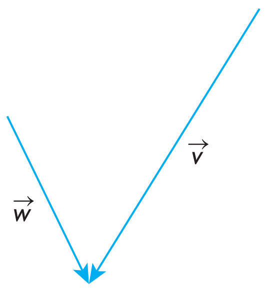
Section 7.2—Velocity

In the previous chapter, we showed that velocity is a vector because it had both magnitude (speed) and direction. In this section, we will demonstrate how two velocities can be combined to determine their resultant velocity.

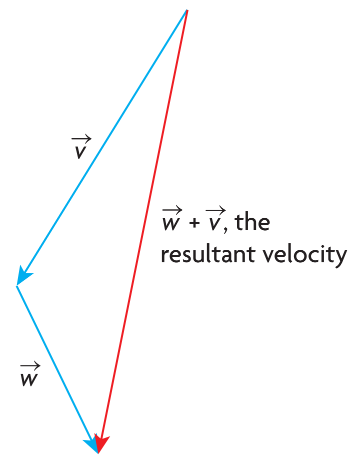
EXAMPLE 1 Representing velocity with diagrams

An airplane has a velocity of $\vec{v}$ (relative to the air) when it encounters a wind having a velocity of $\vec{w}$ (relative to the ground). Draw a diagram showing the possible positions of the velocities and another diagram showing the resultant velocity.

Solution



Position diagram



Vector diagram

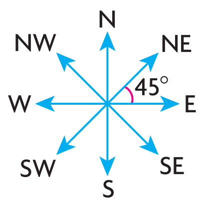
The resultant velocity of any two velocities is their sum. In all calculations involving resultant velocities, it is necessary to draw a triangle showing the velocities so there is a clear recognition of the resultant and its relationship to the other two velocities. When the velocity of the airplane is mentioned, it is understood that we are referring to its air speed. When the velocity of the wind is mentioned, we are referring to its velocity relative to a fixed point, the ground. The resultant velocity of the airplane is the velocity of the airplane relative to the ground and is called the ground velocity of the airplane.

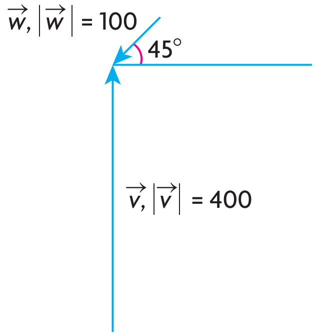
EXAMPLE 2 Selecting a vector strategy to determine ground velocity

A plane is heading due north with an air speed of 400 km/h when it is blown off course by a wind of 100 km/h from the northeast. Determine the resultant ground velocity of the airplane.

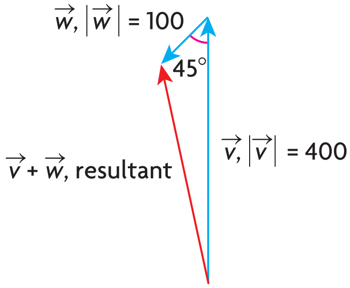
Solution

We start by drawing position and vector diagrams where $\vec{w}$ represents the velocity of the wind and $\vec{v}$ represents the velocity of the airplane in kilometres per hour.





Position diagram



Vector diagram

Use the cosine law to determine the magnitude of the resultant velocity.

$$|\vec{v} + \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}|\cos\theta,\ \theta = 45°,\ |\vec{w}| = 100,\ |\vec{v}| = 400$$

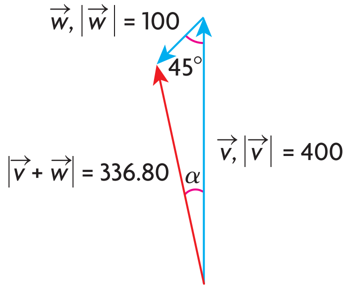
$$|\vec{v} + \vec{w}|^2 = 400^2 + 100^2 - 2(100)(400)\cos 45°$$

$$|\vec{v} + \vec{w}|^2 = 160\,000 + 10\,000 - 80\,000\left(\frac{1}{\sqrt{2}}\right)$$

$$|\vec{v} + \vec{w}|^2 = 170\,000 - \frac{80\,000}{\sqrt{2}}$$

$$|\vec{v} + \vec{w}| \doteq 336.80$$

To state the required velocity, the direction of the resultant vector is needed. Use the sine law to calculate α, the angle between the velocity vector of the plane and the resultant vector.



$$\frac{\sin\alpha}{100} \doteq \frac{\sin 45°}{336.80}$$

$$\sin\alpha \doteq \frac{100\sin 45°}{336.80} \doteq 0.2099$$

$$\alpha \doteq 12.1°$$

Therefore, the resultant velocity is approximately 336.80 km/h, $N12.1°W$ (or $W77.9°N$).

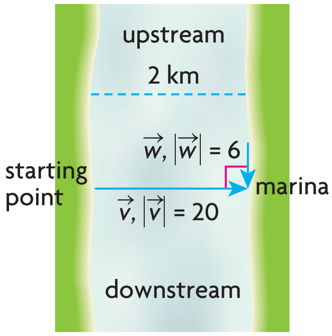
EXAMPLE 3 Using vectors to represent velocities

A river is 2 km wide and flows at 6 km/h. Anna is driving a motorboat, which has a speed of 20 km/h in still water and she heads out from one bank in a direction perpendicular to the current. A marina lies directly across the river from the starting point on the opposite bank.

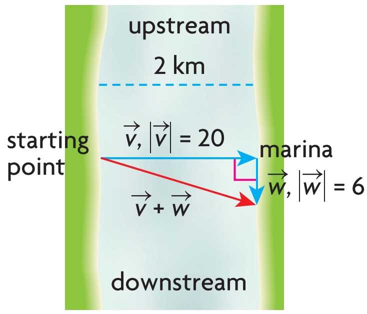
a. How far downstream from the marina will the current push the boat?
b. How long will it take for the boat to cross the river?
c. If Anna decides that she wants to end up directly across the river at the marina, in what direction should she head? What is the resultant velocity of the boat?

Solution

a. As before, we construct a vector and position diagram, where $\vec{w}$ and $\vec{v}$ represent the velocity of the river and the boat, respectively, in kilometres per hour.

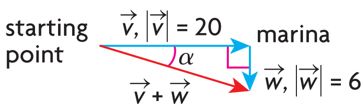


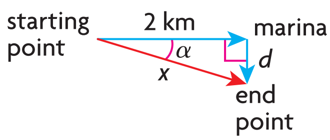
Position diagram



Vector diagram

The distance downstream that the boat lands can be calculated in a variety of ways, but the easiest way is to redraw the velocity triangle from the vector diagram, keeping in mind that the velocity triangle is *similar* to the distance triangle. This is because the distance travelled is directly proportional to the velocity.





Using similar triangles, $\frac{6}{20} = \frac{d}{2}$, $d = 0.6$.

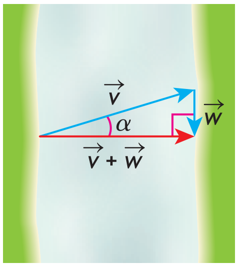
The boat will touch the opposite bank 0.6 km downstream.

b. To calculate the actual distance between the starting and end points, the Pythagorean theorem is used for the distance triangle, with x being the required distance. Thus, $x^2 = 2^2 + (0.6)^2 = 4.36$ and $x \doteq 2.09$, which means that the actual distance the boat travelled was approximately 2.09 km.

To calculate the length of time it took to make the trip, it is necessary to calculate the speed at which this distance was travelled. Again, using similar triangles, $\frac{20}{2} \doteq \frac{|\vec{v} + \vec{w}|}{2.09}$. Solving this proportion, $|\vec{v} + \vec{w}| \doteq 20.9$, so the actual speed of the boat crossing the river was about 20.9 km/h. The actual time taken to cross the river is $t = \frac{d}{v} \doteq \frac{2.09}{20.9} \doteq 0.1$ h, or about 6 min.

Therefore, the boat landed 0.6 km downstream, and it took approximately 6 min to make the crossing.

c. To determine the velocity with which she must travel to reach the marina, we will draw the related vector diagram.



We are given $|\vec{w}| = 6$ and $|\vec{v}| = 20$. To determine the direction in which the boat must travel, let α represent the angle upstream at which the boat heads out.

$$\sin \alpha = \frac{6}{20} \text{ or } \sin^{-1}\left(\frac{6}{20}\right) = \alpha$$

$$\alpha \doteq 17.5°$$

To calculate the magnitude of the resultant velocity, use the Pythagorean theorem.
$|\vec{v}|^2 = |\vec{w}|^2 + |\vec{v} + \vec{w}|^2$ where $|\vec{v}| = 20$ and $|\vec{w}| = 6$

Thus, $20^2 = 6^2 + |\vec{v} + \vec{w}|^2$

$$|\vec{v} + \vec{w}|^2 = 400 - 36$$

$$|\vec{v} + \vec{w}| \doteq 19.08$$

This implies that if Anna wants to travel directly across the river, she will have to travel upstream 17.5° with a speed of approximately 19.08 km/h. The nose of the boat will be headed upstream at 17.5°, but the boat will actually be moving directly across the river at a water speed of 19.08 km/h.

IN SUMMARY

Key Idea

- Problems involving velocities can be solved using strategies involving vectors.

Need to Know

- The velocity of an object is stated relative to a frame of reference. The frame of reference used influences the stated velocity of the object.
- Air speed/water speed is the speed of a plane/boat relative to a person on board. Ground speed is the speed of a plane or boat relative to a person on the ground and includes the effect of wind or current.
- The resultant velocity $\vec{v_r} = \vec{v_1} + \vec{v_2}$.

Exercise 7.2

PART A

1. A woman walks at 4 km/h down the corridor of a train that is travelling at 80 km/h on a straight track.
   a. What is her resultant velocity in relation to the ground if she is walking in the same direction as the train?
   b. If she walks in the opposite direction as the train, what is her resultant velocity?
2. An airplane heading north has an air speed of 600 km/h.
   a. If the airplane encounters a wind from the north at 100 km/h, what is the resultant ground velocity of the plane?
   b. If there is a wind from the south at 100 km/h, what is the resultant ground velocity of the plane?

PART B

3. An airplane has an air speed of 300 km/h and is heading due west. If it encounters a wind blowing south at 50 km/h, what is the resultant ground velocity of the plane?
4. K Adam can swim at the rate of 2 km/h in still water. At what angle to the bank of a river must he head if he wants to swim directly across the river and the current in the river moves at the rate of 1 km/h?
5. A child, sitting in the backseat of a car travelling at 20 m/s, throws a ball at 2 m/s to her brother who is sitting in the front seat.
   a. What is the velocity of the ball relative to the children?
   b. What is the velocity of the ball relative to the road?
6. A boat heads 15° west of north with a water speed of 12 m/s. Determine its resultant velocity, relative to the ground, when it encounters a 5 m/s current from 15° north of east.
7. An airplane is heading due north at 800 km/h when it encounters a wind from the northeast at 100 km/h.
   a. What is the resultant velocity of the airplane?
   b. How far will the plane travel in 1 h?
8. An airplane is headed north with a constant velocity of 450 km/h. The plane encounters a wind from the west at 100 km/h.
   a. In 3 h, how far will the plane travel?
   b. In what direction will the plane travel?

A 9. A small airplane has an air speed of 244 km/h. The pilot wishes to fly to a destination that is 480 km due west from the plane's present location. There is a 44 km/h wind from the south.

a. In what direction should the pilot fly in order to reach the destination?

b. How long will it take to reach the destination?

10. Judy and her friend Helen live on opposite sides of a river that is 1 km wide. Helen lives 2 km downstream from Judy on the opposite side of the river. Judy can swim at a rate of 3 km/h, and the river's current has a speed of 4 km/h. Judy swims from her cottage directly across the river.

a. What is Judy's resultant velocity?

b. How far away from Helen's cottage will Judy be when she reaches the other side?

c. How long will it take Judy to reach the other side?

T 11. An airplane is travelling $N60°E$ with a resultant ground speed of 205 km/h. The nose of the plane is actually pointing east with an airspeed of 212 km/h.

a. What is the wind direction?

b. What is the wind speed?

C 12. Barbara can swim at 4 km/h in still water. She wishes to swim across a river to a point directly opposite from where she is standing. The river is moving at a rate of 5 km/h. Explain, with the use of a diagram, why this is not possible.

PART C

13. Mary leaves a dock, paddling her canoe at 3 m/s. She heads downstream at an angle of 30° to the current, which is flowing at 4 m/s.

a. How far downstream does Mary travel in 10 s?

b. What is the length of time required to cross the river if its width is 150 m?

14. Dave wants to cross a 200 m wide river whose current flows at 5.5 m/s. The marina he wants to visit is located at an angle of S45°W from his starting position. Dave can paddle his canoe at 4 m/s in still water.

a. In which direction should he head to get to the marina?

b. How long will the trip take?

15. A steamboat covers the distance between town A and town B (located downstream) in 5 h without making any stops. Moving upstream from B to A, the boat covers the same distance in 7 h (again making no stops). How many hours does it take a raft moving with the speed of the river current to get from A to B?