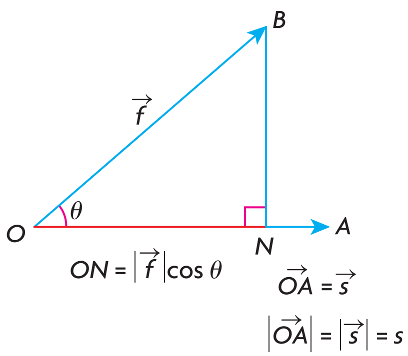
Section 7.7—Applications of the Dot Product and Cross Product

In the previous four sections, the dot product and cross product were discussed in some detail. In this section, some physical and mathematical applications of these concepts will be introduced to give a sense of their usefulness in both physical and mathematical situations.

Physical Application of the Dot Product

When a force is acting on an object so that the object is moved from one point to another, we say that the force has done work. Work is defined as the product of the distance an object has been displaced and the component of the force along the line of displacement.

In the following diagram, $\overrightarrow{OB}$ represents a constant force, $\vec{f}$, acting on an object at O so that this force moves the object from O to A. We will call the distance that the object is displaced s, which is a scalar, where we are assuming that $\vec{s} = \overrightarrow{OA}$ and $s = |\overrightarrow{OA}|$. The scalar projection of $\vec{f}$ on $\overrightarrow{OA}$ equals ON, or $|\vec{f}|\cos\theta$, which is the same calculation for the scalar projection that was done earlier. (This is called the scalar component of $\overrightarrow{OB}$ on $\overrightarrow{OA}$.) The work, W, done by $\vec{f}$ in moving the object is calculated as $W = (|\vec{f}|\cos\theta)(|\overrightarrow{OA}|) = (ON)(s) = \vec{f}\cdot\vec{s}$. As explained before, the force $\vec{f}$ is measured in newtons (N), the displacement is measured in metres (m), and the unit for work is newton-metres, or joules (J). When a 1 N force moves an object 1 m, the amount of work done is 1 J.



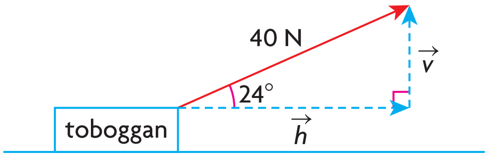
Formula for the Calculation of Work

$W = \vec{f}\cdot\vec{s}$, where $\vec{f}$ is the force acting on an object, measured in newtons (N); $\vec{s}$ is the displacement of the object, measured in metres (m); and W is the work done, measured in joules (J).

EXAMPLE 1 Using the dot product to calculate work

Marianna is pulling her daughter in a toboggan and is exerting a force of 40 N, acting at 24° to the ground. If Marianna pulls the child a distance of 100 m, how much work was done?

Solution



To solve this problem, the 40 N force has been resolved into its vertical and horizontal components. The horizontal component $\vec{h}$ tends to move the toboggan forward, while the vertical component $\vec{v}$ is the force that tends to lift the toboggan.

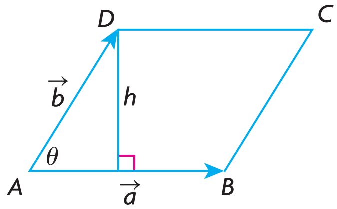
$$\text{From the diagram, } |\vec{h}| = 40\cos 24° \doteq 40(0.9135) \doteq 36.54 \text{ N}$$

The amount of work done is $W \doteq (36.54)(100) \doteq 3654$ J. Therefore, the work done by Marianna is approximately 3654 J.

Geometric Application of the Cross Product

The cross product of two vectors is interesting because calculations involving the cross product can be applied in a number of different ways, giving us results that are important from both a mathematical and physical perspective.

The cross product of two vectors, $\vec{a}$ and $\vec{b}$, can be used to calculate the area of a parallelogram. For any parallelogram, $ABCD$, it is possible to develop a formula for its area, where $\vec{a}$ and $\vec{b}$ are vectors determining its sides and h is its height.



$$\text{Area} = |\vec{a}|h = |\vec{a}|(|\vec{b}|\sin\theta) = |\vec{a}||\vec{b}|\sin\theta$$

where $\sin\theta = \dfrac{h}{|\vec{b}|}$ or $h = |\vec{b}|\sin\theta$

It can be proven that this formula for the area is equal to $|\vec{a} \times \vec{b}|$. That is, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$.

Theorem: For two vectors, $\vec{a}$ and $\vec{b}$, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$, where θ is the angle between the two vectors.

Proof: The formula for the cross product is

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

Therefore, $|\vec{a} \times \vec{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$

The right-hand side is expanded and then factored to give

$$|\vec{a} \times \vec{b}|^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

This formula can be simplified by making the following substitutions:

$|\vec{a}|^2 = a_1^2 + a_2^2 + a_3^2$, $|\vec{b}|^2 = b_1^2 + b_2^2 + b_3^2$, and
$|\vec{a}||\vec{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$

Thus, $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$ (Factor)

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2(1 - \cos^2\theta) = |\vec{a}|^2|\vec{b}|^2\sin^2\theta$$ (Substitution)

$$|\vec{a} \times \vec{b}| = \pm |\vec{a}||\vec{b}|\sin\theta$$

But since $\sin\theta \geq 0$ for $0^\circ \leq \theta \leq 180^\circ$,

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

This gives us the required formula for the area of a parallelogram, which is equivalent to the magnitude of the cross product between the vectors that define the parallelogram.

EXAMPLE 2 Solving area problems using the cross product

a. Determine the area of the parallelogram determined by the vectors $\vec{p} = (-1, 5, 6)$ and $\vec{q} = (2, 3, -1)$.

b. Determine the area of the triangle formed by the points $A(-1, 2, 1)$, $B(-1, 0, 0)$, and $C(3, -1, 4)$.

Solution

a. The cross product is

$$\vec{p} \times \vec{q} = (5(-1) - 3(6), 6(2) - (-1)(-1), -1(3) - 2(5))$$
$$= (-23, 11, -13)$$

The required area is determined by $|\vec{p} \times \vec{q}|$.

$$\sqrt{(-23)^2 + 11^2 + (-13)^2} = \sqrt{529 + 121 + 169} = \sqrt{819}$$
$$\doteq 28.62 \text{ square units}$$

b. We start by constructing position vectors equal to $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Thus,

$$\overrightarrow{AB} = (-1 - (-1), 0 - 2, 0 - 1) = (0, -2, -1) \text{ and } \overrightarrow{AC} = (4, -3, 3)$$

Calculating,

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= (-2(3) - (-1)(-3), -1(4) - 0(3), 0(-3) - (-2)(4)) \\ &= (-9, -4, 8)\end{aligned}$$

And

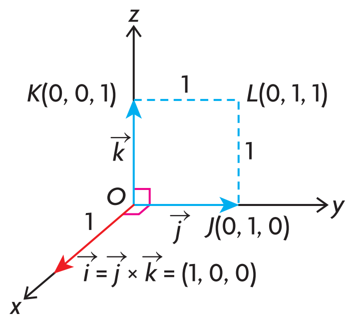
$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-9)^2 + (-4)^2 + (8)^2} = \sqrt{161}$$

Therefore, the area of $\triangle ABC$ is one half of the area of the parallelogram formed by vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, which is $\frac{1}{2}\sqrt{161} \doteq 6.34$ square units.

This connection between the magnitude of the cross product and area allows us further insight into relationships in R^3. This calculation makes a direct and precise connection between the length of the cross product and the area of the parallelogram formed by two vectors. These two vectors can be anywhere in 3-space, not necessarily in the plane. As well, it also allows us to determine, in particular cases, the cross product of two vectors without having to carry out any computation.

EXAMPLE 3 Reasoning about a cross product involving the standard unit vectors

Without calculating, explain why the cross product of $\vec{j}$ and $\vec{k}$ is $\vec{i}$—that is, $\vec{j} \times \vec{k} = \vec{i}$.



Solution

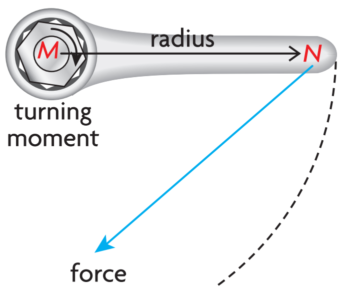
As shown in the diagram, the area of square *OJLK* is 1. The cross product, $\vec{j} \times \vec{k}$, is a vector perpendicular to the plane determined by $\vec{j}$ and $\vec{k}$, and must therefore lie along either the positive or negative x-axis. Using the definition of the cross product, and knowing that these vectors form a right-handed system, the only possibility is that the cross product must then lie along the *positive* x-axis. The length of the cross product must equal the area of the square *OJLK*, which is 1. So, the required cross product is $\vec{i}$ since $|\vec{i}| = 1$.

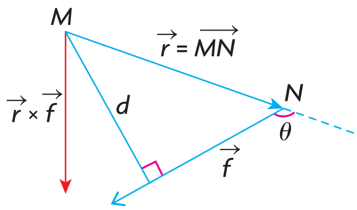
Using the same kind of reasoning, it is interesting to note that $\vec{k} \times \vec{j}$ is $-\vec{i}$, which could be determined by using the definition of a right-handed system and verified by calculation.

Physical Application of the Cross Product

The cross product can also be used in the consideration of forces that involve rotation, or turning about a point or an axis. The rotational or turning effect of a force is something that is commonly experienced in everyday life. A typical example might be the tightening or loosening of a nut using a wrench. A second example is the application of force to a bicycle pedal to make the crank arm rotate. The simple act of opening a door by pushing or pulling on it is a third example of how force can be used to create a turning effect. In each of these cases, there is rotation about either a point or an axis.

In the following situation, a bolt with a right-hand thread is being screwed into a piece of wood by a wrench, as shown. A force $\vec{f}$ is applied to the wrench at point N and is rotating about point M. The vector $\vec{r} = \overrightarrow{MN}$ is the position vector of N with respect to M—that is, it defines the position of N relative to M.





The torque, or the turning effect, of the force $\vec{f}$ about the point M is defined to be the vector $\vec{r} \times \vec{f}$. This vector is perpendicular to the plane formed by the vectors $\vec{r}$ and $\vec{f}$, and gives the direction of the axis through M about which the force tends to twist. In this situation, the vector representing the cross product is directed down as the bolt tightens into the wood and would normally be directed along the axis of the bolt. The magnitude of the torque depends upon two factors: the exerted force, and the distance between the line of the exerted force and the point of rotation, M. The exerted force is $\vec{f}$, and the distance between M and the line of the exerted force is d. The magnitude of the torque is the product of the magnitude of the force (that is, $|\vec{f}|$) and the distance d. Since $d = |\vec{r}|\sin\theta$, the magnitude of the torque $\vec{f}$ about M is $(|\vec{r}|\sin\theta)(|\vec{f}|) = |\vec{r} \times \vec{f}|$. The magnitude of the torque measures the twisting effect of the applied force.

The force $\vec{f}$ is measured in newtons, and the distance d is measured in metres, so the unit of magnitude for torque is (newton)(metres), or joules (J), which is the same unit that work is measured in.

EXAMPLE 4

Using the cross product to calculate torque

A 20 N force is applied at the end of a wrench that is 40 cm in length. The force is applied at an angle of 60° to the wrench. Calculate the magnitude of the torque about the point of rotation M.

Solution

$$|\vec{r} \times \vec{f}| = (|\vec{r}|\sin\theta)|\vec{f}| = (0.40)(20)\frac{\sqrt{3}}{2} \doteq 6.93 \text{ J}$$

One of the implications of calculating the magnitude of torque, $|\vec{r} \times \vec{f}| = |\vec{r}||\vec{f}|\sin\theta$, is that it is maximized when $\sin\theta = 1$ and when the force is applied as far as possible from the turning point—that is, $|\vec{r}|$ is as large as possible. To get the best effect when tightening a bolt, this implies that force should be applied at right angles to the wrench and as far down the handle of the wrench as possible from the turning point.

IN SUMMARY

Key Idea

- Both the dot and cross products have useful applications in geometry and physics.

Need to Know

- $W = \vec{F} \cdot \vec{s}$, where $\vec{F}$ is the force applied to an object, measured in newtons (N); $\vec{s}$ is the displacement of the object, measured in metres (m); and W is work, measured in joules (J).
- $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$
- Area of a parallelogram, with sides $\vec{a}$ and $\vec{b}$, equals $|\vec{a} \times \vec{b}|$
- Area of a triangle, with sides $\vec{a}$ and $\vec{b}$, equals $\frac{1}{2}|\vec{a} \times \vec{b}|$.
- Torque equals $\vec{r} \times \vec{f} = |\vec{r}||\vec{f}|\sin\theta$.
- $|\vec{r} \times \vec{f}|$, the magnitude of the torque, measures the overall twisting effect of applied force.

Exercise 7.7

PART A

C 1. A door is opened by pushing inward. Explain, in terms of torque, why this is most easily accomplished when pushing at right angles to the door as far as possible from the hinge side of the door.

2. a. Calculate $|\vec{a} \times \vec{b}|$, where $\vec{a} = (1, 2, 1)$ and $\vec{b} = (2, 4, 2)$.
   b. If $\vec{a}$ and $\vec{b}$ represent the sides of a parallelogram, explain why your answer for part a. makes sense, in terms of the formula for the area of a parallelogram.

PART B

3. Calculate the amount of work done in each situation.
   a. A stove is slid 3 m across the floor against a frictional force of 150 N.
   b. A 40 kg rock falls 40 m down a slope at an angle of 50° to the vertical.
   c. A wagon is pulled a distance of 250 m by a force of 140 N applied at an angle of 20° to the road.
   d. A lawnmower is pushed 500 m by a force of 100 N applied at an angle of 45° to the horizontal.
4. Determine each of the following by using the method shown in Example 3:
   a. $\vec{i} \times \vec{j}$ b. $-\vec{i} \times \vec{j}$ c. $\vec{i} \times \vec{k}$ d. $-\vec{i} \times \vec{k}$

K

5. Calculate the area of the parallelogram formed by the following pairs of vectors:
   a. $\vec{a} = (1, 1, 0)$ and $\vec{b} = (1, 0, 1)$ b. $\vec{a} = (1, -2, 3)$ and $\vec{b} = (1, 2, 4)$
6. The area of the parallelogram formed by the vectors $\vec{p} = (a, 1, -1)$ and $\vec{q} = (1, 1, 2)$ is $\sqrt{35}$. Determine the value(s) of a for which this is true.
7. In R^3, points $A(-2, 1, 3)$, $B(1, 0, 1)$, and $C(2, 3, 2)$ form the vertices of $\triangle ABC$.
   a. By constructing position vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, determine the area of the triangle.
   b. By constructing position vectors $\overrightarrow{BC}$ and $\overrightarrow{CA}$, determine the area of the triangle.
   c. What conclusion can be drawn?

A

8. A 10 N force is applied at the end of a wrench that is 14 cm long. The force makes an angle of 45° with the wrench. Determine the magnitude of the torque of this force about the other end of the wrench.

T

9. Parallelogram $OBCA$ has its sides determined by $\overrightarrow{OA} = \vec{a} = (4, 2, 4)$ and $\overrightarrow{OB} = \vec{b} = (3, 1, 4)$. Its fourth vertex is point C. A line is drawn from B perpendicular to side AC of the parallelogram to intersect AC at N. Determine the length of BN.

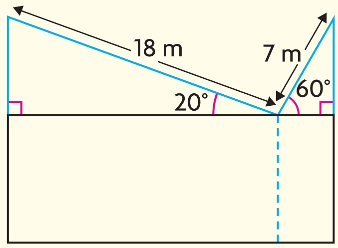
PART C

10. For the vectors $\vec{p} = (1, -2, 3)$, $\vec{q} = (2, 1, 3)$, and $\vec{r} = (1, 1, 0)$, show the following to be true.
    a. The vector $(\vec{p} \times \vec{q}) \times \vec{r}$ can be written as a linear combination of $\vec{p}$ and $\vec{q}$.
    b. $(\vec{p} \times \vec{q}) \times \vec{r} = (\vec{p} \cdot \vec{r})\vec{q} - (\vec{q} \cdot \vec{r})\vec{p}$

CAREER LINK WRAP-UP Investigate and Apply

CHAPTER 7: STRUCTURAL ENGINEERING

A structural engineer is designing a special roof for a building. The roof is designed to catch rainwater and hold solar panels to collect sunlight for electricity. Each angled part of the roof exerts a downward force of 50 kg/m^2, including the loads of the panels and rainwater. The building will need a load-bearing wall at the point where each angled roof meets.



a. Calculate the force of the longer angled roof at the point where the roofs meet.

b. Calculate the force of the shorter angled roof at the point where the roofs meet.

c. Calculate the resultant force that the load-bearing wall must counteract to support the roof.

d. Use the given lengths and angles to calculate the width of the building.

e. If the point where the two roofs meet is moved 2 m to the left, calculate the angles that the sloped roofs will make with the horizontal and the length of each roof. Assume that only the point where the roofs meet can be adjusted and that the height of each roof will not change.

f. Repeat parts a. to c., using the new angles you calculated in part e.

g. Make a conjecture about the angles that the two roofs must make with the horizontal (assuming again that the heights are the same but the point where the roofs meet can be adjusted) to minimize the downward force that the load-bearing wall will have to counteract.

h. Calculate the downward force for the angles you conjectured in part g. Then perform the calculations for other angles to test your conjecture.