Section 8.1—Vector and Parametric Equations of a Line in R^2

In this section, we begin with a discussion about how to find the **vector** and **parametric equations** of a line in R^2. To find the vector and parametric equations of a line, we must be given either two distinct points or one point and a vector that defines the direction of the line. In either situation, a **direction vector** for the line is necessary. A direction vector is defined to be a nonzero vector $\vec{m} = (a, b)$ parallel (collinear) to the given line. The direction vector $\vec{m} = (a, b)$ is represented by a vector with its tail at the origin and its head at the point (a, b). The x and y components of this direction vector are called its **direction numbers**. For the vector (a, b), the direction numbers are a and b.

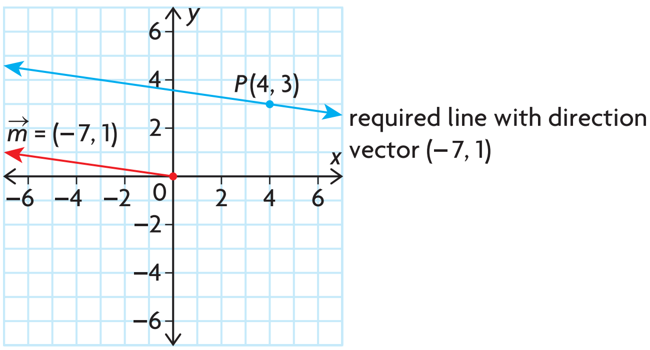
EXAMPLE 1 Representing lines using vectors

a. A line passing through $P(4, 3)$ has $\vec{m} = (-7, 1)$ as its direction vector. Sketch this line.

b. A line passes through the points $A\left(\frac{1}{2}, -3\right)$ and $B\left(\frac{3}{4}, \frac{1}{2}\right)$. Determine a direction vector for this line, and write it using integer components.

Solution

a. The vector $\vec{m} = (-7, 1)$ is a direction vector for the line and is shown on the graph. The required line is parallel to $\vec{m}$ and passes through $P(4, 3)$. This line is drawn through $P(4, 3)$, parallel to $\vec{m}$.



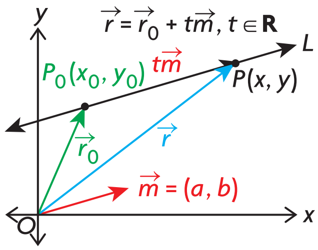
b. When determining a direction vector for the line through $A\left(\frac{1}{2}, -3\right)$ and $B\left(\frac{3}{4}, \frac{1}{2}\right)$, we determine a vector equivalent to either $\overrightarrow{AB}$ or $\overrightarrow{BA}$.

$$\overrightarrow{AB} = \left(\frac{3}{4} - \frac{1}{2}, \frac{1}{2} - (-3)\right) = \left(\frac{1}{4}, \frac{7}{2}\right) \text{ or } \overrightarrow{BA} = \left(-\frac{1}{4}, -\frac{7}{2}\right)$$

Both of these vectors can be multiplied by 4 to ensure that both direction numbers are integers. As a result, either $\vec{m} = (1, 14)$ or $\vec{m} = (-1, -14)$ are the best choices for a direction vector. When we determine the direction vector, any scalar multiple of this vector of the form $t(1, 14)$ is correct, provided that $t \neq 0$. If $t = 0$, $(0, 0)$ would be the direction vector, meaning that the line would not have a defined direction.

Expressing the Equations of Lines Using Vectors

In general, we would like to determine the equation of a line if we have a direction for the line and a point on it. In the following diagram, the given point $P_0(x_0, y_0)$ is on the line L and is associated with vector $\overrightarrow{OP_0}$, designated as $\vec{r_0}$. The direction of the line is given by $\vec{m} = (a, b)$, where $t\vec{m}$, $t \in \mathbf{R}$ is any vector collinear with $\vec{m}$. $P(x, y)$ represents a general point on the line, where $\overrightarrow{OP}$ is the vector associated with this point.



To find the vector equation of line L, the triangle law of addition is used.

In $\triangle OP_0P$, $\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$.

Since $\vec{r} = \overrightarrow{OP}$, $\vec{r_0} = \overrightarrow{OP_0}$, and $t\vec{m} = \overrightarrow{P_0P}$, the vector equation of the line is written as $\vec{r} = \vec{r_0} + t\vec{m}, t \in \mathbf{R}$.

When writing an equation of a line using vectors, the vector form of the line is sometimes modified and put in parametric form. The parametric equations of a line come directly from its vector equation. How to change the equation of a line from vector to parametric form is shown below.

The general vector equation of a line is $\vec{r} = \vec{r_0} + t\vec{m}, t \in \mathbf{R}$.

In component form, this is written as $(x, y) = (x_0, y_0) + t(a, b), t \in \mathbf{R}$. Expanding the right side, $(x, y) = (x_0, y_0) + (ta, tb) = (x_0 + ta, y_0 + tb), t \in \mathbf{R}$. If we equate the respective x and y components, the required parametric form is $x = x_0 + ta$ and $y = y_0 + tb$, $t \in \mathbf{R}$.

Vector and Parametric Equations of a Line in R^2

Vector Equation: $\vec{r} = \vec{r_0} + t\vec{m}, t \in \mathbf{R}$
Parametric Equations: $x = x_0 + ta$, $y = y_0 + tb$, $t \in \mathbf{R}$
where $\vec{r_0}$ is the vector from $(0, 0)$ to the point (x_0, y_0) and $\vec{m}$ is a direction vector with components (a, b).

In either vector or parametric form, t is called a **parameter**. This means that t can be replaced by any real number to obtain the coordinates of points on the line.

EXAMPLE 2 Reasoning about the vector and parametric equations of a line

a. Determine the vector and parametric equations of a line passing through point $A(1, 4)$ with direction vector $\vec{m} = (-3, 3)$.
b. Sketch the line, and determine the coordinates of four points on the line.
c. Is either point $Q(-21, 23)$ or point $R(-29, 34)$ on this line?

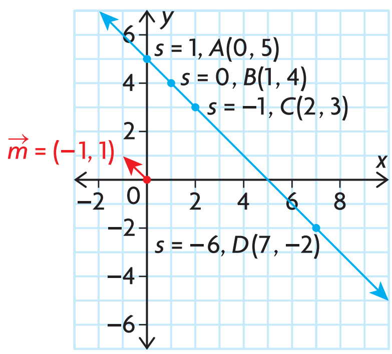
Solution

a. Since $A(1, 4)$ is on the line, $\overrightarrow{OP_0} = \vec{r}_0 = (1, 4)$ and $\vec{m} = (-3, 3)$. The vector equation is $\vec{r} = (1, 4) + t(-3, 3)$, $t \in \mathbf{R}$. The parametric equations are $x = 1 - 3t$, $y = 4 + 3t$, $t \in \mathbf{R}$.

It is also possible to use other scalar multiples of $\vec{m} = (-3, 3)$ as a direction vector, such as $(-1, 1)$, which gives the respective vector and parametric equations $\vec{r} = (1, 4) + s(-1, 1)$, $s \in \mathbf{R}$, and $x = 1 - s$, $y = 4 + s$, $s \in \mathbf{R}$. The vector $(-1, 1)$ has been chosen as our direction vector for the sake of simplicity. Note that we have written the second equation with parameter s to avoid confusion between the two lines. Although the two equations, $\vec{r} = (1, 4) + t(-3, 3)$, $t \in \mathbf{R}$, and $\vec{r} = (1, 4) + s(-1, 1)$, $s \in \mathbf{R}$, appear with different parameters, the lines they represent are identical.

b. To determine the coordinates of points on the line, the parametric equations $x = 1 - s$, $y = 4 + s$, $s \in \mathbf{R}$, were used, with s chosen to be 0, 1, -1, and -6. To find the coordinates of a particular point, such as D, $s = -6$ was substituted into the parametric equations and $x = 1 - (-6) = 7$, $y = 4 + (-6) = -2$.

The required point is $D(7, -2)$. The coordinates of the other points are determined in the same way, using the other values of s.



c. If the point $Q(-21, 23)$ lies on the line, then there must be consistency with the parameter s. We substitute this point into the parametric equations to check for the required consistency. Substituting gives $-21 = 1 - s$ and $23 = 4 + s$.

In the first equation, $s = 22$, and in the second equation, $s = 19$. Since these values are inconsistent, the point Q is not on the line.

If the point $R(-29, 34)$ is on the line, then $-29 = 1 - s$ and $34 = 4 + s$, $s = 30$, for both equations.

Since each of these equations has the same solution, $s = 30$, we conclude that $R(-29, 34)$ is on the line.

Sometimes, the equation of the line must be found when two points are given. This is shown in the following example.

EXAMPLE 3 Connecting vector and parametric equations with two points on a line

a. Determine vector and parametric equations for the line containing points $E(-1, 5)$ and $F(6, 11)$.
b. What are the coordinates of the point where this line crosses the x-axis?
c. Can the equation $\vec{r} = (-15, -7) + t\left(\frac{14}{3}, 4\right)$, $t \in \mathbf{R}$, also represent the line containing points E and F?

Solution

a. A direction vector for the line containing points E and F is $\vec{m} = \overrightarrow{EF} = (6 - (-1), 11 - 5) = (7, 6)$. A vector equation for the line is $\vec{r} = (-1, 5) + s(7, 6)$, $s \in \mathbf{R}$, and its parametric equations are $x = -1 + 7s$, $y = 5 + 6s$, $s \in \mathbf{R}$.

The equation given for this line is not unique. This is because there are an infinite number of choices for the direction vector, and any point on the line could have been used. In writing a second equation for the line, the parametric equations $x = 6 + 7s$, $y = 11 + 6s$, $s \in \mathbf{R}$, would also have been correct because $(6, 11)$ is on the line and the direction vector is $(7, 6)$.

b. The line intersects the x-axis at a point with coordinates of the form $(a, 0)$. At the point of intersection, $y = 0$ and, so, $5 + 6s = 0$, $s = \frac{-5}{6}$. Therefore,

$$\begin{aligned} a &= -1 + 7s \\ &= -1 + 7\left(\frac{-5}{6}\right) \\ &= -\frac{41}{6}, \end{aligned}$$

and the line intersects the x-axis at the point $\left(-\frac{41}{6}, 0\right)$.

c. If this equation represents the same line as the equation in part a., it is necessary for the two lines to have the same direction and contain the same set of points.

The line $\vec{r} = (-15, -7) + t\left(\frac{14}{3}, 4\right), t \in \mathbf{R}$, has $\left(\frac{14}{3}, 4\right)$ as its direction vector.

The two lines will have the same direction vectors because $\frac{3}{2}\left(\frac{14}{3}, 4\right) = (7, 6)$.

The two lines have the same direction, and if these lines have a point in common, then the equations represent the same line. The easiest approach is to substitute $(-15, -7)$ into the first equation to see if this point is on the line. Substituting gives $(-15, -7) = (-1, 5) + s(7, 6)$ or $-15 = -1 + 7s$ and $-7 = 5 + 6s$. Since the solution to both of these equations is $s = -2$, the point $(-15, -7)$ is on the line, and the two equations represent the same line.

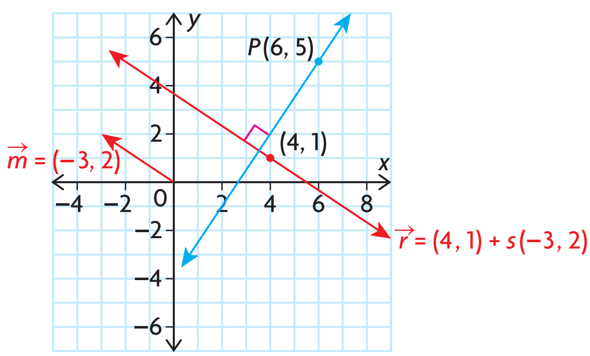
In the next example, vector properties will be used to determine equations for lines that involve perpendicularity.

EXAMPLE 4 Selecting a strategy to determine the vector equation of a perpendicular line

Determine a vector equation for the line that is perpendicular to $\vec{r} = (4, 1) + s(-3, 2), s \in \mathbf{R}$, and passes through point $P(6, 5)$.

Solution

The direction vector for the given line is $\vec{m} = (-3, 2)$, and this line is drawn through $(4, 1)$, as shown in red in the diagram. A sketch of the required line, passing through $(6, 5)$ and perpendicular to the given line, is drawn in blue.



Let the direction vector for the required blue line be $\vec{v} = (a, b)$. Since the direction vector of the given line is perpendicular to that of the required line, $\vec{v} \cdot \vec{m} = 0$.

Therefore, $(a, b) \cdot (-3, 2) = 0$ or $-3a + 2b = 0$.

The simplest integer values for a and b, which satisfy this equation, are $a = 2$ and $b = 3$. This gives the direction vector $(2, 3)$ and the required vector equation for the perpendicular line is $\vec{r} = (6, 5) + t(2, 3),\ t \in \mathbf{R}$.

In this section, the vector and parametric equations of a line in R^2 were discussed. In Section 8.3, the discussion will be extended to R^3, where many of the ideas seen in this section apply to lines in three-space.

The following investigation is designed to aid in understanding the concept of parameter, when dealing with either the vector or parametric equations of a line.

INVESTIGATION

A. i. On graph paper, draw the lines $L_1: \vec{r} = t(0, 1),\ t \in \mathbf{R}$, and $L_2: \vec{r} = p(1, 0)$, $p \in \mathbf{R}$. Make sure that you clearly show a direction vector for each line.

ii. Describe geometrically what each of the two equations represent.

iii. Give a vector equation and corresponding parametric equations for each of the following:

- the line parallel to the x-axis, passing through $P(2, 4)$
- the line parallel to the y-axis, passing through $Q(-2, -1)$

iv. Sketch L_3: $x = -3,\ y = 1 + s,\ s \in \mathbf{R}$, and L_4: $x = 4 + t,\ y = 1,\ t \in \mathbf{R}$, using your own axes.

v. By examining parametric equations of a line, how is it possible to determine by inspection whether the line is parallel to either the x-axis or y-axis?

vi. Write an equation of a line in both vector and parametric form that is parallel to the x-axis.

vii. Write an equation of a line in both vector and parametric form that is parallel to the y-axis.

B. i. Sketch the line $L: \vec{r} = (-3, 0) + s(2, -1), s \in \mathbf{R}$, on graph paper.

ii. On the set of axes used for part i., sketch each of the following:

- $L_1: \vec{r} = (-2, 1) + s(2, -1), s \in \mathbf{R}$
- $L_2: \vec{r} = (-3, 1) + s(2, -1), s \in \mathbf{R}$
- $L_3: \vec{r} = (2, -1) + s(2, -1), s \in \mathbf{R}$
- $L_4: \vec{r} = (4, 2) + s(2, -1), s \in \mathbf{R}$

If you are given the equation $\vec{r} = \vec{r_0} + s(2, -1), s \in \mathbf{R}$, what is the mathematical effect of changing the value of $\vec{r_0}$?

iii. For the line $L_1: \vec{r} = (-2, 1) + s(2, -1), s \in \mathbf{R}$, show that each of the following points are on this line by finding corresponding values of s: $(4, -2), (-4, 2), (198, -99)$, and $(-202, 101)$.

iv. Which part of the equation $\vec{r} = \vec{r_0} + t\vec{m}, t \in \mathbf{R}$, indicates that there are an infinite number of points on this line? Explain your answer.

IN SUMMARY

Key Ideas

- The vector equation of a line in R^2 is $\vec{r} = \vec{r_0} + t\vec{m}$, $t \in \mathbf{R}$, where $\vec{m} = (a, b)$ is the direction vector and $\vec{r_0}$ is the vector from the origin to any point on the line whose general coordinates are (x_0, y_0). This is equivalent to the equation $(x, y) = (x_0, y_0) + t(a, b)$.
- The parametric form of the equation of a line is $x = x_0 + ta$ and $y = y_0 + tb$, $t \in \mathbf{R}$.

Need to Know

- In both the vector and parametric equations, t is a parameter. Every real number for t generates a different point that lies on the line.

Exercise 8.1

PART A

1. A vector equation is given as $\vec{r} = \left(\frac{1}{2}, -\frac{3}{4}\right) + s\left(\frac{1}{3}, \frac{1}{6}\right)$, $s \in \mathbf{R}$. Explain why $\vec{m} = (-2, -1)$, $\vec{m} = (2, 1)$, and $\vec{m} = \left(\frac{2}{7}, \frac{1}{7}\right)$ are acceptable direction vectors for this line.
2. Parametric equations of a line are $x = 1 + 3t$ and $y = 5 - 2t$, $t \in \mathbf{R}$.
   a. Write the coordinates of three points on this line.
   b. Show that the point $P(-14, 15)$ lies on the given line by determining the parameter value of t corresponding to this point.
3. Identify the direction vector and a point on each of the following lines:
   a. $\vec{r} = (3, 4) + t(2, 1)$, $t \in \mathbf{R}$
   b. $x = 1 + 2t$, $y = 3 - 7t$, $t \in \mathbf{R}$
   c. $\vec{r} = (4, 1 + 2t)$, $t \in \mathbf{R}$
   d. $x = -5t$, $y = 6$, $t \in \mathbf{R}$

PART B

4. A line passes through the points $A(2, 1)$ and $B(-3, 5)$. Write two different vector equations for this line.
5. A line is defined by the parametric equations $x = -2 - t$ and $y = 4 + 2t$, $t \in \mathbf{R}$.
   a. Does $R(-9, 18)$ lie on this line? Explain.
   b. Write a vector equation for this line using the given parametric equations.
   c. Write a second vector equation for this line, different from the one you wrote in part b.

6. a. If the equation of a line is $\vec{r} = s(3, 4), s \in \mathbf{R}$, name the coordinates of three points on this line.

   b. Write a vector equation, different from the one given, in part a., that also passes through the origin.

   c. Describe how the line with equation $\vec{r} = (9, 12) + t(3, 4), t \in \mathbf{R}$ relates to the line given in part a.

C

7. A line has $\vec{r} = \left(\frac{1}{3}, \frac{1}{7}\right) + p(-2, 3), p \in \mathbf{R}$, as its vector equation. A student decides to "simplify" this equation by clearing the fractions and multiplies the vector $\left(\frac{1}{3}, \frac{1}{7}\right)$ by 21. The student obtains $\vec{r} = (7, 3) + p(-2, 3), p \in \mathbf{R}$, as a "correct" form of the line. Explain why multiplying a point in this way is incorrect.

8. A line passes through the points $Q(0, 7)$ and $R(0, 9)$.

   a. Sketch this line.

   b. Determine vector and parametric equations for this line.

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9. A line passes through the points $M(4, 5)$ and $N(9, 5)$.

   a. Sketch this line.

   b. Determine vector and parametric equations for this line.

10. For the line $L: \vec{r} = (1, -5) + s(3, 5), s \in \mathbf{R}$, determine the following:

    a. an equation for the line perpendicular to L, passing through $P(2, 0)$

    b. the point at which the line in part a. intersects the y-axis

A

11. The parametric equations of a line are given as $x = -10 - 2s$, $y = 8 + s$, $s \in \mathbf{R}$. This line crosses the x-axis at the point with coordinates $A(a, 0)$ and crosses the y-axis at the point with coordinates $B(0, b)$. If O represents the origin, determine the area of the triangle AOB.

T

12. A line has $\vec{r} = (1, 2) + s(-2, 3)$, $s \in \mathbf{R}$, as its vector equation. On this line, the points A, B, C, and D correspond to parametric values $s = 0, 1, 2$, and 3, respectively. Show that each of the following is true:

    a. $\overrightarrow{AC} = 2\overrightarrow{AB}$ b. $\overrightarrow{AD} = 3\overrightarrow{AB}$ c. $\overrightarrow{AC} = \frac{2}{3}\overrightarrow{AD}$

PART C

13. The line L has $x = 2 + t, y = 9 + t$, $t \in \mathbf{R}$, as its parametric equations. If L intersects the circle with equation $x^2 + y^2 = 169$ at points A and B, determine the following:

    a. the coordinates of points A and B

    b. the length of the chord AB

14. Are the lines $2x - 3y + 15 = 0$ and $(x, y) = (1, 6) + t(6, 4)$ parallel? Explain.