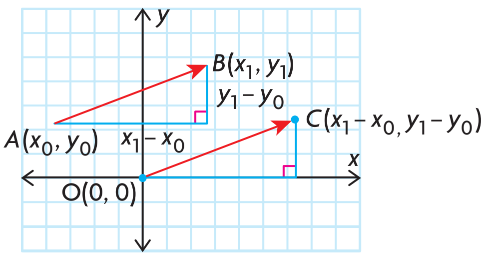
Section 8.2—Cartesian Equation of a Line

In the previous section, we discussed the vector and parametric equations of lines in R^2. In this section, we will show how lines of the form $y = mx + b$ (slope–y-intercept form) and $Ax + By + C = 0$ (Cartesian equation of a line, also called a scalar equation of a line) are related to the vector and parametric equations of the line.

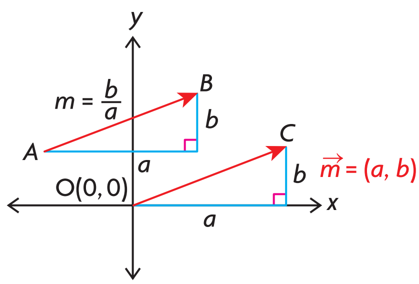
The Relationship between Vector and Scalar Equations of Lines in R^2

The direction, or inclination, of a line can be described in two ways: by its slope and by a direction vector. The slope of the line joining two points $A(x_0, y_0)$ and $B(x_1, y_1)$ is given by the formula $m = \frac{\text{rise}}{\text{run}} = \frac{y_1 - y_0}{x_1 - x_0}$. It is also possible to describe the direction of a line using the vector defined by the two points A and B, $\overrightarrow{AB} = \vec{m} = (x_1 - x_0, y_1 - y_0)$. This direction vector is equivalent to a vector with its tail at the origin and its head at $C(x_1 - x_0, y_1 - y_0)$ and is shown in the diagram below.



Direction Vectors and Slope

In the diagram, a line segment AB with slope $m = \frac{b}{a}$ is shown with a run of a and a rise of b. The vector $\vec{m} = (a, b)$ is used to describe the direction of this line or any line parallel to it, with no restriction on the direction numbers a and b. In practice, a and b can be any two real numbers when describing a direction vector. If the direction vector of a line is $\vec{m} = (a, b)$, this corresponds to a slope of $m = \frac{b}{a}$ except when $a = 0$ (which corresponds to a vertical line).



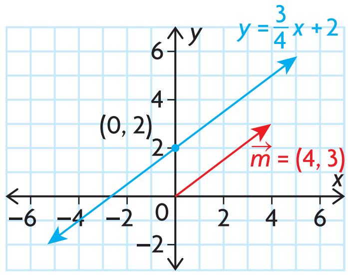
In the following example, we will show how to take a line in slope–y-intercept form and convert it to vector and parametric form.

EXAMPLE 1 Representing the Cartesian equation of a line in vector and parametric form

Determine the equivalent vector and parametric equations of the line $y = \frac{3}{4}x + 2$.

Solution

In the diagram below, the line $y = \frac{3}{4}x + 2$ is drawn. This line passes through $(0, 2)$, has a slope of $m = \frac{3}{4}$, and, as a result, has a direction vector $\vec{m} = (4, 3)$. A vector equation for this line is $\vec{r} = (0, 2) + t(4, 3)$, $t \in \mathbf{R}$, with parametric equations $x = 4t$, $y = 2 + 3t$, $t \in \mathbf{R}$.



In the next example, we will show the conversion of a line in vector form to one in slope–y-intercept form.

EXAMPLE 2 Representing a vector equation of a line in Cartesian form

For the line with equation $\vec{r} = (3, -6) + s(-1, -4)$, $s \in \mathbf{R}$, determine the equivalent slope–y-intercept form.

Solution

Method 1:

The direction vector for this line is $\vec{m} = (-1, -4)$, with slope $m = \frac{-4}{-1} = 4$. This line contains the point $(3, -6)$. If $P(x, y)$ represents a general point on this line, then we can use slope–point form to determine the required equation.

Thus, $\frac{y - (-6)}{x - 3} = 4$

$$4(x - 3) = y + 6$$

$$4x - 12 = y + 6$$

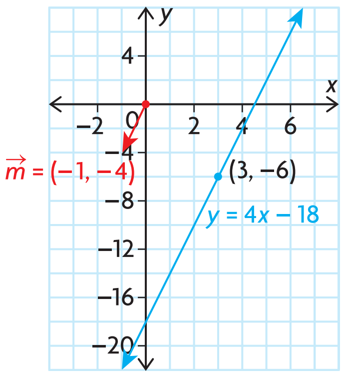
$$4x - 18 = y$$

The required equation for this line is $y = 4x - 18$ in slope–y-intercept form.

Method 2:
We start by writing the given line in parametric form, which is $(x, y) = (3, -6) + s(-1, -4)$ or $(x, y) = (3 - s, -6 - 4s)$. This gives the parametric equations $x = 3 - s$ and $y = -6 - 4s$. To find the required equation, we solve for s in each component. Thus, $s = \frac{x - 3}{-1}$ and $s = \frac{y + 6}{-4}$. Since these equations for s are equal,

$$\frac{x - 3}{-1} = \frac{y + 6}{-4}$$
$$\frac{-4(x - 3)}{-1} = y + 6$$
$$y + 6 = 4(x - 3)$$
$$y = 4x - 18$$

Therefore, the required equation is $y = 4x - 18$, which is the same answer we obtained using Method 1. The graph of this line is shown below.



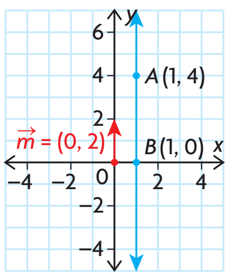
In the example that follows, we examine the situation in which the direction vector of the line is of the form $\vec{m} = (0, b)$.

EXAMPLE 3 Reasoning about equations of vertical lines

Determine the Cartesian form of the line with the equation $\vec{r} = (1, 4) + s(0, 2), s \in \mathbf{R}$.

Solution

The given line passes through the point $(1, 4)$, with direction vector $(0, 2)$, as shown in the diagram below.

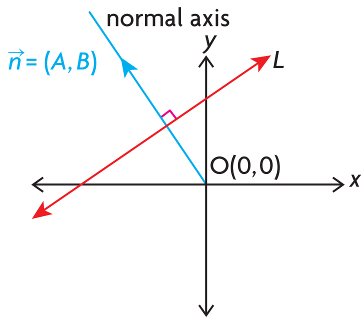


It is not possible, in this case, to calculate the slope because the line has direction vector $(0, 2)$, meaning its slope would be $\frac{2}{0}$, which is undefined. Since the line is parallel to the y-axis, it must have the form $x = a$, where $(a, 0)$ is the point where the line crosses the x-axis. The equation of this line is $x - 1 = 0$ or $x = 1$.

Developing the Cartesian Equation from a Direction Vector

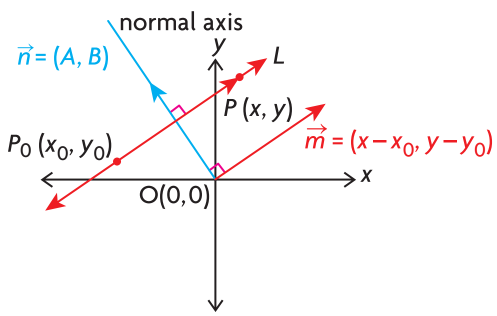
In addition to making the connection between lines in either slope–y-intercept form or Cartesian form with those in vector form, we would like to consider how direction vectors can be used to obtain the equations of lines in Cartesian form.

In the following diagram, the line L represents a general line in R^2. A line has been drawn from the origin, perpendicular to L. This perpendicular line is called the **normal** axis for the line and is the only line that can be drawn from the origin perpendicular to the given line. If the origin is joined to any point on the normal axis, other than itself, the vector formed is described as a normal to the given line. Since there are an infinite number of points on the normal axis, this is a way of saying that any line in R^2 has an infinite number of normals, none of which is the zero vector. A general point on the normal axis is given the coordinates $N(A, B)$, and so a normal vector, denoted by $\vec{n}$, is the vector $\vec{n} = (A, B)$.



The important property of the normal vector is that it is perpendicular to any vector on the given line. This property of normal vectors is what allows us to derive the Cartesian equation of the line.

In the following diagram, the line L is drawn, along with a normal $\vec{n} = (A, B)$, to L. The point $P(x, y)$ represents any point on the line, and the point $P_0(x_0, y_0)$ represents a given point on the line.



To derive the Cartesian equation for this line, we first determine $\overrightarrow{P_0P}$. In coordinate form, this vector is $\overrightarrow{P_0P} = (x - x_0, y - y_0)$, which represents a direction vector for the line. In the diagram, this vector has been shown as $\vec{m} = (x - x_0, y - y_0)$. Since the vectors $\vec{n}$ and $\overrightarrow{P_0P}$ are perpendicular to each other, $\vec{n} \cdot \overrightarrow{P_0P} = 0$.

$(A, B) \cdot (x - x_0, y - y_0) = 0$ (Expand)

$Ax - Ax_0 + By - By_0 = 0$ (Rearrange)

$Ax + By - Ax_0 - By_0 = 0$

Since the point $P_0(x_0, y_0)$ is a point whose coordinates are known, as is $\vec{n} = (A, B)$, we substitute C for the quantity $-Ax_0 - By_0$ to obtain $Ax + By + C = 0$ as the Cartesian equation of the line.

Cartesian Equation of a Line in R^2

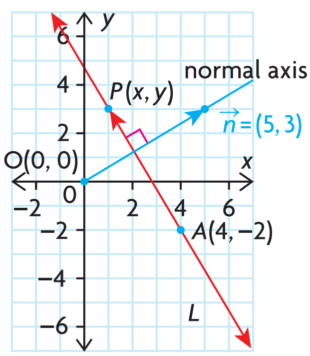
In R^2, the Cartesian equation of a line (or scalar equation) is given by $Ax + By + C = 0$, where a normal to this line is $\vec{n} = (A, B)$. A normal to this line is a vector drawn from the origin perpendicular to the given line to the point $N(A, B)$.

EXAMPLE 4 Connecting the Cartesian equation of a line to its normal

Determine the Cartesian equation of the line passing through $A(4, -2)$, which has $\vec{n} = (5, 3)$ as its normal.

Solution

The required line is sketched by first drawing the normal $\vec{n} = (5, 3)$ and then constructing a line L through $A(4, -2)$ perpendicular to this normal.



Method 1:
Let $P(x, y)$ be any point on the required line L, other than A. Let $\overrightarrow{AP}$ be a vector parallel to L.

$\overrightarrow{AP} = (x - 4, y - (-2)) = (x - 4, y + 2)$.

Since $\vec{n}$ and $\overrightarrow{AP}$ are perpendicular, $\vec{n} \cdot \overrightarrow{AP} = 0$.

Therefore, $(5, 3) \cdot (x - 4, y + 2) = 0$ or $5(x - 4) + 3(y + 2) = 0$

Thus, $5x - 20 + 3y + 6 = 0$ or $5x + 3y - 14 = 0$

Method 2:
Since $\vec{n} = (5, 3)$, the Cartesian equation of the line is of the form $5x + 3y + C = 0$, with C to be determined. Since the point $A(4, -2)$ is a point on this line, it must satisfy the following equation:

$5(4) + 3(-2) + C = 0$

So, $C = -14$, and $5x + 3y - 14 = 0$

Using either method, the required Cartesian equation is $5x + 3y - 14 = 0$.

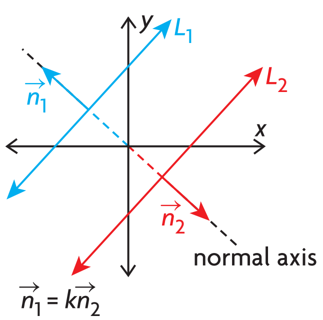
Since it has been established that the line with equation $Ax + By + C = 0$ has a normal vector of $\vec{n} = (A, B)$, this now provides an easy test to determine whether lines are parallel or perpendicular.

Parallel and Perpendicular Lines and their Normals

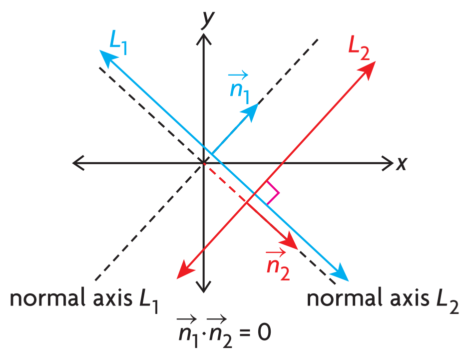
If the lines L_1 and L_2 have normals $\vec{n_1}$ and $\vec{n_2}$, respectively, we know the following:

1. The two lines are parallel if and only if their normals are scalar multiples. $\vec{n_1} = k\vec{n_2}, k \in \mathbf{R}, k \neq 0$ It follows that the lines direction vectors are also scalar multiples in this case.
2. The two lines are perpendicular if and only if their dot product is zero. $\vec{n_1} \cdot \vec{n_2} = 0$ It follows that dot product of the direction vectors is also zero in this case.

Two parallel lines



Two perpendicular lines



The next examples demonstrate these ideas.

EXAMPLE 5 Reasoning about parallel and perpendicular lines in R^2

a. Show that the lines $L_1: 3x - 4y - 6 = 0$ and $L_2: 6x - 8y + 12 = 0$ are parallel and non-coincident.
b. For what value of k are the lines $L_3: kx + 4y - 4 = 0$ and $L_4: 3x - 2y - 3 = 0$ perpendicular lines?

Solution

a. The lines are parallel because when the two normals, $\vec{n_1} = (3, -4)$ and $\vec{n_2} = (6, -8)$, are compared, the two vectors are scalar multiples $\vec{n_2} = (6, -8) = 2(3, -4) = 2\vec{n_1}$. The lines are non-coincident, since there is no value of t such that $6x - 8y + 12 = t(3x - 4y - 6)$. In simple terms, lines can only be coincident if their equations are scalar multiples of each other.
b. If the lines are perpendicular, then the normal vectors $\vec{n_3} = (k, 4)$ and $\vec{n_4} = (3, -2)$ have a dot product equal to zero—that is $(k, 4) \cdot (3, -2) = 0$ or $3k - 8 = 0$, $k = \frac{8}{3}$. This implies that the lines $3x - 2y - 3 = 0$ and $\frac{8}{3}x + 4y - 4 = 0$ are perpendicular.

The following investigation helps in understanding the relationship between normals and perpendicular lines.

EXAMPLE 6 Selecting a strategy to determine the angle between two lines in R^2

Determine the acute angle formed at the point of intersection created by the following pair of lines:
L_1: $(x, y) = (2, 2) + s(-1, 3)$, $s \in \mathbf{R}$
L_2: $(x, y) = (5, 1) + t(3, 4)$, $t \in \mathbf{R}$

Solution

The direction of each line is determined by their respective direction vectors, so the angle formed at the point of intersection is equivalent to the angle formed by the direction vectors when drawn tail to tail. For L_1 its direction vector is $\vec{a} = (-1, 3)$ and for L_2 its direction vectors is $\vec{b} = (3, 4)$. These lines are clearly not parallel as their direction vectors are not scalar multiples. They are also not perpendicular because the dot product of their direction vectors is a nonzero value. The angle between two vectors is determined by:

$$\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}\right) \quad \text{(Substitute)}$$

$$\theta = \cos^{-1}\left(\frac{(-1, 3) \cdot (3, 4)}{(\sqrt{(-1)^2 + (3)^2})(\sqrt{(3)^2 + (4)^2})}\right) \quad \text{(Simplify)}$$

$$\theta = \cos^{-1}\left(\frac{-3 + 12}{(\sqrt{10})(\sqrt{25})}\right)$$

$$\theta = \cos^{-1}\left(\frac{9}{5\sqrt{10}}\right)$$ (Evaluate)

$$\theta \doteq 55.3°$$

The acute angle formed at the point of intersection of the given lines is about 55.3°.

INVESTIGATION

A. A family of lines has $kx - 2y - 4 = 0$ as its equation. On graph paper, sketch the three members of this family when $k = 1$, $k = -1$, and $k = 2$.

B. What point do the three lines you sketched in part A have in common?

C. A second family of lines has $4x - ty - 8 = 0$ as its equation. Sketch the three members of this family used in part A for $t = -2$, $t = 2$, and $t = -4$.

D. What points do the lines in part C have in common?

E. Select the three pairs of perpendicular lines from the two families. Verify that you are correct by calculating the dot products of their respective normals.

F. By selecting different values for k and t, determine another pair of lines that are perpendicular.

G. In general, if you are given a line in R^2, how many different lines is it possible to draw through a particular point perpendicular to the given line? Explain your answer.

IN SUMMARY

Key Idea

- The Cartesian (or scalar) equation of a line in R^2 is $Ax + By + C = 0$, where $\vec{n} = (A, B)$ is a normal to the line.

Need to Know

- Two planes whose normals are $\vec{n_1}$ and $\vec{n_2}$:
  - are parallel if and only if $\vec{n_1} = k\vec{n_2}$, where k is any nonzero real number.
  - are perpendicular if and only if $\vec{n_1} \cdot \vec{n_2} = 0$.
- The angle between two lines is defined by the angle between their direction vectors, $\vec{a}$ and $\vec{b}$, where $\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$.

Exercise 8.2

PART A

1. A line has $y = -\frac{5}{6}x + 9$ as its equation.
   a. Give a direction vector for a line that is parallel to this line.
   b. Give a direction vector for a line that is perpendicular to this line.
   c. Give the coordinates of a point on the given line.
   d. In both vector and parametric form, give the equations of the line parallel to the given line and passing through $A(7, 9)$.
   e. In both vector and parametric form, give the equations of the line perpendicular to the given line and passing through $B(-2, 1)$.
2. a. Sketch the line defined by the equation $\vec{r} = (2, 1) + s(-2, 5), s \in \mathbf{R}$.
   b. On the same axes, sketch the line $\vec{q} = (-2, 5) + t(2, 1),\ t \in \mathbf{R}$.
   c. Discuss the impact of switching the components of the direction vector with the coordinates of the point on the line in the vector equation of a line in R^2.
3. For each of the given lines, determine the vector and parametric equations.

   a. $y = \frac{7}{8}x - 6$ b. $y = \frac{3}{2}x + 5$ c. $y = -1$ d. $x = 4$

C 4. Explain how you can show that the lines with equations $x - 3y + 4 = 0$ and $6x - 18y + 24 = 0$ are coincident.

5. Two lines have equations $2x - 3y + 6 = 0$ and $4x - 6y + k = 0$.
   a. Explain, with the use of normal vectors, why these lines are parallel.
   b. For what value of k will these lines be coincident?

PART B

6. Determine the Cartesian equation for the line with a normal vector of $(4, 5)$, passing through the point $A(-1, 5)$.
7. A line passes through the points $A(-3, 5)$ and $B(-2, 4)$. Determine the Cartesian equation of this line.
8. A line is perpendicular to the line $2x - 4y + 7 = 0$ and that passes through the point $P(7, 2)$. Determine the equation of this line in Cartesian form.

K 9. A line has parametric equations $x = 3 - t$, $y = -2 - 4t$, $t \in \mathbf{R}$.
   a. Sketch this line.
   b. Determine a Cartesian equation for this line.

10. For each pair of lines, determine the size of the acute angle, to the nearest degree, that is created by the intersection of the lines.

a. $(x, y) = (3, 6) + t(2, -5)$ and $(x, y) = (-3, 4) - t(-4, -1)$

b. $x = 2 - 5t, y = 3 + 4t$ and $x = -1 + t, y = 2 - 6t$

c. $y = 0.5x + 6$ and $y = -0.75x - 1$

d. $(x, y) = (-1, -1) + t(2, 4)$ and $2x - 4y = 8$

e. $x = 2t, y = 1 - 5t$ and $(x, y) = (4, 0) + t(-4, 1)$

f. $x = 3$ and $5x - 10y + 20 = 0$

11. The angle between any pair of lines in Cartesian form is also the angle between their normal vectors. For the lines $x - 3y + 6 = 0$ and $x + 2y - 7 = 0$, do the following:

a. Sketch the lines.

b. Determine the acute and obtuse angles between these two lines.

12. The line segment joining $A(-3, 2)$ and $B(8, 4)$ is the hypotenuse of a right triangle. The third vertex, C, lies on the line with the vector equation $(x, y) = (-6, 6) + t(3, -4)$.

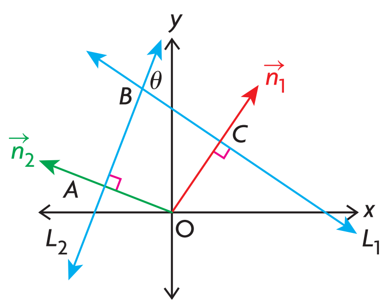
a. Determine the coordinates of C.

b. Illustrate with a diagram.

c. Use vectors to show that $\angle ACB = 90°$.

PART C

13. Lines L_1 and L_2 have $\vec{n_1}$ and $\vec{n_2}$ as their respective normals. Prove that the angle between the two lines is the same as the angle between the two normals.



(*Hint:* Show that $\angle AOC = \theta$ by using the fact that the sum of the angles in a quadrilateral is 360°.)

14. The lines $x - y + 1 = 0$ and $x + ky - 3 = 0$ have an angle of 60° between them. For what values of k is this true?