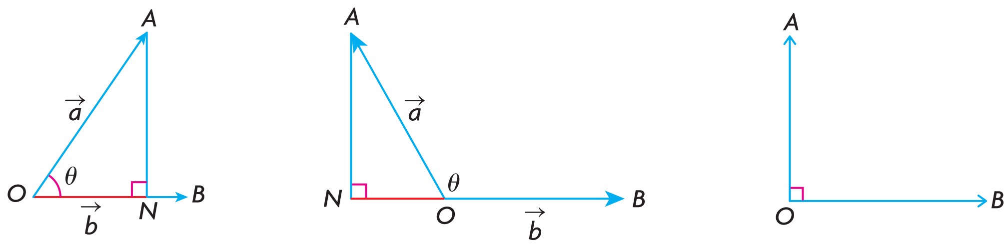
Section 7.5—Scalar and Vector Projections

In the last two sections, the concept of the dot product was discussed, first in geometric form and then in algebraic form. In this section, the dot product will be used along with the concept of **projections**. These concepts are closely related, and each has real significance from both a practical and theoretical point of view.

When two vectors, $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$, are placed tail to tail, and θ is the angle between the vectors, $0° \leq \theta \leq 180°$, the **scalar projection** of $\vec{a}$ on $\vec{b}$ is ON, as shown in the following diagram. The scalar projection can be determined using right triangle trigonometry and can be applied to either geometric or algebraic vectors equally well.

Scalar Projection of $\vec{a}$ on $\vec{b}$

The scalar projection of vector $\vec{a}$ onto $\vec{b}$ is ON, where $ON = |\vec{a}|\cos\theta$.



$0 \leq \theta < 90°$ $\qquad$ $90° < \theta \leq 180°$ $\qquad$ $\theta = 90°$

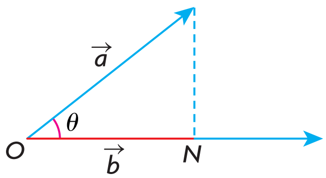
Observations about the Scalar Projection

A number of observations should be made about scalar projections. The scalar projection of $\vec{a}$ on $\vec{b}$ is obtained by drawing a line from the head of vector $\vec{a}$ perpendicular to $\overrightarrow{OB}$, or an extension of $\overrightarrow{OB}$. If the point where this line meets the vector is labelled N, then the scalar projection $\vec{a}$ on $\vec{b}$ is ON. Since ON is a real number, or scalar, and also a projection, it is called a scalar projection. If the angle between two given vectors is such that $0° \leq \theta < 90°$, then the scalar projection is positive; otherwise, it is negative for $90° < \theta \leq 180°$ and 0 if $\theta = 90°$.

The sign of scalar projections should not be surprising, since it corresponds exactly to the sign convention for dot products that we saw in the previous two sections. An important point is that the scalar projection between perpendicular vectors is always 0 because the angle between the vectors is 90° and $\cos 90° = 0$. Another important point is that it is not possible to take the scalar projection of the vector $\vec{a}$ on $\vec{0}$. This would result in a statement involving division by 0, which is meaningless.

Another observation should be made about scalar projections that is not immediately obvious from the given definition. The scalar projection of vector $\vec{a}$ on vector $\vec{b}$ is in general not equal to the scalar projection of vector $\vec{b}$ on vector $\vec{a}$, which can be seen from the following.

Calculating the scalar projection of $\vec{a}$ on $\vec{b}$:



Projection of $\vec{a}$ onto $\vec{b}$

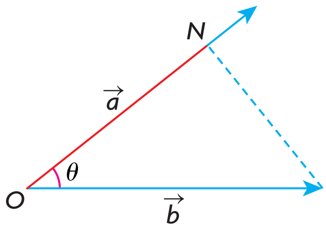
When calculating this projection, what is needed is to solve for $|\vec{a}|\cos\theta$ in the dot product formula.

We know that $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

Rewrite this formula as $\vec{a}\cdot\vec{b} = \left(|\vec{a}|\cos\theta\right)|\vec{b}|$.

Solving for $|\vec{a}|\cos\theta$ gives $|\vec{a}|\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

Calculating the scalar projection of $\vec{b}$ on $\vec{a}$:

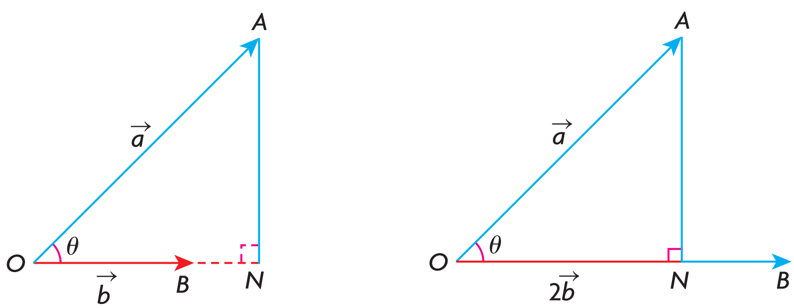


Projection of $\vec{b}$ onto $\vec{a}$

To find the scalar projection of $\vec{b}$ on $\vec{a}$, it is necessary to solve for $|\vec{b}|\cos\theta$ in the dot product formula. This is done in exactly the same way as above, and we find that $|\vec{b}|\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

From this, we can see that, in general, $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \neq \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$. It is correct to say, however, that these scalar projections are equal if $|\vec{a}| = |\vec{b}|$.

Another observation to make about scalar projections is that the scalar projection of $\vec{a}$ on $\vec{b}$ is independent of the length of $\vec{b}$. This is demonstrated in the following diagram:



From the diagram, we can see that the scalar projection of vector $\vec{a}$ on vector $\vec{b}$ equals ON. If we take the scalar projection of $\vec{a}$ on $2\vec{b}$, this results in the exact same line segment ON.

EXAMPLE 1 Reasoning about the characteristics of the scalar projection

a. Show algebraically that the scalar projection of $\vec{a}$ on $\vec{b}$ is identical to the scalar projection of $\vec{a}$ on $2\vec{b}$.

b. Show algebraically that the scalar projection of $\vec{a}$ on $\vec{b}$ is not the same as $\vec{a}$ on $-2\vec{b}$.

Solution

a. The scalar projection of $\vec{a}$ on $\vec{b}$ is given by the formula $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$.

The scalar projection of $\vec{a}$ on $2\vec{b}$ is $\frac{\vec{a} \cdot 2\vec{b}}{|2\vec{b}|}$. If we use the properties of the dot product and the fact that $|2\vec{b}| = 2|\vec{b}|$, this quantity can be written as $\frac{2(\vec{a} \cdot \vec{b})}{2|\vec{b}|}$, and then simplified to $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$.

From this, we see that what was shown geometrically is verified algebraically.

b. As before, the scalar projection of $\vec{a}$ on $\vec{b}$ is given by the formula $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$.

The scalar projection of $\vec{a}$ on $-2\vec{b}$ is $\frac{\vec{a} \cdot (-2\vec{b})}{|-2\vec{b}|}$. Using the same approach as above and recognizing that $|-2\vec{b}| = 2|\vec{b}|$, this can be rewritten as

$$\frac{\vec{a} \cdot (-2\vec{b})}{|-2\vec{b}|} = \frac{-2\vec{a} \cdot \vec{b}}{2|\vec{b}|} = \frac{-(\vec{a} \cdot \vec{b})}{|\vec{b}|}.$$

In this case, the direction of the vector $-2\vec{b}$ changes the scalar projection to the opposite sign from the projection of $\vec{a}$ on $\vec{b}$.

The following example shows how to calculate scalar projections involving algebraic vectors. All the properties applying to geometric vectors also apply to algebraic vectors.

EXAMPLE 2 Selecting a strategy to calculate the scalar projection involving algebraic vectors

For the vectors $\vec{a} = (-3, 4, 5\sqrt{3})$ and $\vec{b} = (-2, 2, -1)$, calculate each of the following scalar projections:

a. $\vec{a}$ on $\vec{b}$

b. $\vec{b}$ on $\vec{a}$

Solution

a. The required scalar projection is $|\vec{a}|\cos\theta$ and, since $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, as before, $|\vec{a}|\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

We start by calculating $\vec{a}\cdot\vec{b}$.

$$\begin{aligned}\vec{a}\cdot\vec{b} &= -3(-2) + 4(2) + 5\sqrt{3}(-1)\\ &= 14 - 5\sqrt{3}\\ &\doteq 5.34\end{aligned}$$

Since $|\vec{b}| = \sqrt{(-2)^2 + (2)^2 + (-1)^2} = 3$,

$$|\vec{a}|\cos\theta \doteq \frac{5.34}{3} \doteq 1.78$$

The scalar projection of $\vec{a}$ on $\vec{b}$ is approximately 1.78.

b. In this case, the required scalar projection is $|\vec{b}|\cos\theta$. Solving as in the solution to part a. $|\vec{b}|\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

Since $|\vec{a}| = (-3)^2 + (4)^2 + (5\sqrt{3})^2 = 10$,

$$|\vec{b}|\cos\theta \doteq \frac{5.34}{10} \doteq 0.53$$

The scalar projection of $\vec{b}$ on $\vec{a}$ is approximately 0.53.

Calculating Scalar Projections

The scalar projection of $\vec{a}$ on $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

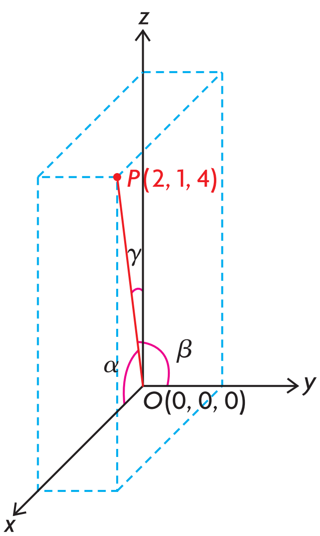
The scalar projection of $\vec{b}$ on $\vec{a}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

In general, $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} \neq \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

Scalar projections are sometimes used to calculate the angle that a position vector $\overrightarrow{OP}$ makes with each of the positive coordinate axes. This concept is illustrated in the next example.

EXAMPLE 3 Selecting a strategy to determine the direction angles of a vector in R^3

Determine the angle that the vector $\overrightarrow{OP} = (2, 1, 4)$ makes with each of the coordinate axes.



Solution

To calculate the required **direction angles**, it is necessary to project $\overrightarrow{OP}$ on each of the coordinate axes. To carry out the calculation, we use the standard basis vectors $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, and $\vec{k} = (0, 0, 1)$ so that $\overrightarrow{OP}$ can be projected along the x-axis, y-axis, and z-axis, respectively. We define α as the angle between $\overrightarrow{OP}$ and the positive x-axis, β as the angle between $\overrightarrow{OP}$ and the positive y-axis, and γ as the angle between $\overrightarrow{OP}$ and the positive z-axis.

Calculating α:
To calculate the angle that $\overrightarrow{OP}$ makes with the x-axis, we start by writing $\overrightarrow{OP} \cdot \vec{i} = |\overrightarrow{OP}||\vec{i}| \cos \alpha$, which implies $\cos \alpha = \dfrac{\overrightarrow{OP} \cdot \vec{i}}{|\overrightarrow{OP}||\vec{i}|}$.
Since $|\overrightarrow{OP}| = \sqrt{21}$ and $|\vec{i}| = 1$, we substitute to find

$$\cos \alpha = \frac{\overrightarrow{OP} \cdot \vec{i}}{|\overrightarrow{OP}||\vec{i}|}$$

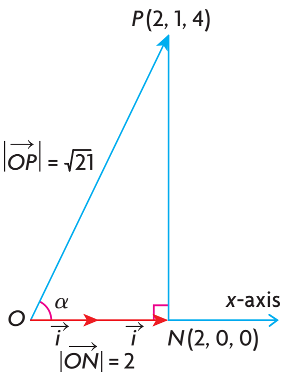
$$\cos \alpha = \frac{(2, 1, 4) \cdot (1, 0, 0)}{\sqrt{21}(1)}$$

$$\cos \alpha = \frac{2}{\sqrt{21}}$$

Thus, $\alpha = \cos^{-1}\left(\dfrac{2}{\sqrt{21}}\right)$ and $\alpha \doteq 64.1°$.

Therefore, the angle that $\overrightarrow{OP}$ makes with the x-axis is approximately $64.1°$. In its simplest terms, the cosine of the required angle α is the scalar projection of $\overrightarrow{OP}$ on $\vec{i}$, divided by $|\overrightarrow{OP}|$—that is, $\cos \alpha = \dfrac{\overrightarrow{OP} \cdot \vec{i}}{|\overrightarrow{OP}|} = \dfrac{2}{\sqrt{21}}$.

This angle is illustrated in the following diagram:



Calculating β and γ:
If we use the same procedure, we can also calculate β and γ, the angles that $\overrightarrow{OP}$ makes with the y-axis and z-axis, respectively.

Thus, $\cos\beta = \frac{(2, 1, 4)\cdot(0, 1, 0)}{\sqrt{21}} = \frac{1}{\sqrt{21}}$

$$\beta = \cos^{-1}\left(\frac{1}{\sqrt{21}}\right), \beta \doteq 77.4^\circ$$

Similarly, $\cos\gamma = \frac{4}{\sqrt{21}}, \gamma \doteq 29.2^\circ$

Therefore, $\overrightarrow{OP}$ makes angles of 64.1°, 77.4°, and 29.2° with the positive x-axis, y-axis and z-axis, respectively.

In our example, specific numbers were used, but the calculation is identical if we consider $\overrightarrow{OP} = (a, b, c)$ and develop a formula for the required direction angles. The cosines of the angles are referred to as the **direction cosines** of α, β, and γ.

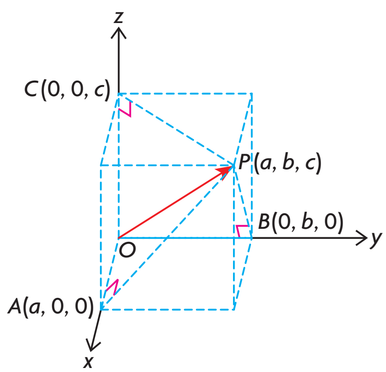
Direction Cosines for $\overrightarrow{OP} = (a, b, c)$

If α, β, and γ are the angles that $\overrightarrow{OP}$ makes with the positive x-axis, y-axis, and z-axis, respectively, then

$$\cos\alpha = \frac{(a, b, c)\cdot(1, 0, 0)}{|\overrightarrow{OP}|} = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos\beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}} \text{ and } \cos\gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

These angles can be visualized by constructing a rectangular box and drawing in the appropriate projections. If we are calculating α, the angle that $\overrightarrow{OP}$ makes with the positive x-axis, the projection of $\overrightarrow{OP}$ on the x-axis is just a, and $|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$, so $\cos\alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$.
We calculate $\cos\beta$ and $\cos\gamma$ in the same way.



EXAMPLE 4 Calculating a specific direction angle

For the vector $\overrightarrow{OP} = (-2\sqrt{2}, 4, -5)$, determine the direction cosine and the corresponding angle that this vector makes with the positive z-axis.

Solution

We can use the formula to calculate γ.

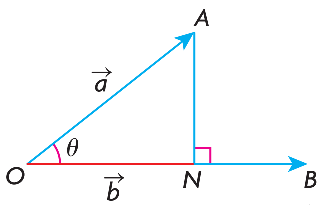
$$\cos\gamma = \frac{-5}{\sqrt{(-2\sqrt{2})^2 + (4)^2 + (-5)^2}} = \frac{-5}{\sqrt{49}} = \frac{-5}{7} \doteq -0.7143$$

and $\gamma \doteq 135.6^\circ$

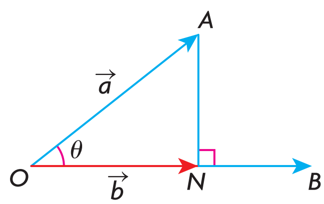
Examining Vector Projections

Thus far, we have calculated scalar projections of a vector onto a vector. This computation can be modified slightly to find the corresponding vector projection of a vector on a vector.

The calculation of the vector projection of $\vec{a}$ on $\vec{b}$ is just the corresponding scalar projection of $\vec{a}$ on $\vec{b}$ multiplied by $\frac{\vec{b}}{|\vec{b}|}$. The expression $\frac{\vec{b}}{|\vec{b}|}$ is a unit vector pointing in the direction of $\vec{b}$.



Scalar projection of $\vec{a}$ on $\vec{b}$



Vector projection of $\vec{a}$ on $\vec{b}$

Vector Projection of $\vec{a}$ on $\vec{b}$

vector projection of $\vec{a}$ on $\vec{b}$

$= $ (scalar projection of $\vec{a}$ on $\vec{b}$) (unit vector in the direction of $\vec{b}$)

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\left(\frac{\vec{b}}{|\vec{b}|}\right)$$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\vec{b}, \vec{b} \neq \vec{0}$$

EXAMPLE 5 Connecting a scalar projection to its corresponding vector projection

Find the vector projection of $\overrightarrow{OA} = (4, 3)$ on $\overrightarrow{OB} = (4, -1)$.

Solution

The formula for the scalar projection of $\overrightarrow{OA}$ on $\overrightarrow{OB}$ is $\dfrac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OB}|}$.

$$ON = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OB}|} = \frac{(4, 3) \cdot (4, -1)}{\sqrt{(4)^2 + (-1)^2}}$$

$$= \frac{13}{\sqrt{17}}$$

The vector projection, $\overrightarrow{ON}$, is found by multiplying ON by the unit vector $\dfrac{\overrightarrow{OB}}{|\overrightarrow{OB}|}$.

Since $|\overrightarrow{OB}| = \sqrt{(4)^2 + (-1)^2} = \sqrt{17}$,

$$\frac{\overrightarrow{OB}}{|\overrightarrow{OB}|} = \frac{1}{\sqrt{17}}(4, -1)$$

The required vector projection is

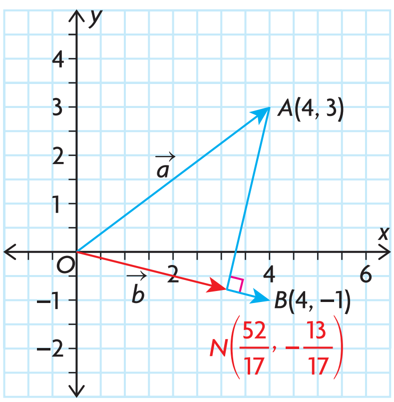
$\overrightarrow{ON} = (ON)$(a unit vector in the same direction as $\overrightarrow{OB}$)

$$\overrightarrow{ON} = \frac{13}{\sqrt{17}}\left(\frac{1}{\sqrt{17}}(4, -1)\right)$$

$$= \frac{13}{17}(4, -1)$$

$$= \left(\frac{52}{17}, -\frac{13}{17}\right)$$

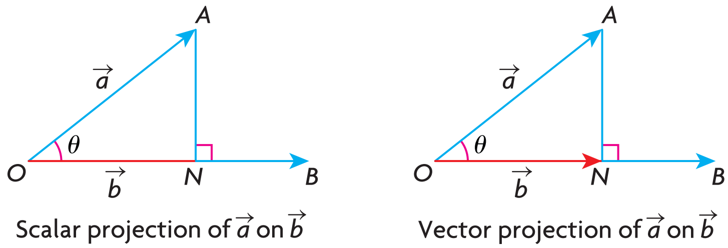
The vector projection $\overrightarrow{ON}$ is shown in red in the following diagram:



IN SUMMARY

Key Idea

- A projection of one vector onto another can be either a scalar or a vector. The difference is the vector projection has a direction.



Need to Know

- The scalar projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = |\vec{a}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.
- The vector projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} = \left(\dfrac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\vec{b}$
- The direction cosines for $\overrightarrow{OP} = (a, b, c)$ are
  $\cos\alpha = \dfrac{a}{\sqrt{a^2 + b^2 + c^2}}$, $\cos\beta = \dfrac{b}{\sqrt{a^2 + b^2 + c^2}}$,
  $\cos\gamma = \dfrac{c}{\sqrt{a^2 + b^2 + c^2}}$, where α, β, and γ are the direction angles between the position vector $\overrightarrow{OP}$ and the positive x-axis, y-axis and z-axis, respectively.

Exercise 7.5

PART A

1. a. The vector $\vec{a} = (2, 3)$ is projected onto the x-axis. What is the scalar projection? What is the vector projection?

   b. What are the scalar and vector projections when $\vec{a}$ is projected onto the y-axis?

2. Explain why it is not possible to obtain either a scalar projection or a vector projection when a nonzero vector $\vec{x}$ is projected on $\vec{0}$.

3. Consider two nonzero vectors, $\vec{a}$ and $\vec{b}$, that are perpendicular to each other. Explain why the scalar and vector projections of $\vec{a}$ on $\vec{b}$ must be 0 and $\vec{0}$, respectively. What are the scalar and vector projections of $\vec{b}$ on $\vec{a}$?

4. Draw two vectors, $\vec{p}$ and $\vec{q}$. Draw the scalar and vector projections of $\vec{p}$ on $\vec{q}$. Show, using your diagram, that these projections are not necessarily the same as the scalar and vector projections of $\vec{q}$ on $\vec{p}$.

5. Using the formulas in this section, determine the scalar and vector projections of $\overrightarrow{OP} = (-1, 2, -5)$ on $\vec{i}$, $\vec{j}$, and $\vec{k}$. Explain how you could have arrived at the same answer without having to use the formulas.

PART B

6. a. For the vectors $\vec{p} = (3, 6, -22)$ and $\vec{q} = (-4, 5, -20)$, determine the scalar and vector projections of $\vec{p}$ on $\vec{q}$.

   b. Determine the direction angles for $\vec{p}$.

K 7. For each of the following, determine the scalar and vector projections of $\vec{x}$ on $\vec{y}$.

   a. $\vec{x} = (1, 1), \vec{y} = (1, -1)$

   b. $\vec{x} = (2, 2\sqrt{3}), \vec{y} = (1, 0)$

   c. $\vec{x} = (2, 5), \vec{y} = (-5, 12)$

8. a. Determine the scalar and vector projections of $\vec{a} = (-1, 2, 4)$ on each of the three axes.

   b. What are the scalar and vector projections of $m(-1, 2, 4)$ on each of the three axes?

C 9. a. Given the vector $\vec{a}$, show with a diagram that the vector projection of $\vec{a}$ on $\vec{a}$ is $\vec{a}$ and that the scalar projection of $\vec{a}$ on $\vec{a}$ is $|\vec{a}|$.

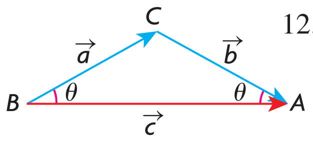
   b. Using the formulas for scalar and vector projections, explain why the results in part a. are correct if we use $\theta = 0°$ for the angle between the two vectors.

10. a. Using a diagram, show that the vector projection of $-\vec{a}$ on $\vec{a}$ is $-\vec{a}$.

   b. Using the formula for determining scalar projections, show that the result in part a. is true.

A 11. a. Find the scalar and vector projections of $\overrightarrow{AB}$ along each of the axes if A has coordinates $(1, 2, 2)$ and B has coordinates $(-1, 3, 4)$.

   b. What angle does $\overrightarrow{AB}$ make with the y-axis?



12. In the diagram shown, $\triangle ABC$ is an isosceles triangle where $|\vec{a}| = |\vec{b}|$.
    a. Draw the scalar projection of $\vec{a}$ on $\vec{c}$.
    b. Relocate $\vec{b}$, and draw the scalar projection of $\vec{b}$ on $\vec{c}$.
    c. Explain why the scalar projection of $\vec{a}$ on $\vec{c}$ is the same as the scalar projection of $\vec{b}$ on $\vec{c}$.
    d. Does the vector projection of $\vec{a}$ on $\vec{c}$ equal the vector projection of $\vec{b}$ on $\vec{c}$?
13. Vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 10$ and $|\vec{b}| = 12$, and the angle between them is $135°$.
    a. Show that the scalar projection of $\vec{a}$ on $\vec{b}$ does not equal the scalar projection of $\vec{b}$ on $\vec{a}$.
    b. Draw diagrams to illustrate the corresponding vector projections associated with part a.
14. You are given the vector $\overrightarrow{OD} = (-1, 2, 2)$ and the three points, $A(-2, 1, 4)$, $B(1, 3, 3)$, and $C(-6, 7, 5)$.
    a. Calculate the scalar projection of $\overrightarrow{AB}$ on $\overrightarrow{OD}$.
    b. Verify computationally that the scalar projection of $\overrightarrow{AB}$ on $\overrightarrow{OD}$ added to the scalar projection of $\overrightarrow{BC}$ on $\overrightarrow{OD}$ equals the scalar projection of $\overrightarrow{AC}$ on $\overrightarrow{OD}$.
    c. Explain why this same result is also true for the corresponding vector projections.

T

15. a. If α, β, and γ represent the direction angles for vector $\overrightarrow{OP}$, prove that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.
    b. Determine the coordinates of a vector $\overrightarrow{OP}$ that makes an angle of $30°$ with the y-axis, $60°$ with the z-axis, and $90°$ with the x-axis.
    c. In Example 3, it was shown that, in general, the direction angles do not always add to $180°$—that is, $\alpha + \beta + \gamma \neq 180°$. Under what conditions, however, must the direction angles always add to $180°$?

PART C

16. A vector in R^3 makes equal angles with the coordinate axes. Determine the size of each of these angles if the angles are
    a. acute
    b. obtuse
17. If α, β, and γ represent the direction angles for vector $\overrightarrow{OP}$, prove that $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$.
18. Vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are not collinear. The sum of the direction angles of each vector is $180°$. Draw diagrams to illustrate possible positions of points A and B.