of the second tugboat as $\vec{f_2}$. They are denoted as vectors because these forces have both magnitude and direction. θ is the angle between the two forces shown in the diagram, where the vectors are placed tail to tail.

In considering this situation, a number of assumptions have been made:

1. The direction of the force exerted by each of the tugboats is indicated by the direction of the arrows.
2. The magnitude of the force exerted by each of the two tugboats is proportional to the length of the corresponding force vector. This means that the longer the force vector, the greater the exerted force.
3. The forces that have been exerted have been applied at a common point on the ship.

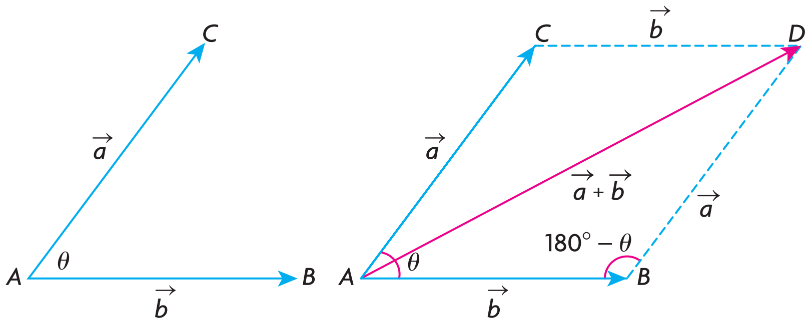
What we want to know is whether we can predict the direction the ship will move and with what force. Intuitively, we know that the ship will move in a direction somewhere between the direction of the forces, but because $|\vec{f_2}| > |\vec{f_1}|$ (the magnitude of the second force is greater than that of the first force), the boat should move closer to the direction of $\vec{f_2}$ rather than $\vec{f_1}$. The combined magnitude of the two forces should be greater than either of $|\vec{f_1}|$ or $|\vec{f_2}|$ but not equal to their sum, because they are pulling at an angle of θ to each other, i.e. they are not pulling in exactly the same direction.
There are several other observations to be made in this situation. The actions of the two tug boats are going to pull the ship in a way that combines the force vectors. The ship is going to be towed in a constant direction with a certain force, which, in effect, means the two smaller force vectors can be replaced with just

one vector. To find this single vector to replace $\vec{f_1}$ and $\vec{f_2}$, the parallelogram determined by these vectors is constructed. The main diagonal of the parallelogram is called the **resultant** or sum of these two vectors and represents the combined effect of the two vectors. The resultant of $\vec{f_1}$ and $\vec{f_2}$ has been shown in the following diagram as the diagonal, $\overrightarrow{OG}$, of the parallelogram.



The length (or magnitude) of each vector representing a force is proportional to the actual force exerted. After the tugboats exert their forces, the ship will head in the direction of $\overrightarrow{OG}$ with a force proportional to the length of $\overrightarrow{OG}$.

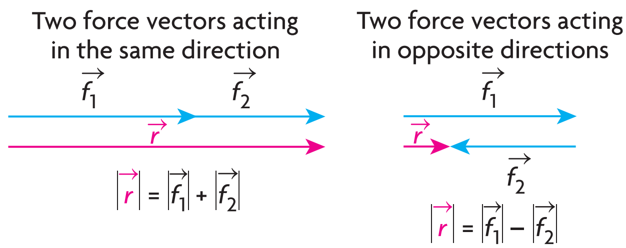
The Parallelogram Law for Adding Two Vectors



To determine the sum of the two vectors $\vec{a}$ and $\vec{b}$, complete the parallelogram formed by these two vectors when placed tail to tail. Their sum is the vector $\overrightarrow{AD}$, the diagonal of the constructed parallelogram,
$\vec{a} + \vec{b} = \overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$.

Consider the triangle formed by vectors $\vec{a}$, $\vec{b}$ and $\vec{a} + \vec{b}$. It is important to note that $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$. This means that the magnitude of the sum $\vec{a} + \vec{b}$ is less than or equal to the combined magnitudes of $\vec{a}$ and $\vec{b}$. The magnitude of $\vec{a} + \vec{b}$ is equal to the sum of the magnitudes of $\vec{a}$ and $\vec{b}$ only when these three vectors lie in the same direction.

In the tugboat example, this means the overall effect of the two tugboats is less than the sum of their individual efforts. If the tugs pulled in the same direction, the overall magnitude would be equal to the sum of their individual magnitudes. If they pulled in opposite directions, the overall magnitude would be their difference.

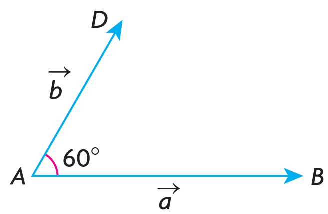


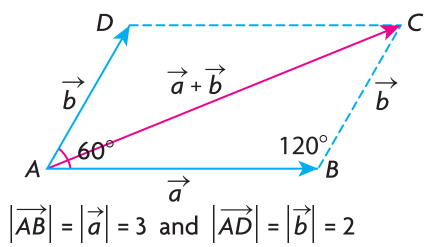
EXAMPLE 1 Selecting a strategy to determine the magnitude of a resultant vector

Given vectors $\vec{a}$ and $\vec{b}$ such that the angle between the two vectors is 60°, $|\vec{a}| = 3$, and $|\vec{b}| = 2$, determine $|\vec{a} + \vec{b}|$.

Solution

If it is stated that the angle between the vectors is θ, this means that the vectors are placed tail to tail and the angle between the vectors is θ. In this problem, the angle between the vectors is given to be 60°, so the vectors are placed tail to tail as shown.





To calculate the value of $|\vec{a} + \vec{b}|$, draw the diagonal of the related parallelogram. From the diagram, $\overrightarrow{AB} + \overrightarrow{BC} = \vec{a} + \vec{b} = \overrightarrow{AC}$. Note that the angle between $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is 120°, the supplement of 60°.

$$\text{Now, } |\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos(\angle ABC) \qquad \text{(Cosine law)}$$

$$|\vec{a} + \vec{b}|^2 = 3^2 + 2^2 - 2(3)(2)\cos 120° \qquad \text{(Substitution)}$$

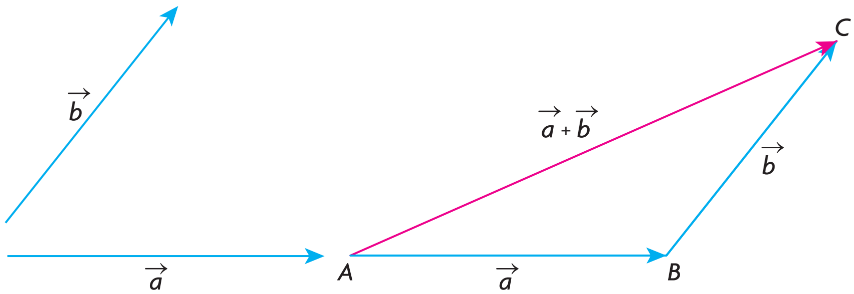
$$|\vec{a} + \vec{b}|^2 = 13 - 2(3)(2)\left(\frac{-1}{2}\right)$$

$$|\vec{a} + \vec{b}|^2 = 19$$

Therefore, $|\vec{a} + \vec{b}| = \sqrt{19} \doteq 4.36$.

When finding the sum of two or more vectors, it is not necessary to draw a parallelogram each time. In the following, we show how to add vectors using the triangle law of addition.

Triangle Law of Addition

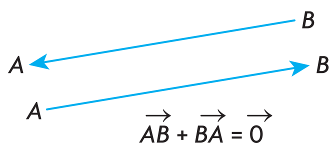


In the diagram, the sum of the vectors $\vec{a}$ and $\vec{b}$, $\vec{a} + \vec{b}$, is found by translating the tail of vector $\vec{b}$ to the head of vector $\vec{a}$. This could also have been done by translating $\vec{a}$ so that its tail was at the head of $\vec{b}$. In either case, the sum of the vectors $\vec{a}$ and $\vec{b}$ is $\overrightarrow{AC}$.

A way of thinking about the sum of two vectors is to imagine that, if you start at point A and walk to point B and then to C, you end up in the exact location as if you walked directly from point A to C. Thus, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

The Zero Vector

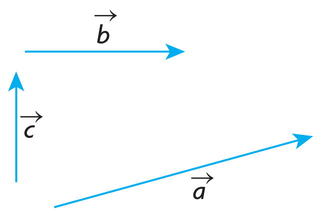
An observation that comes directly from the triangle law of addition is that when two opposite vectors are added, the resultant is the zero vector. This means that the combined effect of a vector and its opposite is the zero vector. In symbols, $\overrightarrow{AB} + \overrightarrow{BA} = \vec{0}$.



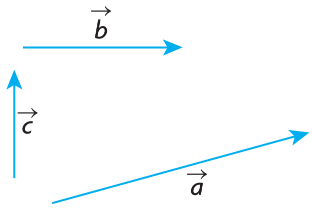
The zero vector has a magnitude of 0, i.e., $|\vec{0}| = 0$, and no defined direction.

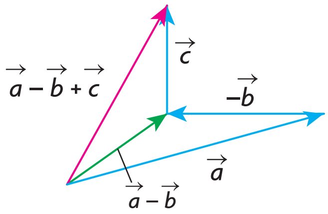
EXAMPLE 2 Representing a combination of three vectors using the triangle law of addition

Suppose you are given the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ as shown below. Using these three vectors, sketch $\vec{a} - \vec{b} + \vec{c}$.



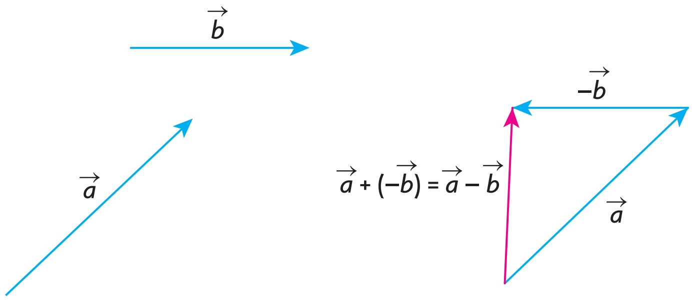
Solution





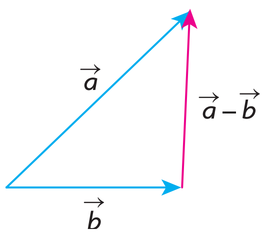
To draw the required vector, first draw $-\vec{b}$, the opposite of $\vec{b}$, and then place the vectors head to tail as shown. It should be emphasized that $\vec{a} - \vec{b}$ actually means $\vec{a} + (-\vec{b})$. Note that the required resultant vector $\vec{a} - \vec{b} + \vec{c}$ is also the resultant vector of $(\vec{a} - \vec{b}) + \vec{c}$ by the triangle law of addition.

The Difference of Two Vectors, $\vec{a} - \vec{b}$



In the diagram above, the difference between vectors $\vec{a}$ and $\vec{b}$ is found by adding the opposite vector $\vec{b}$ to $\vec{a}$ using the triangle law of addition.

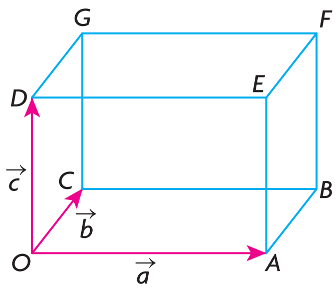
Another way to think about $\vec{a} - \vec{b}$ is to arrange the vectors tail to tail. In this case, $\vec{a} - \vec{b}$ is the vector that must be added to $\vec{b}$ to get $\vec{a}$. This is illustrated in the following diagram. Using the vectors above, the difference vector is the same as the one produced by adding the opposite.



The concept of addition and subtraction is applied in Example 3.

EXAMPLE 3 Representing a single vector as a combination of vectors

In the rectangular box shown below, $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OC} = \vec{b}$, and $\overrightarrow{OD} = \vec{c}$.



Express each of the following vectors in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

a. $\overrightarrow{BC}$ b. $\overrightarrow{GF}$ c. $\overrightarrow{OB}$ d. $\overrightarrow{AC}$ e. $\overrightarrow{BG}$ f. $\overrightarrow{OF}$

Solution

a. $\overrightarrow{BC}$ is the opposite of $\vec{a}$, so $\overrightarrow{BC} = -\vec{a}$.

b. $\overrightarrow{GF}$ is the same as $\vec{a}$, so $\overrightarrow{GF} = \vec{a}$.

c. In rectangle $OABC$, $\overrightarrow{OB}$ is the diagonal of the rectangle, so $\overrightarrow{OB} = \vec{a} + \vec{b}$.

d. Since $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$ and $\overrightarrow{AO} = -\vec{a}$, $\overrightarrow{AC} = -\vec{a} + \vec{b}$ or $\overrightarrow{AC} = \vec{b} - \vec{a}$.

e. Since $\overrightarrow{BG} = \overrightarrow{BC} + \overrightarrow{CG}$, $\overrightarrow{BC} = -\vec{a}$, and $\overrightarrow{CG} = \vec{c}$, $\overrightarrow{BG} = -\vec{a} + \vec{c}$ or $\overrightarrow{BG} = \vec{c} - \vec{a}$.

f. Since $\overrightarrow{OF} = \overrightarrow{OB} + \overrightarrow{BF}$, $\overrightarrow{OB} = \vec{a} + \vec{b}$, and $\overrightarrow{BF} = \vec{c}$, $\overrightarrow{OF} = (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + \vec{b} + \vec{c}$.

In the next example, we demonstrate how vectors might be used in a situation involving velocity.

EXAMPLE 4 Solving a problem using vectors

An airplane heads due south at a speed of 300 km/h and meets a wind from the west at 100 km/h. What is the resultant velocity of the airplane (relative to the ground)?

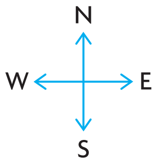
Solution

Let $\vec{v}$ represent the air speed of the airplane (velocity of the airplane without the wind).

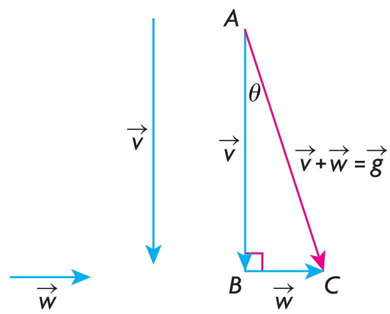
Let $\vec{w}$ represent the velocity of the wind.

Let $\vec{g}$ represent the ground speed of the airplane (the resultant velocity of the airplane with the wind taken into account relative to a fixed point on the ground).

The vectors are drawn so that their lengths are proportionate to their speed. That is to say, $|\vec{v}| = 3|\vec{w}|$.



In order to calculate the resultant ground velocity, the vectors are arranged head-to-tail as shown. The resultant ground velocity, $\vec{v} + \vec{w}$, is the hypotenuse of a right triangle.



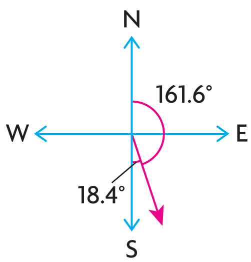
So, $|\vec{g}|^2 = |\vec{v} + \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$

$$|\vec{v} + \vec{w}|^2 = 300^2 + 100^2$$
$$|\vec{v} + \vec{w}|^2 = 100\,000$$
$$|\vec{v} + \vec{w}| = \sqrt{100\,000}$$
$$\doteq 316.23$$

Since we are calculating the resultant ground velocity, we must also determine the new direction of the airplane. To do so, we must determine θ.

Thus, $\tan\theta = \frac{|\vec{w}|}{|\vec{v}|} = \frac{100}{300} = \frac{1}{3}$ and $\theta = \tan^{-1}\left(\frac{1}{3}\right) \doteq 18.4°$.

This means that the airplane is heading S18.4°E at a speed of 316.23 km/h. The wind has not only thrown the airplane off course, but it has also caused it to speed up. When we say the new direction of the airplane is S18.4°E, this means that the airplane is travelling in a south direction, 18.4° toward the east. This is illustrated in the following diagram.



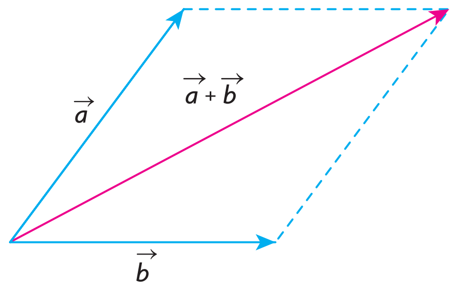
Other ways of stating this would be E71.6°S or a **bearing** of 161.6° (i.e., 161.6° rotated clockwise from due North).

In calculating the velocity of an object, such as an airplane, the velocity must always be calculated relative to some fixed object or some frame of reference. For example, if you are walking forward in an airplane at 5 km/h, your velocity relative to the airplane is 5 km/h in the same direction as the airplane, but relative to the ground, your velocity is 5 km/h in the same direction as the airplane plus the velocity of the airplane relative to the ground. If the airplane is travelling forward at 800 km/h, then your velocity relative to the ground is 805 km/h in the same direction as the airplane. In our example, the velocities given are measured relative to the ground, as is the final velocity. This is often referred to as the ground velocity.

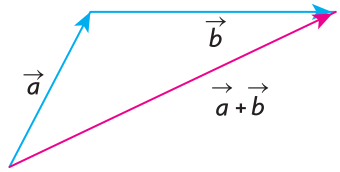
IN SUMMARY

Key Ideas

- To determine the sum of any two vectors $\vec{a}$ and $\vec{b}$, arranged tail-to-tail, complete the parallelogram formed by the two vectors. Their sum is the vector that is the diagonal of the constructed parallelogram.

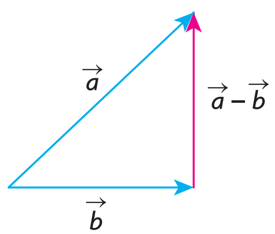


- The sum of the vectors $\vec{a}$ and $\vec{b}$ is also found by translating the tail of vector $\vec{b}$ to the head of vector $\vec{a}$. The resultant is the vector from the tail of $\vec{a}$ to the head of $\vec{b}$.



Need to Know

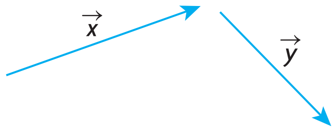
- When two opposite vectors are added, the resultant is the zero vector.
- The zero vector has a magnitude of 0 and no defined direction.
- To think about $\vec{a}-\vec{b}$, arrange the vectors tail to tail. $\vec{a}-\vec{b}$ is the vector that must be added to $\vec{b}$ to get $\vec{a}$. This is the vector from the head of $\vec{b}$ to the head of $\vec{a}$. This vector is also equivalent to $\vec{a}+\left(-\vec{b}\right)$.



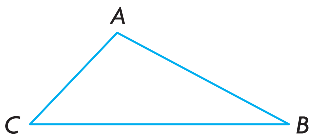
Exercise 6.2

PART A

1. The vectors $\vec{x}$ and $\vec{y}$ are drawn as shown below. Draw a vector equivalent to each of the following.



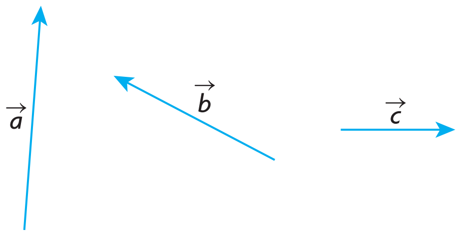
a. $\vec{x} + \vec{y}$ b. $\vec{x} - \vec{y}$ c. $\vec{y} - \vec{x}$ d. $(-\vec{y}) + (-\vec{x})$



2. Given $\triangle ABC$, draw appropriate arrows on the sides of the triangle and give a single vector equivalent for each of the following:

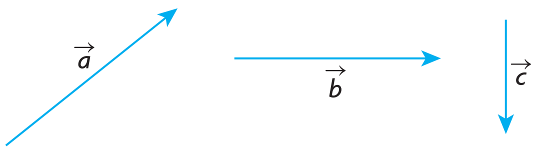
a. $\overrightarrow{BC} + \overrightarrow{CA}$ b. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$ c. $\overrightarrow{AB} - \overrightarrow{AC}$ d. $-\overrightarrow{BC} + \overrightarrow{BA}$

3. Given the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, construct vectors equivalent to each of the following.



a. $\vec{a} + \vec{b} + \vec{c}$ b. $\vec{a} + \vec{b} - \vec{c}$ c. $\vec{a} - \vec{b} - \vec{c}$ d. $\vec{a} - \vec{b} + \vec{c}$

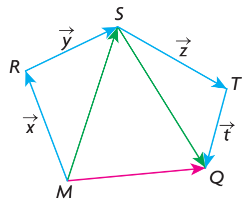
4. Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are as shown.

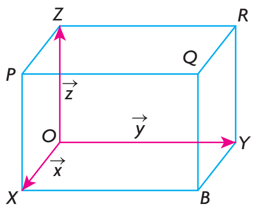


a. Construct $\vec{a} + \left(\vec{b} + \vec{c}\right)$.
b. Construct $\left(\vec{a} + \vec{b}\right) + \vec{c}$.
c. Compare your results from parts a. and b.

5. Each of the following vector expressions can be simplified and written as a single vector. Write the single vector corresponding to each expression and illustrate your answer with a sketch.
   a. $\overrightarrow{PQ} - \overrightarrow{RQ} + \overrightarrow{RS}$
   b. $\overrightarrow{PS} + \overrightarrow{RQ} - \overrightarrow{RS} - \overrightarrow{PQ}$

C 6. Explain why $(\vec{x} + \vec{y}) + (\vec{z} + \vec{t})$ equals $\overrightarrow{MQ}$ in the following diagram.

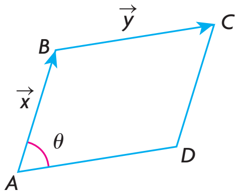




7. The rectangular box shown is labelled with $\overrightarrow{OX} = \vec{x}$, $\overrightarrow{OY} = \vec{y}$, and $\overrightarrow{OZ} = \vec{z}$.

   Express each of the following vectors in terms of $\vec{x}$, $\vec{y}$, and $\vec{z}$.
   a. $\overrightarrow{BY}$ b. $\overrightarrow{XB}$ c. $\overrightarrow{OB}$ d. $\overrightarrow{XY}$ e. $\overrightarrow{OQ}$ f. $\overrightarrow{QZ}$ g. $\overrightarrow{XR}$ h. $\overrightarrow{PO}$

8. In the diagram, $\vec{x}$ and $\vec{y}$ represent adjacent sides of a parallelogram.



   a. Draw vectors that are equivalent to $\vec{x} - \vec{y}$ and $\vec{y} - \vec{x}$.
   b. To calculate $|\vec{x} - \vec{y}|$, the formula $|\vec{x} - \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 - 2|\vec{x}||\vec{y}|\cos\theta$ is used. Show, by drawing the vector $\vec{y} - \vec{x}$, that the formula for calculating $|\vec{y} - \vec{x}|$ is the same.

PART B

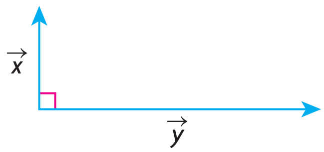
9. In still water, Maria can paddle at the rate of 7 km/h. The current in which she paddles has a speed of 4 km/h.
   a. At what velocity does she travel downstream?
   b. Using vectors, draw a diagram that illustrates her velocity going downstream.
   c. If Maria changes her direction and heads upstream instead, what is her speed? Using vectors, draw a diagram that illustrates her velocity going upstream.

10. a. In the example involving a ship being towed by the two tugboats, draw $\vec{f_1}, \vec{f_2}, \theta$, and $\vec{f_1} + \vec{f_2}$.

b. Show that $|\vec{f_1} + \vec{f_2}| = \sqrt{|\vec{f_1}|^2 + |\vec{f_2}| + 2|\vec{f_1}||\vec{f_2}|\cos\theta}$.

A 11. A small airplane is flying due north at 150 km/h when it encounters a wind of 80 km/h from the east. What is the resultant ground velocity of the airplane?

K 12. $|\vec{x}| = 7$ and $|\vec{y}| = 24$. If the angle between these vectors is 90°, determine $|\vec{x} + \vec{y}|$ and calculate the angle between $\vec{x}$ and $\vec{x} + \vec{y}$.

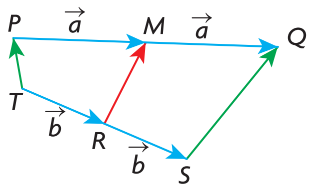


13. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are two unit vectors (vectors with magnitude 1) with an angle of 150° between them. Calculate $|\overrightarrow{AB} + \overrightarrow{AC}|$.

T 14. $ABCD$ is a parallelogram whose diagonals BD and AC meet at the point E. Prove that $\overrightarrow{EA} + \overrightarrow{EB} + \overrightarrow{EC} + \overrightarrow{ED} = \vec{0}$.

PART C

15. M is the midpoint of line segment PQ, and R is the midpoint of TS. If $\overrightarrow{PM} = \overrightarrow{MQ} = \vec{a}$ and $\overrightarrow{TR} = \overrightarrow{RS} = \vec{b}$, as shown, prove that $2\overrightarrow{RM} = \overrightarrow{TP} + \overrightarrow{SQ}$.



16. Two nonzero vectors, $\vec{a}$ and $\vec{b}$, are such that $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$. Show that $\vec{a}$ and $\vec{b}$ must represent the sides of a rectangle.

17. The three medians of $\triangle PQR$ meet at a common point G. The point G divides each median in a 2:1 ratio. Prove that $\overrightarrow{GP} + \overrightarrow{GQ} + \overrightarrow{GR} = \vec{0}$.