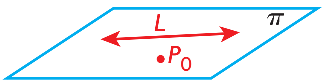
Section 8.4—Vector and Parametric Equations of a Plane

In the previous section, the vector, parametric, and symmetric equations of lines in R^3 were developed. In this section, we will develop vector and parametric equations of planes in R^3. Planes are flat surfaces that extend infinitely far in all directions. To represent planes, parallelograms are used to represent a small part of the plane and are designated with the Greek letter π. This is the usual method for representing planes. In real life, part of a plane might be represented by the top of a desk, by a wall, or by the ice surface of a hockey rink.

Before developing the equation of a plane, we start by showing that planes can be determined in essentially four ways. That is, a plane can be determined if we are given any of the following:

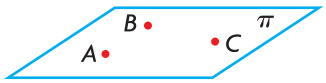
a. a line and a point not on the line
b. three noncollinear points (three points not on a line)
c. two intersecting lines
d. two parallel and non-coincident lines

a.



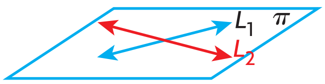
A line and a point not on the line

c.



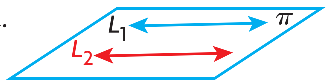
Three noncollinear points

b.



Two intersecting lines

d.



Two parallel and non-coincident lines

If we are given any one of these conditions, we are guaranteed that we can form a plane, and the plane formed will be unique. For example, in condition a, we are given line L and point P_0 not on this line; there is just one plane that can be formed using this point and this line.

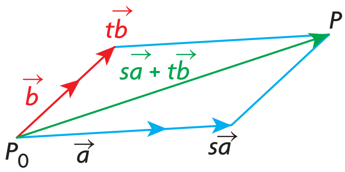
Linear Combinations and their Relationship to Planes

The ideas of linear combination and spanning sets are the two concepts needed to understand how to obtain the vector and parametric equations of planes. For example, suppose that vectors $\vec{a} = (1, 2, -1)$ and $\vec{b} = (0, 2, 1)$ and a linear combination of these vectors—that is, $\overrightarrow{P_0P} = s(1, 2, -1) + t(0, 2, 1),\ s, t \in \mathbf{R}$— are formed. As different values are chosen for s and t, a new vector is formed. Different values for these parameters have been selected, and the corresponding calculations have been done in the table shown, with vector $\overrightarrow{P_0P}$ also calculated.

s	t	$s(1, 2, -1) + t(0, 2, 1)$, $s, t \in \mathbf{R}$	$\overrightarrow{P_0P}$
-2	1	$-2(1, 2, -1) + 1(0, 2, 1)$	$(-2, -4, 2) + (0, 2, 1) = (-2, -2, 3)$
4	-3	$4(1, 2, -1) - 3(0, 2, 1)$	$(4, 8, -4) + (0, -6, -3) = (4, 2, -7)$
10	-7	$10(1, 2, -1) - 7(0, 2, 1)$	$(10, 20, -10) + (0, -14, -7) = (10, 6, -17)$
-2	-1	$-2(1, 2, -1) - 1(0, 2, 1)$	$(-2, -4, 2) + (0, -2, -1) = (-2, -6, 1)$

$\overrightarrow{P_0P}$ is on the plane determined by the vectors $\vec{a}$ and $\vec{b}$, as is its head. The parameters s and t are chosen from the set of real numbers, meaning that there are an infinite number of vectors formed by selecting all possible combinations of s and t. Each one of these vectors is different, and every point on the plane can be obtained by choosing appropriate parameters. This observation is used in developing the vector and parametric equations of a plane.

In the following diagram, two noncollinear vectors, $\vec{a}$ and $\vec{b}$, are given. The linear combinations of these vectors, $s\vec{a} + t\vec{b}$, form a diagonal of the parallelogram determined by $s\vec{a}$ and $t\vec{b}$.

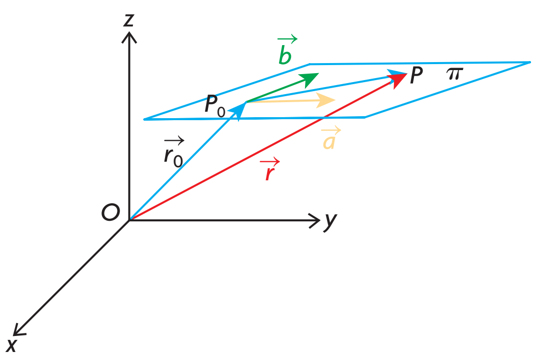


EXAMPLE 1 **Developing the vector and parametric equations of a plane**

Two noncollinear vectors, $\vec{a}$ and $\vec{b}$, are given, and the point P_0. Determine the vector and parametric equations of the plane π formed by taking all linear combinations of these vectors.

Solution

The vectors $\vec{a}$ and $\vec{b}$ can be translated anywhere in R^3. When drawn tail to tail they form and infinite number of parallel planes, but only one such plane contains the point P_0. We start by drawing a parallelogram to represent part of this plane π. This plane contains P_0, $\vec{a}$, and $\vec{b}$.



From the diagram, it can be seen that vector $\vec{r_0}$ represents the vector for a particular point P_0 on the plane, and $\vec{r}$ represents the vector for any point P on the plane. $\overrightarrow{P_0P}$ is on the plane and is a linear combination of $\vec{a}$ and $\vec{b}$—that is, $\overrightarrow{P_0P} = s\vec{a} + t\vec{b}$, $s, t \in \mathbf{R}$. Using the triangle law of addition in $\triangle OP_0P$, $\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$. Thus, $\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}$, $s, t \in \mathbf{R}$.

The vector equation for the plane is $\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}$, $s, t \in \mathbf{R}$, and can be used to generate parametric equations for the plane.

If $\vec{r} = (x, y, z)$, $\vec{r_0} = (x_0, y_0, z_0)$, $\vec{a} = (a_1, a_2, a_3)$, and $\vec{b} = (b_1, b_2, b_3)$, these expressions can be substituted into the vector equation to obtain $(x, y, z) = (x_0, y_0, z_0) + s(a_1, a_2, a_3) + t(b_1, b_2, b_3)$, $s, t \in \mathbf{R}$.

Expanding, $(x, y, z) = (x_0, y_0, z_0) + (sa_1, sa_2, sa_3) + (tb_1, tb_2, tb_3)$

Simplifying, $(x, y, z) = (x_0 + sa_1 + tb_1, y_0 + sa_2 + tb_2, z_0 + sa_3 + tb_3)$

Equating the respective components gives the parametric equations $x = x_0 + sa_1 + tb_1$, $y = y_0 + sa_2 + tb_2$, $z = z_0 + sa_3 + tb_3$, $s, t \in \mathbf{R}$.

Vector and Parametric Equations of a Plane in R^3

In R^3, a plane is determined by a vector $\vec{r_0} = (x_0, y_0, z_0)$ where (x_0, y_0, z_0) is a point on the plane, and two noncollinear vectors vector $\vec{a} = (a_1, a_2, a_3)$ and vector $\vec{b} = (b_1, b_2, b_3)$.

Vector Equation of a Plane: $\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}$, $s, t \in \mathbf{R}$ or equivalently $(x, y, z) = (x_0, y_0, z_0) + s(a_1, a_2, a_3) + t(b_1, b_2, b_3)$.

Parametric Equations of a Plane: $x = x_0 + sa_1 + tb_1$,
$y = y_0 + sa_2 + tb_2$,
$z = z_0 + sa_3 + tb_3$, $s, t \in \mathbf{R}$

The vectors $\vec{a}$ and $\vec{b}$ are the direction vectors for the plane. When determining the equation of a plane, it is necessary to have two direction vectors. As will be seen in the examples, any pair of noncollinear vectors are coplanar, so they can be used as direction vectors for a plane. The vector equation of a plane always requires two parameters, s and t, each of which are real numbers. Because two parameters are required to define a plane, the plane is described as two-dimensional. Earlier, we saw that the vector equation of a line, $\vec{r} = \vec{r_0} + t\vec{m}$, $t \in \mathbf{R}$, required just one parameter. For this reason, a line is described as one-dimensional. A second observation about the vector equation of the plane is that there is a one-to-one correspondence between the chosen parametric values (s, t) and points on the plane. Each time values for s and t are selected, this generates a different point on the plane, and because s and t can be any real number, this will generate all points on the plane.

After deriving vector and parametric equations of lines, a symmetric form was also developed. Although it is possible to derive vector and parametric equations of planes, it is not possible to derive a corresponding symmetric equation of a plane.

The next example shows how to derive an equation of a plane passing through three points.

EXAMPLE 2 Selecting a strategy to represent the vector and parametric equations of a plane

a. Determine a vector equation and the corresponding parametric equations for the plane that contains the points $A(-1, 3, 8)$, $B(-1, 1, 0)$, and $C(4, 1, 1)$.
b. Do either of the points $P(14, 1, 3)$ or $Q(14, 1, 5)$ lie on this plane?

Solution

a. In determining the required vector equation, it is necessary to have two direction vectors for the plane. The following shows the calculations for each of the direction vectors.

Direction Vector 1:
When calculating the first direction vector, any two points can be used and a position vector determined. If the points $A(-1, 3, 8)$ and $B(-1, 1, 0)$ are used, then $\overrightarrow{AB} = (-1 - (-1), 1 - 3, 0 - 8) = (0, -2, -8)$.

Since, $(0, -2, -8) = -2(0, 1, 4)$, a possible first direction vector is $\vec{a} = (0, 1, 4)$.

Direction Vector 2:
When finding the second direction vector, any two points (other than A and B) can be chosen. If B and C are used, then
$\overrightarrow{BC} = (4 - (-1), 1 - 1, 1 - 0) = (5, 0, 1)$.

A second direction vector is $\vec{b} = (5, 0, 1)$.

To determine the equation of the plane, any of the points A, B, or C can be used. An equation for the plane is $\vec{r} = (-1, 3, 8) + s(0, 1, 4) + t(5, 0, 1), s, t \in \mathbf{R}$.

Writing the vector equation in component form will give the parametric equations. Thus, $(x, y, z) = (-1, 3, 8) + (0, s, 4s) + (5t, 0, t)$.

The parametric equations are $x = -1 + 5t, y = 3 + s$, and $z = 8 + 4s + t$, $s, t \in \mathbf{R}$.

b. If the point $P(14, 1, 3)$ lies on the plane, there must be parameters that correspond to this point. To find these parameters, $x = 14$ and $y = 1$ are substituted into the corresponding parametric equations.

Thus, $14 = -1 + 5t$ and $1 = 3 + s$.

Solving for s and t, we find that $14 + 1 = 5t$, or $t = 3$ and $1 - 3 = s$, or $s = -2$. Using these values, consistency will be checked with the z component. If $s = -2$ and $t = 3$ are substituted into $z = 8 + 4s + t$, then $z = 8 + 4(-2) + 3 = 3$. Since the z value for the point is also 3, this tells us that the point with coordinates $P(14, 1, 3)$ is on the given plane.

From this, it can be seen that the parametric values used for the x and y components, $s = -2$ and $t = 3$, will not produce consistent values for $z = 5$. So, the point $Q(14, 1, 5)$ is not on the plane.

In the following example, we will show how to use vector and parametric equations to find the point of intersection of planes with the coordinate axes.

EXAMPLE 3 Solving a problem involving a plane

A plane π has $\vec{r} = (6, -2, -3) + s(1, 3, 0) + t(2, 2, -1), s, t \in \mathbf{R}$, as its equation. Determine the point of intersection between π and the z-axis.

Solution

We start by writing this equation in parametric form. The parametric equations of the plane are $x = 6 + s + 2t$, $y = -2 + 3s + 2t$, and $z = -3 - t$.

The plane intersects the z-axis at a point with coordinates of the form $P(0, 0, c)$—that is, where $x = y = 0$. Substituting these values into the parametric equations for x and y gives $0 = 6 + s + 2t$ and $0 = -2 + 3s + 2t$. Simplifying gives the following system of equations:

① $s + 2t = -6$

② $3s + 2t = 2$

Subtracting equation ① from equation ② gives $2s = 8$, so $s = 4$.

The value of t is found by substituting into either of the two equations. Using equation ①, $4 + 2t = -6$, or $t = -5$.

Solving for z using the equation of the third component, we find that $z = -3 - (-5) = 2$.

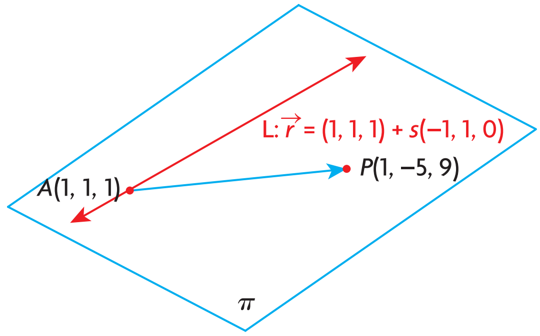
Thus, the plane cuts the z-axis at the point $P(0, 0, 2)$.

EXAMPLE 4 Representing the equations of a plane from a point and a line

Determine the vector and parametric equations of the plane containing the point $P(1, -5, 9)$ and the line $L: \vec{r} = (1, 1, 1) + s(-1, 1, 0), s \in \mathbf{R}$.

Solution

In the following diagram, a representation of the line L and the point P are given.



To find the equation of the plane, it is necessary to find two direction vectors and a point on the plane. The line $L: \vec{r} = (1, 1, 1) + s(-1, 1, 0), s \in \mathbf{R}$, gives a point and one direction vector, so all that is required is a second direction vector. Using $A(1, 1, 1)$ and $P(1, -5, 9)$, $\overrightarrow{AP} = (1 - 1, -5 - 1, 9 - 1) = (0, -6, 8) = -2(0, 3, -4)$. The equation of the plane is $\vec{r} = (1, 1, 1) + s(-1, 1, 0) + t(0, 3, -4),\ s, t \in \mathbf{R}$.

The corresponding parametric equations are $x = 1 - s$, $y = 1 + s + 3t$, and $z = 1 - 4t$, $s, t \in \mathbf{R}$.

IN SUMMARY

Key Idea

- In R^3, if $\vec{r_0} = (x_0, y_0, z_0)$ is determined by a point on a plane and $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ are direction vectors, then
  - the vector equation of the plane is $\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}$, $s, t \in \mathbf{R}$ or equivalently $(x, y, z) = (x_0, y_0, z_0) + s(a_1, a_2\ a_3) + t(b_1, b_2, b_3)$
  - the parametric form of the equation of the plane is $x = x_0 + sa_1 + tb_1$, $y = y_0 + sa_2 + tb_2$, $z = z_0 + sa_3 + tb_3$, $s, t \in \mathbf{R}$
  - there are no symmetric equations of the plane

Need to Know

- Replacing the parameters in the vector and parametric equations of a plane with numbers generates points on the plane. Because there are an infinite number of real numbers that can be used for s and t, there are an infinite number of points that lie on a plane.

Exercise 8.4

PART A

1. State which of the following equations define lines and which define planes. Explain how you made your decision.
   a. $\vec{r} = (1, 2, 3) + s(1, 1, 0) + t(3, 4, -6), s, t \in \mathbf{R}$
   b. $\vec{r} = (-2, 3, 0) + m(3, 4, 7),\ m \in \mathbf{R}$
   c. $x = -3 - t, y = 5, z = 4 + t, t \in \mathbf{R}$
   d. $\vec{r} = m(4, -1, 2) + t(4, -1, 5), m, t \in \mathbf{R}$
2. A plane has vector equation $\vec{r} = (2, 1, 3) + s\left(\frac{1}{3}, -2, \frac{3}{4}\right) + t(6, -12, 30)$, $s, t \in \mathbf{R}$.
   a. Express the first direction vector with only integers.
   b. Reduce the second direction vector.
   c. Write a new equation for the plane using the calculations from parts a. and b.
3. A plane has $x = 2m, y = -3m + 5n, z = -1 - 3m - 2n, m, n \in \mathbf{R}$, as its parametric equations.
   a. By inspection, identify the coordinates of a point that is on this plane.
   b. What are the direction vectors for this plane?
   c. What point corresponds to the parameter values of $m = -1$ and $n = -4$?
   d. What are the parametric values corresponding to the point $A(0, 15, -7)$?
   e. Using your answer for part d., explain why the point $B(0, 15, -8)$ cannot be on this plane.
4. A plane passes through the points $P(-2, 3, 1)$, $Q(-2, 3, 2)$, and $R(1, 0, 1)$.
   a. Using $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ as direction vectors, write a vector equation for this plane.
   b. Using $\overrightarrow{QR}$ and one other direction vector, write a second vector equation for this plane.

C 5. Explain why the equation $\vec{r} = (-1, 0, -1) + s(2, 3, -4) + t(4, 6, -8)$, $s, t \in \mathbf{R}$, does not represent the equation of a plane. What does this equation represent?

PART B

6. Determine vector equations and the corresponding parametric equations of each plane.
   a. the plane with direction vectors $\vec{a} = (4, 1, 0)$ and $\vec{b} = (3, 4, -1)$, passing through the point $A(-1, 2, 7)$
   b. the plane passing through the points $A(1, 0, 0)$, $B(0, 1, 0)$, and $C(0, 0, 1)$
   c. the plane passing through points $A(1, 1, 0)$ and $B(4, 5, -6)$, with direction vector $\vec{a} = (7, 1, 2)$

7. a. Determine parameters corresponding to the point $P(5, 3, 2)$, where P is a point on the plane with equation $\pi: \vec{r} = (2, 0, 1) + s(4, 2, -1) + t(-1, 1, 2)$, $s, t \in \mathbf{R}$.
   b. Show that the point $A(0, 5, -4)$ does not lie on π.
8. A plane has $\vec{r} = (-3, 5, 6) + s(-1, 1, 2) + v(2, 1, -3)$, $s, v \in \mathbf{R}$ as its equation.
   a. Give the equations of two intersecting lines that lie on this plane.
   b. What point do these two lines have in common?

A

9. Determine the coordinates of the point where the plane with equation $\vec{r} = (4, 1, 6) + s(11, -1, 3) + t(-7, 2, -2)$, $s, t \in \mathbf{R}$, crosses the z-axis.
10. Determine the equation of the plane that contains the point $P(-1, 2, 1)$ and the line $\vec{r} = (2, 1, 3) + s(4, 1, 5)$, $s \in \mathbf{R}$.
11. Determine the equation of the plane that contains the point $A(-2, 2, 3)$ and the line $\vec{r} = m(2, -1, 7)$, $m \in \mathbf{R}$.
12. a. Determine two pairs of direction vectors that can be used to represent the xy-plane in R^3.
    b. Write a vector and parametric equations for the xy-plane in R^3.

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13. a. Determine a vector equation for the plane containing the points $O(0, 0, 0)$, $A(-1, 2, 5)$, and $C(3, -1, 7)$.
    b. Determine a vector equation for the plane containing the points $P(-2, 2, 3)$, $Q(-3, 4, 8)$, and $R(1, 1, 10)$.
    c. How are the planes found in parts a. and b. related? Explain your answer.
14. Show that the following equations represent the same plane:
    $\vec{r} = u(-3, 2, 4) + v(-4, 7, 1)$, $u, v \in \mathbf{R}$, and
    $\vec{r} = s(-1, 5, -3) + t(-1, -5, 7)$, $s, t \in \mathbf{R}$
    (*Hint*: Express each direction vector in the first equation as a linear combination of the direction vectors in the second equation.)

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15. The plane with equation $\vec{r} = (1, 2, 3) + m(1, 2, 5) + n(1, -1, 3)$ intersects the y- and z-axes at the points A and B, respectively. Determine the equation of the line that contains these two points.

PART C

16. Suppose that the lines L_1 and L_2 are defined by the equations $\vec{r} = \overrightarrow{OP_0} + s\vec{a}$ and $\vec{r} = \overrightarrow{OP_0} + t\vec{b}$, respectively, where $s, t \in \mathbf{R}$, and $\vec{a}$ and $\vec{b}$ are noncollinear vectors. Prove that the plane defined by the equation $\vec{r} = \overrightarrow{OP_0} + s\vec{a} + t\vec{b}$ contains both of these lines.