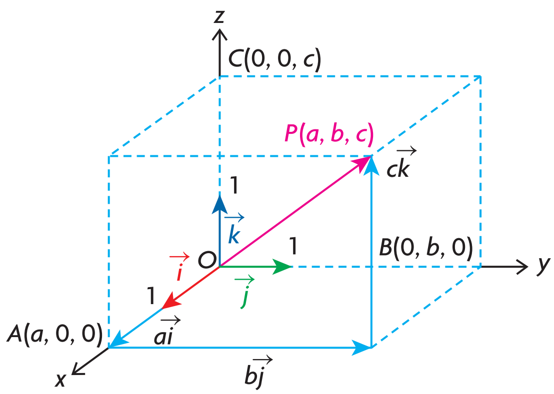
# *Section 6.7*—Operations With Vectors in $R^3$

The most important applications of vectors occur in $R^3$. In this section, results will be developed that will allow us to begin to apply ideas in $R^3$.

## Defining a Vector in $R^3$ in Terms of Unit Vectors

In $R^2$, the vectors $\vec{i}$ and $\vec{j}$ were chosen as basis vectors. In $R^3$, the vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ were chosen as basis vectors. These are vectors that each have magnitude 1, but now we introduce $\vec{k}$ as a vector that lies along the positive $z$-axis. If we use the same reasoning applied for two dimensions, then it can be seen that each vector $\overrightarrow{OP} = (a, b, c)$ can be written as $\overrightarrow{OP} = a\vec{i} + b\vec{j} + c\vec{k}$. Each of the vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ are shown below, as well as $\overrightarrow{OP} = (a, b, c)$.




From the diagram, $\overrightarrow{OA} = a\vec{i}$, $\overrightarrow{OB} = b\vec{j}$, and $\overrightarrow{OC} = c\vec{k}$. Using the triangle law of addition, $\overrightarrow{OP} = a\vec{i} + b\vec{j} + c\vec{k}$. Since $\overrightarrow{OP} = (a, b, c)$, we conclude that $\overrightarrow{OP} = a\vec{i} + b\vec{j} + c\vec{k} = (a, b, c)$. This result is analogous to the result derived for $R^2$.

**Representation of Vectors in $R^3$**

The position vector, $\overrightarrow{OP}$, whose tail is at the origin and whose head is located at point $P$, can be represented as either $\overrightarrow{OP} = (a, b, c)$ or $\overrightarrow{OP} = a\vec{i} + b\vec{j} + c\vec{k}$, where $O(0, 0, 0)$ is the origin, $P(a, b, c)$ is a point in $R^3$, and $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the standard unit vectors along the $x$-, $y$- and $z$- axes, respectively. This means that $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, and $\vec{k} = (0, 0, 1)$. Every vector in $R^3$ can be expressed uniquely in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

## EXAMPLE 1 Representing vectors in $R^3$ in two equivalent forms

a. Write each of the vectors $\overrightarrow{OP} = (2, 1, -3)$, $\overrightarrow{OQ} = (-3, 1, -5)$, $\overrightarrow{OR} = (0, -2, 0)$, and $\overrightarrow{OS} = (3, 0, 0)$ using the standard unit vectors.

b. Express each of the following vectors in component form: $\overrightarrow{OP} = \vec{i} - 2\vec{j} - \vec{k}$, $\overrightarrow{OS} = 3\vec{k}$, $\overrightarrow{OM} = 2\vec{i} - 6\vec{k}$, and $\overrightarrow{ON} = \vec{i} - \vec{j} - 7\vec{k}$.

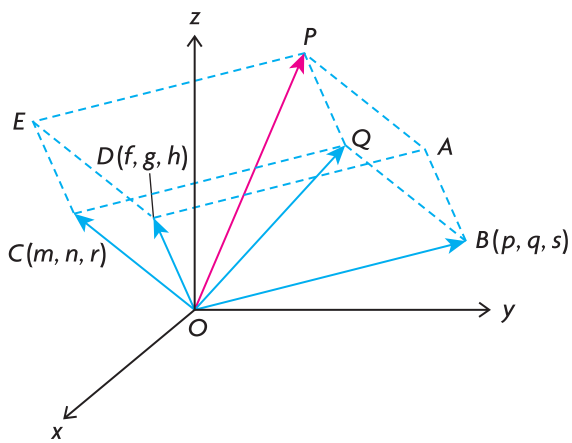
### Solution

a. $\overrightarrow{OP} = 2\vec{i} + \vec{j} - 3\vec{k}$, $\overrightarrow{OQ} = -3\vec{i} + \vec{j} - 5\vec{k}$, $\overrightarrow{OR} = -2\vec{j}$, and $\overrightarrow{OS} = 3\vec{i}$

b. $\overrightarrow{OP} = (1, -2, -1)$, $\overrightarrow{OS} = (0, 0, 3)$, $\overrightarrow{OM} = (2, 0, -6)$, and $\overrightarrow{ON} = (1, -1, -7)$

In $R^2$, we showed how to add two algebraic vectors. The result in $R^3$ is analogous to this result.

### Addition of Three Vectors in $R^3$




Writing each of the three given position vectors in terms of the standard basis vectors, $\overrightarrow{OB} = p\vec{i} + q\vec{j} + s\vec{k}$, $\overrightarrow{OC} = m\vec{i} + n\vec{j} + r\vec{k}$, and $\overrightarrow{OD} = f\vec{i} + g\vec{j} + h\vec{k}$.

Using the parallelogram law of addition, $\overrightarrow{OP} = \overrightarrow{OD} + \overrightarrow{OQ}$ and $\overrightarrow{OQ} = \overrightarrow{OB} + \overrightarrow{OC}$.

Substituting, $\overrightarrow{OP} = \overrightarrow{OD} + (\overrightarrow{OB} + \overrightarrow{OC})$.

Therefore,

$$\begin{aligned}
\overrightarrow{OP} &= (f\vec{i} + g\vec{j} + h\vec{k}) + ((p\vec{i} + q\vec{j} + s\vec{k}) + (m\vec{i} + n\vec{j} + r\vec{k})) && \text{(Commutative and associative properties of vector addition)}\\
&= (f\vec{i} + p\vec{i} + m\vec{i}) + (g\vec{j} + q\vec{j} + n\vec{j}) + (h\vec{k} + s\vec{k} + r\vec{k}) \\
&= (f + p + m)\vec{i} + (g + q + n)\vec{j} + (h + s + r)\vec{k} && \text{(Distributive property of scalars)}\\
&= (f + p + m, g + q + n, h + s + r)
\end{aligned}$$

This result demonstrates that the method for adding algebraic vectors in $R^3$ is the same as in $R^2$. Adding two vectors means adding their respective components. It should also be noted that the result for the subtraction of vectors in $R^3$ is analogous to the result in $R^2$. If $\overrightarrow{OA} = (a_1, a_2, a_3)$ and $\overrightarrow{OB} = (b_1, b_2, b_3)$, then $\overrightarrow{OA} - \overrightarrow{OB} = (a_1, a_2, a_3) - (b_1, b_2, b_3) = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$.

In $R^3$, the shape that was used to generate the result for the addition of three vectors was not a parallelogram but a *parallelepiped*, which is a box-like shape with pairs of opposite faces being identical parallelograms. From our diagram, it can be seen that parallelograms *ODAB* and *CEPQ* are copies of each other. It is also interesting to note that the parallelepiped is completely determined by the components of the three position vectors $\overrightarrow{OB}$, $\overrightarrow{OC}$, and $\overrightarrow{OD}$. That is to say, the coordinates of *all* the vertices of the parallelepiped can be determined by the repeated application of the Triangle Law of Addition.

For vectors in $R^2$, we showed that the multiplication of an algebraic vector by a scalar was produced by multiplying each component of the vector by the scalar. In $R^3$, this result also holds, i.e., $m\overrightarrow{OP} = m(a, b, c) = (ma, mb, mc), m \in \mathbf{R}$.

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## EXAMPLE 2 Selecting a strategy to determine a combination of vectors in $R^3$

Given $\vec{a} = -\vec{i} + 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{j} - 3\vec{k}$, and $\vec{c} = \vec{i} - 3\vec{j} + 2\vec{k}$, determine each of the following:

a. $2\vec{a} - \vec{b} + \vec{c}$ b. $\vec{a} + \vec{b} + \vec{c}$

### Solution

a. *Method 1* (Standard Unit Vectors)

$$\begin{aligned} 2\vec{a} - \vec{b} + \vec{c} &= 2(-\vec{i} + 2\vec{j} + \vec{k}) - (2\vec{j} - 3\vec{k}) + (\vec{i} - 3\vec{j} + 2\vec{k}) \\ &= -2\vec{i} + 4\vec{j} + 2\vec{k} - 2\vec{j} + 3\vec{k} + \vec{i} - 3\vec{j} + 2\vec{k} \\ &= -2\vec{i} + \vec{i} + 4\vec{j} - 2\vec{j} - 3\vec{j} + 2\vec{k} + 3\vec{k} + 2\vec{k} \\ &= -\vec{i} - \vec{j} + 7\vec{k} \\ &= (-1, -1, 7) \end{aligned}$$

*Method 2* (Components)

Converting to component form, we have $\vec{a} = (-1, 2, 1)$, $\vec{b} = (0, 2, -3)$, and $\vec{c} = (1, -3, 2)$.

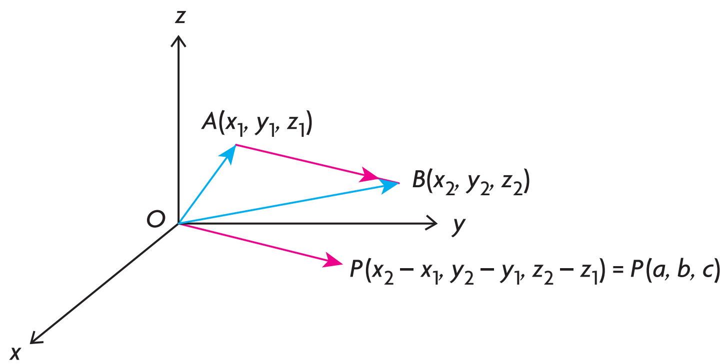
$$\begin{aligned} \text{Therefore, } 2\vec{a} - \vec{b} + \vec{c} &= 2(-1, 2, 1) - (0, 2, -3) + (1, -3, 2) \\ &= (-2, 4, 2) + (0, -2, 3) + (1, -3, 2) \\ &= (-2 + 0 + 1, 4 - 2 - 3, 2 + 3 + 2) \\ &= (-1, -1, 7) \\ &= -\vec{i} - \vec{j} + 7\vec{k} \end{aligned}$$

b. Using components, $\vec{a} + \vec{b} + \vec{c} = (-1, 2, 1) + (0, 2, -3) + (1, -3, 2)$

$$= (-1 + 0 + 1, 2 + 2 + (-3), 1 + (-3) + 2)$$
$$= (0, 1, 0) = \vec{j}$$

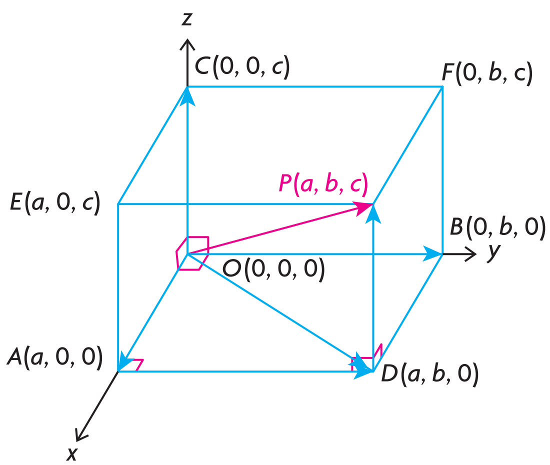
## Vectors in $R^3$ Defined by Two Points

Position vectors and their magnitude in $R^3$ are calculated in a manner similar to $R^2$.




To determine the components of $\overrightarrow{AB}$, the same method is used in $R^3$ as was used in $R^2$, i.e., $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$, which implies that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, or, in component form, $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$. Thus, the components of the algebraic vector $\overrightarrow{AB}$ can be found by subtracting the coordinates of point $A$ from the coordinates of point $B$.

If $P$ has coordinates $(a, b, c)$, we can calculate the magnitude of $\overrightarrow{OP}$.




From the diagram, we first note that $|\overrightarrow{OA}| = |a|$, $|\overrightarrow{OB}| = |b|$, and $|\overrightarrow{OC}| = |c|$. We also observe, using the Pythagorean theorem, that $|\overrightarrow{OP}|^2 = |\overrightarrow{OD}|^2 + |\overrightarrow{OC}|^2$ and, since $|\overrightarrow{OD}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2$, substitution gives $|\overrightarrow{OP}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 + |\overrightarrow{OC}|^2$.

Writing this expression in its more familiar coordinate form, we get $|\overrightarrow{OP}|^2 = |a|^2 + |b|^2 + |c|^2$ or $|\overrightarrow{OP}| = \sqrt{|a|^2 + |b|^2 + |c|^2}$. The use of the absolute value signs in the formula guarantees that the components are positive before they are squared. Because squaring components guarantees the result will be positive, it would have been just as easy to write the formula as $|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$, which gives an identical result.

**Position Vectors and Magnitude in $R^3$**

If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points, then the vector $|\overrightarrow{AB}| = (x_2 - x_1, y_2 - y_1, z_2 - z_1) = (a, b, c)$ is equivalent to the related position vector, $\overrightarrow{OP}$, and

$$|\overrightarrow{AB}| = |\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

## EXAMPLE 3 Connecting vectors in $R^3$ with their components

If $A(7, -11, 13)$ and $B(4, -7, 25)$ are two points in $R^3$, determine each of the following:

a. $|\overrightarrow{OA}|$ b. $|\overrightarrow{OB}|$ c. $\overrightarrow{AB}$ d. $|\overrightarrow{AB}|$

### Solution

a. $|\overrightarrow{OA}| = \sqrt{7^2 + (-11)^2 + 13^2} = \sqrt{339} \doteq 18.41$

b. $|\overrightarrow{OB}| = \sqrt{4^2 + (-7)^2 + 25^2} = \sqrt{690} \doteq 26.27$

c. $\overrightarrow{AB} = (4 - 7, -7 - (-11), 25 - 13) = (-3, 4, 12)$

d. $|\overrightarrow{AB}| = \sqrt{(-3)^2 + 4^2 + 12^2} = \sqrt{169} = 13$

In this section, we developed further properties of algebraic vectors. In the next section, we will demonstrate how these properties can be used to understand the geometry of $R^3$.

## IN SUMMARY

### Key Ideas

- In $R^3$, $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, $\vec{k} = (0, 0, 1)$. All are unit vectors along the *x*-, *y*- and *z*-axes, respectively.
- $\overrightarrow{OP} = (a, b, c) = a\vec{i} + b\vec{j} + c\vec{k}$, $|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$
- The vector between two points with its tail at $A(x_1, y_1, z_1)$ and head at $B(x_1, y_1, z_1)$ is determined as follows:
  $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2, y_2, z_2) - (x_1, y_1, z_1) = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$
- The vector $\overrightarrow{AB}$ is equivalent to the position vector $\overrightarrow{OP}$ since their directions and magnitude are the same.
  $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

### Need to Know

- If $\overrightarrow{OA} = (a, b, c)$ and $\overrightarrow{OD} = (d, e, f)$, then $\overrightarrow{OA} + \overrightarrow{OD} = (a + d, b + e, c + f)$.
- $m\overrightarrow{OP} = m(a, b, c) = (ma, mb, mc)$, $m \in \mathbf{R}$

# Exercises 6.7

## PART A

1. a. Write the vector $\overrightarrow{OA} = (-1, 2, 4)$ using the standard unit vectors.
   b. Determine $|\overrightarrow{OA}|$.

2. Write the vector $\overrightarrow{OB} = 3\vec{i} + 4\vec{j} - 4\vec{k}$ in component form and calculate its magnitude.

3. If $\vec{a} = (1, 3, -3)$, $\vec{b} = (-3, 6, 12)$, and $\vec{c} = (0, 8, 1)$, determine $\left|\vec{a} + \frac{1}{3}\vec{b} - \vec{c}\right|$.

4. For the vectors $\overrightarrow{OA} = (-3, 4, 12)$ and $\overrightarrow{OB} = (2, 2, -1)$, determine the following:
   a. the components of vector $\overrightarrow{OP}$, where $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB}$
   b. $|\overrightarrow{OA}|$, $|\overrightarrow{OB}|$, and $|\overrightarrow{OP}|$
   c. $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$. What does $\overrightarrow{AB}$ represent?

## PART B

K 5. Given $\vec{x} = (1, 4, -1)$, $\vec{y} = (1, 3, -2)$, and $\vec{z} = (-2, 1, 0)$, determine a vector equivalent to each of the following:

a. $\vec{x} - 2\vec{y} - \vec{z}$

b. $-2\vec{x} - 3\vec{y} + \vec{z}$

c. $\frac{1}{2}\vec{x} - \vec{y} + 3\vec{z}$

d. $3\vec{x} + 5\vec{y} + 3\vec{z}$

6. Given $\vec{p} = 2\vec{i} - \vec{j} + \vec{k}$ and $\vec{q} = -\vec{i} - \vec{j} + \vec{k}$, determine the following in terms of the standard unit vectors.

a. $\vec{p} + \vec{q}$

b. $\vec{p} - \vec{q}$

c. $2\vec{p} - 5\vec{q}$

d. $-2\vec{p} + 5\vec{q}$

7. If $\vec{m} = 2\vec{i} - \vec{k}$ and $\vec{n} = -2\vec{i} + \vec{j} + 2\vec{k}$, calculate each of the following:

a. $|\vec{m} - \vec{n}|$

b. $|\vec{m} + \vec{n}|$

c. $|2\vec{m} + 3\vec{n}|$

d. $|-5\vec{m}|$

8. Given $\vec{x} + \vec{y} = -\vec{i} + 2\vec{j} + 5\vec{k}$ and $\vec{x} - \vec{y} = 3\vec{i} + 6\vec{j} - 7\vec{k}$, determine $\vec{x}$ and $\vec{y}$.

C 9. Three vectors, $\overrightarrow{OA} = (a, b, 0)$, $\overrightarrow{OB} = (a, 0, c)$, and $\overrightarrow{OC} = (0, b, c)$, are given.

a. In a sentence, describe what each vector represents.

b. Write each of the given vectors using the standard unit vectors.

c. Determine a formula for each of $|\overrightarrow{OA}|$, $|\overrightarrow{OB}|$, and $|\overrightarrow{OC}|$.

d. Determine $\overrightarrow{AB}$. What does $\overrightarrow{AB}$ represent?

10. Given the points $A(-2, -6, 3)$ and $B(3, -4, 12)$, determine each of the following:

a. $|\overrightarrow{OA}|$

b. $|\overrightarrow{OB}|$

c. $\overrightarrow{AB}$

d. $|\overrightarrow{AB}|$

e. $\overrightarrow{BA}$

f. $|\overrightarrow{BA}|$

11. The vertices of quadrilateral $ABCD$ are given as $A(0, 3, 5)$, $B(3, -1, 17)$, $C(7, -3, 15)$, and $D(4, 1, 3)$. Prove that $ABCD$ is a parallelogram.

12. Given $2\vec{x} + \vec{y} - 2\vec{z} = \vec{0}$, $\vec{x} = (-1, b, c)$, $\vec{y} = (a, -2, c)$, and $\vec{z} = (-a, 6, c)$, determine the value of the unknowns.

A 13. A parallelepiped is determined by the vectors $\overrightarrow{OA} = (-2, 2, 5)$, $\overrightarrow{OB} = (0, 4, 1)$, and $\overrightarrow{OC} = (0, 5, -1)$.

a. Draw a sketch of the parallelepiped formed by these vectors.

b. Determine the coordinates of all of the vertices for the parallelepiped.

T 14. Given the points $A(-2, 1, 3)$ and $B(4, -1, 3)$, determine the coordinates of the point on the $x$-axis that is equidistant from these two points.

## PART C

15. Given $|\vec{a}| = 3$, $|\vec{b}| = 5$, and $|\vec{a} + \vec{b}| = 7$, determine $|\vec{a} - \vec{b}|$.