# *Section 7.1*—Vectors as Forces

The concept of **force** is something that everyone is familiar with. When we think of force, we usually think of it associated with effort or muscular exertion. This is experienced when an object is moved from one place to another. Examples of activities that involve forces are pulling a toboggan, lifting a book, shooting a basketball, or pedalling a bicycle. Each of these activities involves the use of muscular action that exerts a force. There are, however, many other examples of force in which muscular action is not present. For example, the attraction of the Moon to Earth, the attraction of a piece of metal to a magnet, the thrust exerted by an engine when gasoline combusts in its cylinders, or the force exerted by shock absorbers in cars to reduce vibration.

## Force as a Vector Quantity

Force can be considered something that either pushes or pulls an object. When a large enough force is applied to an object at rest, the object tends to move. When a push or pull is applied to a body that is already in motion, the motion of the body tends to change. Generally speaking, force can be defined as that which changes, or tends to change, the state of rest, or uniform motion of a body.

When describing certain physical quantities, there is little value in describing them with magnitude alone. For example, if we are describing the velocity of wind, it is not very practical to say that the wind has a speed of 30 km/h without specifying the direction of the wind. It makes more sense to say that a wind has a speed of 30 km/h travelling south. Similarly, the description of a force without specifying its magnitude and direction has little practical value. Because force is described by both magnitude and direction, it is a vector. The rules that apply to vectors also apply to forces.

Before we consider situations involving the calculation of force, it is necessary to describe the unit in which force is measured. On Earth, force is defined as the product between the mass of an object and the acceleration due to gravity ($9.8\ \text{m/s}^2$). So a 1 kg mass exerts a downward force of $1\ \text{kg} \times 9.8\ \text{m/s}^2$ or $9.8\ \text{kg} \cdot \text{m/s}^2$. This unit of measure is called a newton and is abbreviated as N. Because of Earth's gravitational field, which acts downward, we say that a 1 kg mass exerts a force of 9.8 N. Thus, the force exerted by a 2 kg mass at Earth's surface is about 19.6 N. A person having a mass of 60 kg would exert approximately $60 \times 9.8$, or 588 N, on the surface of Earth. So weight, expressed in newtons, is a force acting with a downward direction.

In problems involving forces, it is often the case that two or more forces act simultaneously on an object. To better understand the effect of these forces, it is useful to be able to find the single force that would produce exactly the same effect as all the forces acting together produced. This single force is called the **resultant**, or **sum**, of all the forces working on the object. It is important that we be able to determine the direction and magnitude of this single force. When we find the resultant of several forces, this resultant may be substituted for the individual forces, and the separate forces need not be considered further. The process of finding the resultant of all the forces acting on an object is called the **composition of forces**.

**Resultant and Composition of Forces**

The resultant of several forces is the single force that can be used to represent the combined effect of all the forces. The individual forces that make up the resultant are referred to as the components of the resultant.

If several forces are acting on an object, it is often advantageous to find a single force which, when applied to the object, would prevent any further motion that these original forces tended to produce. This single force is called the **equilibrant** because it would keep the object in a state of equilibrium.

**Equilibrant of Several Forces**

The equilibrant of a number of forces is the single force that opposes the resultant of the forces acting on an object. When the equilibrant is applied to the object, this force maintains the object in a state of equilibrium.

In the first example, we will consider collinear forces and demonstrate how to calculate their resultant and equilibrant. Collinear forces are those forces that act along the same straight line (in the same or opposite direction).

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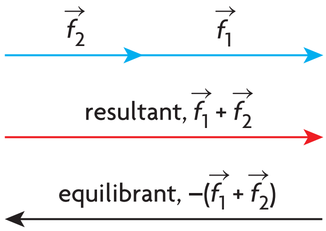
## EXAMPLE 1 Representing force using vectors

Two children, James and Fred, are pushing on a rock. James pushes with a force of 80 N in an easterly direction, and Fred pushes with a force of 60 N in the same direction. Determine the resultant and equilibrant of these two forces.

### Solution

To visualize the first force, we represent it with a horizontal line segment measuring 8 cm, pointing east. We represent the 60 N force with a line segment of 6 cm, also pointing east. The vectors used to represent forces are proportionate in length to the magnitude of the forces they represent.

**Resultant and Equilibrant of the Two Collinear Force Vectors**




The resultant of these forces, $\vec{f_1} + \vec{f_2}$, is the single vector pointing east with a magnitude of 140 N. The combined effort of James and Fred working together exerts a force on the rock of magnitude 140 N in an easterly direction. The equilibrant of these forces is the vector, $-(\vec{f_1} + \vec{f_2})$, which has a magnitude of 140 N pointing in the opposite direction, west. In general, the resultant and equilibrant are two vectors having the same magnitude but pointing in opposite directions.

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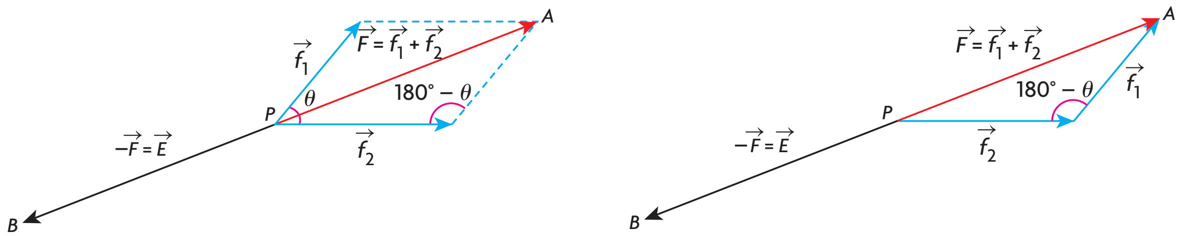
It is not typical that forces acting on an object are collinear. In the following diagram, the two noncollinear forces, $\vec{f_1}$ and $\vec{f_2}$, are applied at the point $P$ and could be thought of as two forces applied to an object in an effort to move it.

The natural question is, how do we determine the resultant of these two forces? Since forces are vectors, it follows from our work in the previous chapter that the resultant of two noncollinear forces is represented by either the diagonal of the parallelogram determined by these two vectors when placed tail to tail or the third side of the triangle formed when the vectors are placed head to tail. In the following diagrams, vector $\overrightarrow{PA} = \vec{F}$ is the resultant of $\vec{f_1}$ and $\vec{f_2}$, while the vector $\overrightarrow{PB} = \vec{E}$ is the equilibrant of $\vec{f_1}$ and $\vec{f_2}$.

**Resultant and Equilibrant of the Force Vectors $\vec{f_1}$ and $\vec{f_2}$**



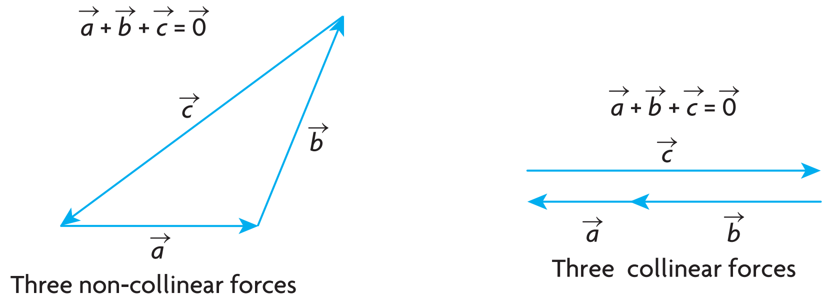
The resultant vector $\vec{F}$ and the equilibrant vector $\vec{E}$ are examples of two vectors that are in a state of equilibrium. When both these forces are applied to an object at point $P$, the object does not move. Since these vectors have the same magnitude but opposite directions it follows that $\vec{F} + \vec{E} = \vec{F} + (-\vec{F}) = \vec{0}$.

### Vectors in a State of Equilibrium

When three noncollinear vectors are in a state of equilibrium, these vectors will always lie in the same plane and form a linear combination. When the three vectors are arranged head to tail, the result is a triangle because the resultant of two of the forces is opposed by the third force. This means that if three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are in equilibrium, such that $\vec{c}$ is the equilibrant of $\vec{a}$ and $\vec{b}$, then $-\vec{c} = \vec{a} + \vec{b}$ or $\vec{a} + \vec{b} + \vec{c} = (-\vec{c}) + (\vec{c}) = \vec{0}$.

It is important to note that it also is possible for three vectors to be in equilibrium when the three forces are collinear. As with noncollinear vectors, one of the three forces is balanced by the resultant of the two other forces. In this case, the three forces do not form a triangle in the traditional sense. Instead, the sides of the "triangle" lie along the same straight line.

**Forces in Equilibrium**




In the following example, the resultant of two noncollinear forces is calculated.

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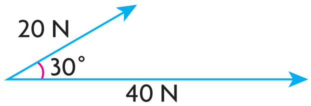
## EXAMPLE 2 Connecting the resultant force to vector addition

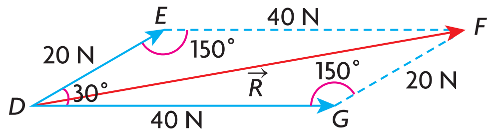
Two forces of 20 N and 40 N act at an angle of 30° to each other. Determine the resultant of these two forces.

### Solution

We start the solution to this problem by drawing both a position diagram and a vector diagram. A position diagram indicates the actual position of the given vectors, and a vector diagram takes the information given in the position diagram and puts it in a form that allows for the determination of the resultant vector using either the triangle or parallelogram law. As before, the position diagram is drawn approximately to scale, and the side lengths of the parallelogram are labelled. The resultant of the two given vectors is $\overrightarrow{DF} = \vec{R}$, and the supplement of $\angle EDG$ is $\angle FED$, which measures 150°.




Position diagram




Vector diagram

In $\triangle DEF$, $|\vec{R}|^2 = 20^2 + 40^2 - 2(20)(40)\cos 150°$ (Cosine law)

$$|\vec{R}|^2 = 20^2 + 40^2 - 2(20)(40)\left(\frac{-\sqrt{3}}{2}\right)$$

$$|\vec{R}|^2 = 2000 + 800\sqrt{3}$$

$$|\vec{R}|^2 \doteq 3385.64$$

Therefore, $|\vec{R}| \doteq 58.19$ N. If we let $\vec{E}$ represent the equilibrant, then $|\vec{E}| \doteq 58.19$ N.

Since we are asked to calculate the resultant and equilibrant of the two forces, we must also calculate angles so that we can state each of their relative positions. To do this, we use the sine law.

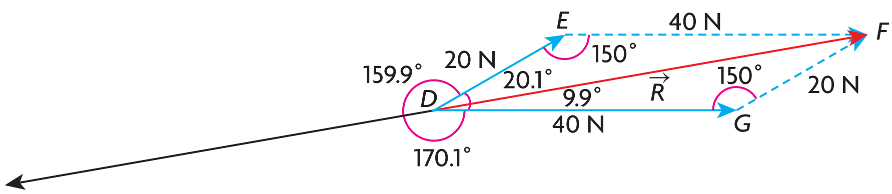
In $\triangle DEF$, $\dfrac{\sin \angle DEF}{|\vec{R}|} = \dfrac{\sin \angle EDF}{|\overrightarrow{EF}|}$ (Sine law)

$$\frac{\sin 150°}{58.19} \doteq \frac{\sin \angle EDF}{40}$$

$$\sin \angle EDF \doteq \frac{40(\sin 150°)}{58.19}$$

$$\sin \angle EDF \doteq 0.3437$$

$$\text{Thus, } \angle EDF \doteq 20.1°$$



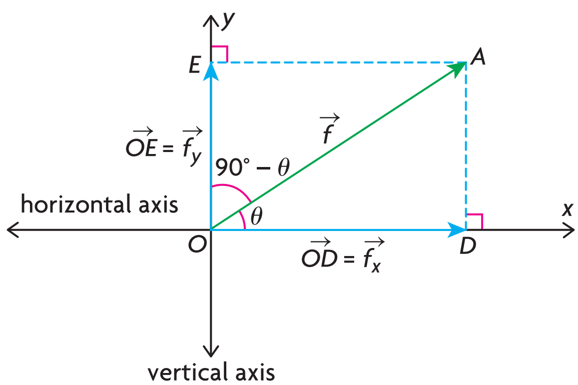

The resultant and equilibrant are forces, each having a magnitude of approximately 58.19 N. The resultant makes an angle of 20.1° with the 20 N force and 9.9° with the 40 N force. The equilibrant makes an angle of 159.9° with the 20 N force and an angle of 170.1° with the 40 N force.

We have shown that if we take any two forces that act at the same point, acting at an angle of $\theta$ to each other, the forces may be composed to obtain the resultant of these two forces. Furthermore, the resultant of any two forces is unique because there is only one parallelogram that can be formed with these two forces.

### Resolving a Vector into Its Components

In many situations involving forces, we are interested in a process that is the opposite of composition. This process is called **resolution**, which means taking a single force and decomposing it into two components. When we resolve a force into two components, it is possible to do this in an infinite number of ways because there are infinitely many parallelograms having a particular single force as the diagonal. However, the most useful and important way to resolve a force vector occurs when this vector is resolved into two components that are at right angles to each other. These components are usually referred to as the horizontal and vertical components.

In the following diagram, we demonstrate how to resolve the force vector $\vec{f}$ into its horizontal and vertical components.




The vector resolved into components is the vector $\overrightarrow{OA}$, or vector $\vec{f}$. From $A$, the head of the vector, perpendicular lines are drawn to meet the $x$-axis and $y$-axis at points $D$ and $E$, respectively. The vectors $\overrightarrow{OD}$ and $\overrightarrow{OE}$ are called the horizontal and vertical components of the vector $\overrightarrow{OA}$, where the angle between $\vec{f}$ and the $x$-axis is labelled $\theta$.

To calculate $|\overrightarrow{OD}|$, we use the cosine ratio in the right triangle $OAD$.

$$\text{In } \triangle OAD, \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{|\overrightarrow{OD}|}{|\overrightarrow{OA}|}$$

$$\text{Therefore, } |\overrightarrow{OD}| = |\overrightarrow{OA}|\cos\theta$$

This means that the vector $\overrightarrow{OD}$, the horizontal component of $\overrightarrow{OA}$, has magnitude $|\overrightarrow{OA}|\cos\theta$.

The magnitude of the vertical component of $\overrightarrow{OA}$ is calculated in the same way using $\triangle OEA$.

In $\triangle OEA$, $\cos(90° - \theta) = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{|\overrightarrow{OE}|}{|\overrightarrow{OA}|}$

$$\text{Therefore, } |\overrightarrow{OE}| = |\overrightarrow{OA}|\cos(90° - \theta)$$

$$\text{Since } \sin\theta = \cos(90° - \theta),$$

$$|\overrightarrow{OE}| = |\overrightarrow{OA}|\sin\theta$$

What we have shown is that $|\overrightarrow{OD}| = |\overrightarrow{OA}|\cos\theta$ and $|\overrightarrow{OE}| = |\overrightarrow{OA}|\sin\theta$. If we replace $\overrightarrow{OA}$ with $\vec{f}$, this would imply that $|\vec{f}_x| = |\vec{f}|\cos\theta$ and $|\vec{f}_y| = |\vec{f}|\sin\theta$, where $\vec{f}_x$ and $\vec{f}_y$ represent the horizontal and vertical components of $\vec{f}$, respectively.

**Resolution of a Vector into Horizontal and Vertical Components**

If the vector $\vec{f}$ is resolved into its respective horizontal and vertical components, $\vec{f}_x$ and $\vec{f}_y$, then $|\vec{f}_x| = |\vec{f}|\cos\theta$ and $|\vec{f}_y| = |\vec{f}|\sin\theta$, where $\theta$ is the angle that $\vec{f}$ makes with the $x$-axis.

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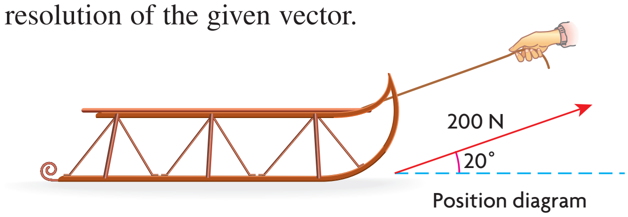
## EXAMPLE 3 Connecting forces to the components of a given vector

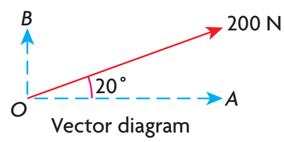
Kayla pulls on a rope attached to her sleigh with a force of 200 N. If the rope makes an angle of 20° with the horizontal, determine:

a. the force that pulls the sleigh forward
b. the force that tends to lift the sleigh

### Solution

In this problem, we are asked to resolve the force vector into its two rectangular components. We start by drawing a position diagram and, beside it, show the resolution of the given vector.




From the diagram, the vector $\overrightarrow{OA}$ is the horizontal component of the given force vector that pulls the sleigh forward. The vector $\overrightarrow{OB}$ is the vertical component of the given force vector that tends to lift the sleigh. To calculate their magnitudes, we directly apply the formulas developed.

$$\left|\overrightarrow{OA}\right| = 200(\cos 20°) \doteq 200(0.9397) \doteq 187.94 \text{ N}$$

and

$$\left|\overrightarrow{OB}\right| = 200(\sin 20°) \doteq 200(0.3420) \doteq 68.40 \text{ N}$$

The sleigh is pulled forward with a force of approximately 187.94 N, and the force that tends to lift it is approximately 68.40 N.

In the following example, we will use two different methods to solve the problem. In the first solution, a triangle of forces will be used. In the second solution, the concept of resolution of forces will be used.

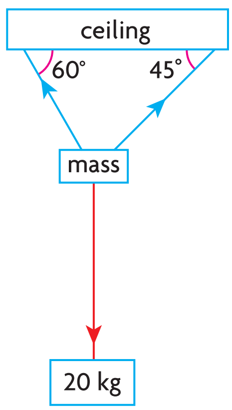
## EXAMPLE 4 Selecting a strategy to solve a problem involving several forces

A mass of 20 kg is suspended from a ceiling by two lengths of rope that make angles of 60° and 45° with the ceiling. Determine the tension in each of the ropes.

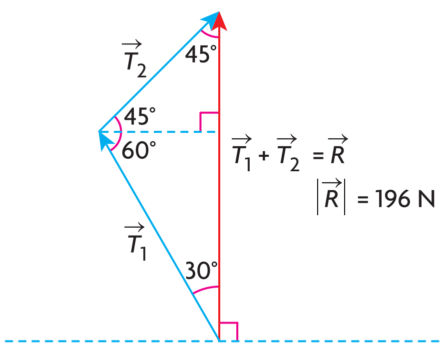
### Solution

*Method 1* Triangle of forces

First, recall that the downward force exerted per kilogram is 9.8 N. So the 20 kg mass exerts a downward force of 196 N. Draw a position diagram and a vector diagram. Let the tension vectors for the two pieces of rope be $\overrightarrow{T_1}$ and $\overrightarrow{T_2}$, and let their resultant be $\overrightarrow{R}$. The magnitude of the resultant force created by the tensions in the ropes must equal the magnitude of the downward force on the mass caused by gravity since the system is in a state of equilibrium.




Position diagram




Vector diagram

To calculate the required tensions, it is necessary to use the sine law in the vector diagram.

Thus, $\dfrac{\left|\overrightarrow{T_1}\right|}{\sin 45°} = \dfrac{\left|\overrightarrow{T_2}\right|}{\sin 30°} = \dfrac{196}{\sin 105°}$

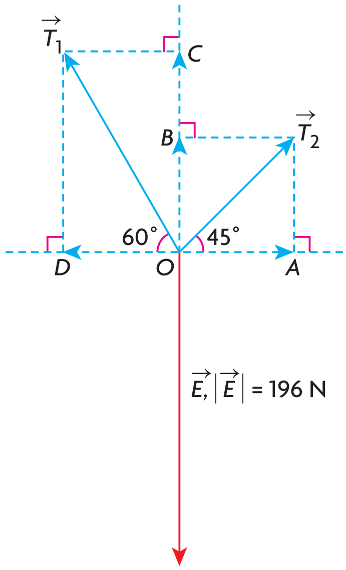
$$|\vec{T_1}|\sin 105° = 196(\sin 45°) \quad \text{and} \quad |\vec{T_2}|\sin 105° = 196(\sin 30°)$$

$$|\vec{T_1}| = \frac{196(0.7071)}{0.9659} \doteq 143.48 \text{ N} \quad \text{and} \quad |\vec{T_2}| = \frac{196(0.5)}{0.9659} \doteq 101.46 \text{ N}$$

Therefore, the tensions in the two ropes are approximately 143.48 N and 101.46 N.

*Method 2* Resolution of Forces

We start by drawing a diagram showing the tension vectors, $\vec{T_1}$ and $\vec{T_2}$, and the equilibrant, $\vec{E}$. The tension vectors are shown in their resolved form.




For the tension vectors, the magnitudes of their components are calculated.

*Horizontal components:*

$$|\vec{OA}| = \cos 45°|\vec{T_2}| \doteq 0.7071|\vec{T_2}| \text{ and } |\vec{OB}| \doteq 0.7071|\vec{T_2}|;$$

*Vertical components:*

$$|\vec{OC}| = \sin 60°|\vec{T_1}| \doteq 0.8660|\vec{T_1}| \text{ and } |\vec{OD}| = 0.5|\vec{T_1}|$$

For the system to be in equilibrium, the magnitudes of the horizontal and vertical components must balance each other.

*Horizontal components:* $|\vec{OA}| = |\vec{OD}|$ or $0.7071|\vec{T_2}| \doteq 0.5|\vec{T_1}|$

*Vertical components:* $|\vec{OB}| + |\vec{OC}| = |\vec{E}|$ or $0.7071|\vec{T_2}| + 0.8660|\vec{T_1}| \doteq 196$

This gives the following system of two equations in two unknowns.

① $0.7071|\vec{T_2}| \doteq 0.5|\vec{T_1}|$

② $0.7071|\vec{T_2}| + 0.8660|\vec{T_1}| \doteq 196$

In equation ①, $|\vec{T_1}| \doteq \dfrac{0.7071|\vec{T_2}|}{0.5}$ or $|\vec{T_1}| \doteq 1.4142|\vec{T_2}|$.

If we substitute this into equation ②, we obtain

$$0.7071|\vec{T_2}| + 0.8660[(1.4142)(|\vec{T_2}|)] \doteq 196$$

$$1.9318|\vec{T_2}| \doteq 196$$

$$|\vec{T_2}| \doteq \frac{196}{1.9318}$$

$$\doteq 101.46 \text{ N}$$

$$\text{Since } |\vec{T_1}| \doteq 1.4142|\vec{T_2}|,$$

$$|\vec{T_1}| \doteq 1.4142(101.46)$$

$$\doteq 143.48 \text{ N}$$

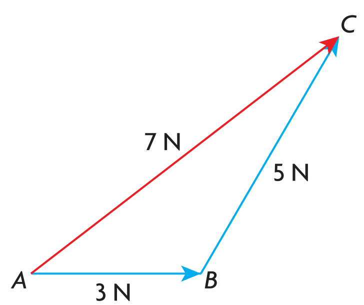
Therefore, the tensions in the ropes are 143.48 N and 101.46 N, as before.

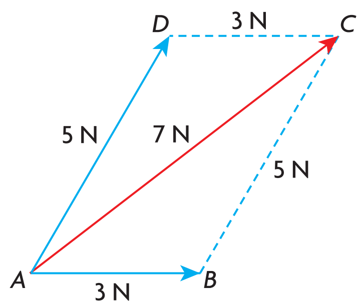
## EXAMPLE 5 Reasoning about equilibrium in a system involving three forces

a. Is it possible for three forces of 15 N, 18 N, and 38 N to keep a system in a state of equilibrium?
b. Three forces having magnitudes 3 N, 5 N, and 7 N are in a state of equilibrium. Calculate the angle between the two smaller forces.

### Solution

a. For a system to be in equilibrium, it is necessary that a triangle be formed having lengths proportional to 15, 18, and 38. Since $15 + 18 < 38$, a triangle cannot be formed because the triangle inequality states that for a triangle to be formed, the sum of any two sides must be greater than or equal to the third side. Therefore, three forces of 15 N, 18 N, and 38 N cannot keep a system in a state of equilibrium.
b. We start by drawing the triangle of forces and the related parallelogram.




Using $\triangle ABC$,

$$|\vec{AC}|^2 = |\vec{AB}|^2 + |\vec{BC}|^2 - 2|\vec{AB}||\vec{BC}|\cos\angle CBA \qquad \text{(Cosine law)}$$

$$7^2 = 3^2 + 5^2 - 2(3)(5)\cos\angle CBA$$

$$
\begin{aligned}
49 &= 34 - 30 \cos \angle CBA \\
\frac{-1}{2} &= \cos \angle CBA \\
120° &= \cos^{-1}(-0.5) \angle CBA
\end{aligned}
$$

The angle that is required is $\angle DAB$, the supplement of $\angle CBA$. $\angle DAB = 60°$, and the angle between the 3 N and 5 N force is 60°.

## IN SUMMARY

### Key Ideas

- Problems involving forces can be solved using strategies involving vectors.
- When two or more forces are applied to an object, the net effect of the forces can be represented by the resultant vector determined by adding the vectors that represent each of the forces.
- A system is in a state of equilibrium when the net effect of all the forces acting on an object causes no movement of the object.

### Need to Know

- $\vec{F} = \vec{F_1} + \vec{F_2}$ is the resultant of $\vec{F_1}$ and $\vec{F_2}$.
- $-\vec{F} = -(\vec{F_1} + \vec{F_2})$ is the equilibrant of $\vec{F_1}$ and $\vec{F_2}$.
- If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then $\vec{a}$, $\vec{b}$, and $\vec{c}$ are in a state of equilibrium.

# Exercise 7.1

### PART A

1. a. Name some common household items that have approximate weights of 10 N, 50 N, and 100 N.

   b. What is your weight in newtons?

2. Three forces of 10 N, 20 N, and 30 N are in a state of equilibrium.

   a. Draw a sketch of these three forces.

   b. What is the angle between the equilibrant and each of the smaller forces?

3. Two forces of 10 N and 20 N are acting on an object. How should these forces be arranged to produce the largest possible resultant?

4. Explain in your own words why three forces must lie in the same plane if they are acting on an object in equilibrium.

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## PART B

5. Determine the resultant and equilibrant of each pair of forces acting on an object.
   a. $\vec{f_1}$ has a magnitude of 5 N acting due east, and $\vec{f_2}$ has a magnitude of 12 N acting due north.
   b. $\vec{f_1}$ has a magnitude of 9 N acting due west, and $\vec{f_2}$ has a magnitude of 12 N acting due south.
6. Which of the following sets of forces acting on an object could produce equilibrium?
   a. 2 N, 3 N, 4 N
   b. 9 N, 40 N, 41 N
   c. $\sqrt{5}$ N, 6 N, 9 N
   d. 9 N, 10 N, 19 N
7. Using a vector diagram, explain why it is easier to do chin-ups when your hands are 30 cm apart instead of 90 cm apart. (Assume that the force exerted by your arms is the same in both cases.)
8. A force, $\vec{f_1}$, of magnitude 6 N acts on particle P. A second force, $\vec{f_2}$, of magnitude 8 N acts at 60° to $\vec{f_1}$. Determine the resultant and equilibrant of $\vec{f_1}$ and $\vec{f_2}$.
9. Resolve a force of 10 N into two forces perpendicular to each other, such that one component force makes an angle of 15° with the 10 N force.

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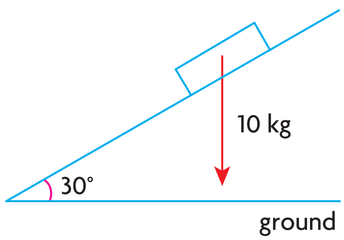
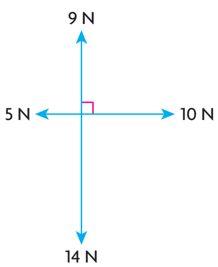
10. A 10 kg block lies on a smooth ramp that is inclined at 30°. What force, parallel to the ramp, would prevent the block from moving? (Assume that 1 kg exerts a force of 9.8 N.)




11. Three forces, with magnitudes 13 N, 7 N, and 8 N, are in a state of equilibrium.
    a. Draw a sketch of these three forces.
    b. Determine the angle between the two smallest forces.
12. Four forces of magnitude 5 N, 9 N, 10 N, and 14 N are arranged as shown in the diagram at the left. Determine the resultant of these forces.

13. Two forces, $\vec{f_1}$ and $\vec{f_2}$, act at right angles to each other. The magnitude of the resultant of these two forces is 25 N, and $|\vec{f_1}| = 24$ N.

  a. Determine $|\vec{f_2}|$.

  b. Determine the angle between $\vec{f_1}$ and the resultant, and the angle between $\vec{f_1}$ and the equilibrant.

C

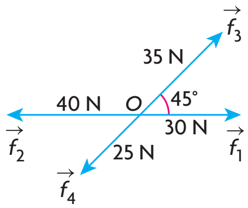
14. Three forces, each having a magnitude of 1 N, are arranged to produce equilibrium.

  a. Draw a sketch showing an arrangement of these forces, and demonstrate that the angle between the resultant and each of the other two forces is 60°.

  b. Explain how to determine the angle between the equilibrant and the other two vectors.

T

15. Four forces, $\vec{f_1}$, $\vec{f_2}$, $\vec{f_3}$, and $\vec{f_4}$, are acting on an object and lie in the same plane, as shown. The forces $\vec{f_1}$ and $\vec{f_2}$ act in an opposite direction to each other, with $|\vec{f_1}| = 30$ N and $|\vec{f_2}| = 40$ N. The forces $\vec{f_3}$ and $\vec{f_4}$ also act in opposite directions, with $|\vec{f_3}| = 35$ N and $|\vec{f_4}| = 25$ N. If the angle between $\vec{f_1}$ and $\vec{f_3}$ is 45°, determine the resultant of these four forces.




16. A mass of 20 kg is suspended from a ceiling by two lengths of rope that make angles of 30° and 45° with the ceiling. Determine the tension in each of the ropes.

17. A mass of 5 kg is suspended by two strings, 24 cm and 32 cm long, from two points that are 40 cm apart and at the same level. Determine the tension in each of the strings.

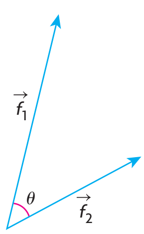
## PART C

18. Two tugs are towing a ship. The smaller tug is 15° off the port bow, and the larger tug is 20° off the starboard bow. The larger tug pulls twice as hard as the smaller tug. In what direction will the ship move?

19. Three forces of 5 N, 8 N, and 10 N act from the corner of a rectangular solid along its three edges.

  a. Calculate the magnitude of the equilibrant of these three forces.

  b. Determine the angle that the equilibrant makes with each of the three forces.




20. Two forces, $\vec{f_1}$ and $\vec{f_2}$, make an angle $\theta$ with each other when they are placed tail to tail, as shown. Prove that $|\vec{f_1} + \vec{f_2}| = \sqrt{|\vec{f_1}|^2 + |\vec{f_2}|^2 + 2|\vec{f_1}||\vec{f_2}|\cos\theta}$.