Section 8.6—Sketching Planes in R^3

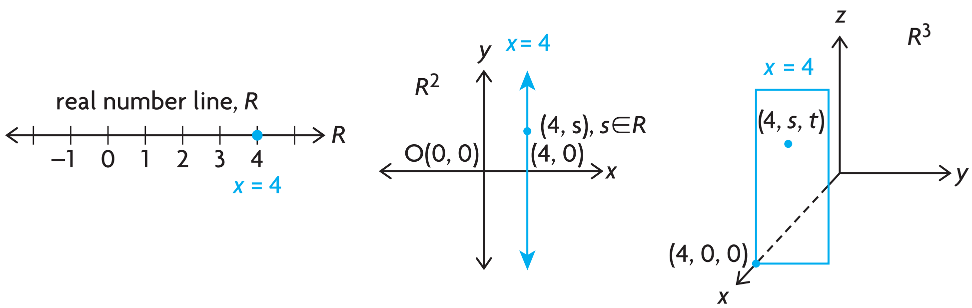
In previous sections, we developed methods for finding the equation of planes in both vector and Cartesian form. In this section, we examine how to sketch a plane if the equation is given in Cartesian form. When graphing planes in R^3, many of the same methods used for graphing a line in R^2 will be used.

An important first observation is that, if we are given an equation such as $x = 4$ and are asked to find its related graph, then a different graph is produced depending on the dimension in which we are working.

1. On the real number line, this equation refers to a point at $x = 4$.
2. In R^2, this equation represents a line parallel to the y-axis and 4 units to its right.
3. In R^3, the equation $x = 4$ represents a plane that intersects the x-axis at $(4, 0, 0)$ and is 4 units in front of the plane formed by the y- and z-axes.

We can see that the equation $x = 4$ results in a different graph depending on whether it is drawn on the number line, in R^2, or in R^3.

Different interpretations of the graph with equation $x = 4$



Varying the Coefficients in the Cartesian Equation

In the following situations, the graph of $Ax + By + Cz + D = 0$ in R^3 is considered for different cases.

Case 1– The equation contains one variable

Case 1a: Two of A, B, or C equal zero, and D equals zero.

In this case, the resulting equation will be of the form $x = 0, y = 0$, or $z = 0$. If $x = 0$, for example, this equation represents the yz-plane, since every point on this plane has an x-value equal to 0. Similarly, $y = 0$ represents the xz-plane, and $z = 0$ represents the xy-plane.

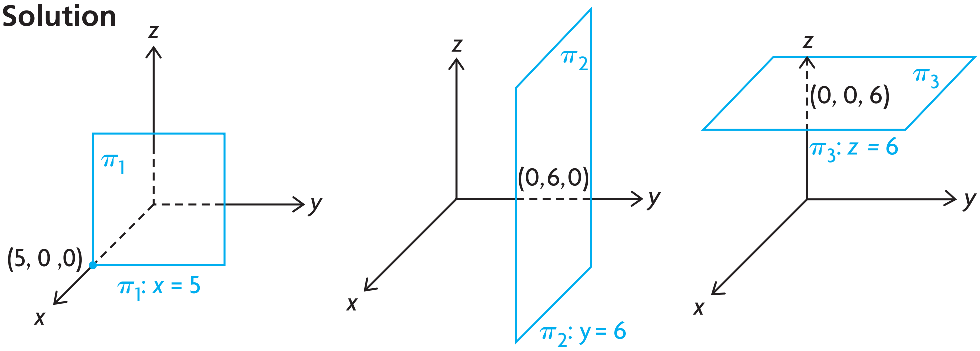
Case 1b: Two of A, B, or C equal zero, and D does not equal zero.

If two of the three coefficients are equal to zero, the resulting equation will be of the form $x = a$, $y = b$, or $z = c$. The following examples show that these equations represent planes parallel to the yz-, xz-, and xy-planes, respectively.

EXAMPLE 1 Representing the graphs of planes in R^3 whose Cartesian equations involve one variable

Draw the planes with equations $\pi_1 : x = 5$, $\pi_2 : y = 6$, and $\pi_3 : z = 6$.

Solution



Descriptions of the planes in Example 1 are given in the following table.

Plane	Description	Generalization
π_1: $x = 5$	A plane parallel to the yz-plane crosses the x-axis at (5, 0, 0). This plane has an x-intercept of 5.	A plane with equation $x = a$ is parallel to the yz-plane and crosses the x-axis at the point $(a, 0, 0)$. The plane $x = 0$ is the yz-plane.
π_2: $y = 6$	A plane parallel to the xz-plane crosses the y-axis at (0, 6, 0). This plane has a y-intercept of 6.	A plane with equation $y = b$ is parallel to the xz-plane and crosses the y-axis at the point $(0, b, 0)$. The plane $y = 0$ is the xz-plane.
π_3: $z = 6$	A plane parallel to the xy-plane crosses the x-axis at the point (0, 0, 6). This plane has a z-intercept of 6.	A plane with equation $z = c$ is parallel to the xy-plane and crosses the z-axis at the point $(0, 0, c)$. The plane $z = 0$ is the xy-plane.

Case 2– The equation contains two variables

Case 2a: One of A, B, or C equals zero, and D equals zero.

In this case, the resulting equation will have the form $Ax + By = 0$, $Ax + Cz = 0$, or $By + Cz = 0$. The following example demonstrates how to graph an equation of this type.

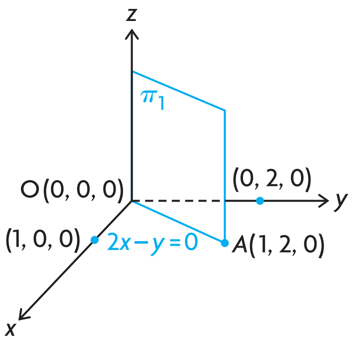
EXAMPLE 2 Representing the graph of a plane in R^3 whose Cartesian equation involves two variables, $D = 0$

Sketch the plane defined by the equation $\pi_1 : 2x - y = 0$.

Solution

For $\pi_1 : 2x - y = 0$, we note that the origin $O(0, 0, 0)$ lies on the plane, and it also contains the z-axis. We can see that π_1 contains the z-axis because, if it is written in the form $2x - y + 0z = 0$, $(0, 0, t)$ is on the plane because $2(0) - 0 + 0(t) = 0$. Since $(0, 0, t)$, $t \in \mathbf{R}$, represents any point on the z-axis, this means that the plane contains the z-axis.

In addition, the plane cuts the xy-plane along the line $2x - y = 0$. All that is necessary to graph this line is to select a point on the xy-plane that satisfies the equation and join that point to the origin. Since the point with coordinates $A(1, 2, 0)$ satisfies the equation, we draw the parallelogram determined by the z-axis and the line joining $O(0, 0, 0)$ to $A(1, 2, 0)$, and we have a sketch of the plane $2x - y = 0$.



EXAMPLE 3 Describing planes whose Cartesian equations involve two variables, $D = 0$

Write descriptions of the planes $\pi_1 : 2x - z = 0$ and $\pi_2 : y + 2z = 0$.

Solution

These equations can be written as $\pi_1 : 2x + 0y - z = 0$ and $\pi_2 : 0x + y + 2z = 0$.

Using the same reasoning as above, this implies that π_1 contains the origin and the y-axis, and cuts the xz-plane along the line with equation $2x - z = 0$. Similarly, π_2 contains the origin and the x-axis, and cuts the yz-plane along the line with equation $y + 2z = 0$.

Case 2b: One of A, B, or C equals zero, and D does not equal zero.

If one of the coefficients equals zero and $D \neq 0$, the resulting equations can be written as $Ax + By + D = 0$, $Ax + Cz + D = 0$, or $By + Cz + D = 0$.

EXAMPLE 4 Graphing planes whose Cartesian equations involve two variables, $D \neq 0$

Sketch the plane defined by the equation $\pi_1: 2x - 5y - 10 = 0$.

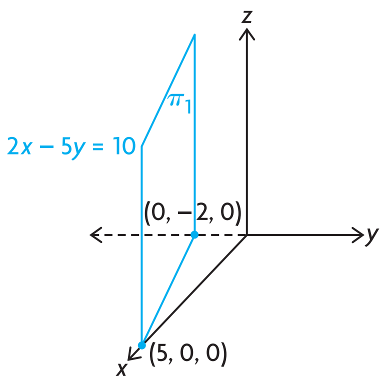
Solution

It is best to write this equation as $2x - 5y = 10$ so that we can easily calculate the intercepts.

x-intercept: We calculate the *x*-intercept for this plane in exactly the same way that we would calculate the *x*-intercept for the line $2x - 5y = 10$. If we let $y = 0$, then $2x - 5(0) = 10$ or $x = 5$. This means that the plane has an *x*-intercept of 5 and that it crosses the *x*-axis at $(5, 0, 0)$.

y-intercept: To calculate the *y*-intercept, we let $x = 0$. Thus, $-5y = 10, y = -2$. This means that the plane has a *y*-intercept of -2 and it crosses the *y*-axis at $(0, -2, 0)$.

To complete the analysis for the plane, we write the equation as $2x - 5y + 0z = 10$. If the plane did cross the *z*-axis, it would do so at a point where $x = y = 0$. Substituting these values into the equation, we obtain $2(0) - 5(0) + 0z = 10$ or $0z = 10$. This implies that the plane has no *z*-intercept because there is no value of *z* that will satisfy the equation. Thus, the plane passes through the points (5, 0, 0) and $(0, -2, 0)$ and is parallel to the *z*-axis. The plane is sketched in the diagram below. Possible direction vectors for the plane are $\overrightarrow{m_1} = (5 - 0, 0 - (-2), 0 - 0) = (5, 2, 0)$ and $\overrightarrow{m_2} = (0, 0, 1)$.



Using the same line of reasoning as above, if *A*, *C* and *D* are nonzero when $B = 0$, the resulting plane is parallel to the *y*-axis. If *B*, *C* and *D* are nonzero when $A = 0$, the resulting plane is parallel to the *x*-axis.

Case 3– The equation contains three variables

Case 3b: *A*, *B*, and *C* do not equal zero, and *D* equals zero.

This represents an equation of the form, $Ax + By + Cz = 0$, which is a plane most easily sketched using the fact that a plane is uniquely determined by three points. The following example illustrates this approach.

EXAMPLE 5 Graphing planes whose Cartesian equations involve three variables, $D = 0$

Sketch the plane $\pi_1 : x + 3y - z = 0$.

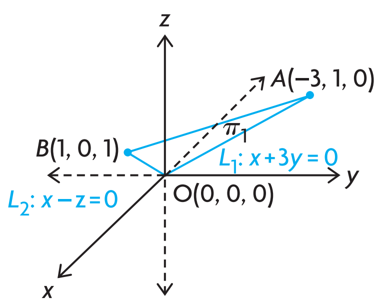
Solution

Since there is no constant in the equation, the point $(0, 0, 0)$ is on the plane. To sketch the plane, we require two other points. We first find a point on the xy-plane and a second point on the xz-plane.

Point on xy-plane: Any point on the xy-plane has $z = 0$. If we first substitute $z = 0$ into $x + 3y - z = 0$, then $x + 3y - 0 = 0$, or $x + 3y = 0$, which means that the given plane cuts the xy-plane along the line $x + 3y = 0$. Using this equation, we can now select convenient values for x and y to obtain the coordinates of a point on this line. The easiest values are $x = -3$ and $y = 1$, implying that the point $A(-3, 1, 0)$ is on the plane.

Point on xz-plane: Any point on the xz-plane has $y = 0$. If we substitute $y = 0$ into $x + 3y - z = 0$, then $x + 3(0) - z = 0$, or $x = z$, which means that the given plane cuts the xz-plane along the line $x = z$. As before, we choose convenient values for x and z. The easiest values are $x = z = 1$, implying that $B(1, 0, 1)$ is a point on the plane.

Since three points determine a plane, we locate these points in R^3 and form the related triangle. This triangle, OAB, represents part of the required plane.



Case 3b: A, B, and C do not equal zero, and D does not equal zero.

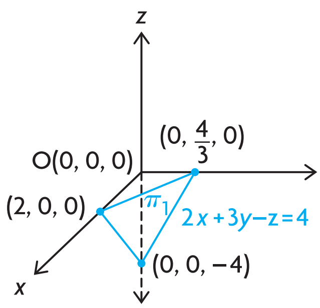
This represents the plane with equation $Ax + By + Cz + D = 0$, which is most easily sketched by finding its intercepts. Since we know that three noncollinear points determine a plane, knowing these three intercepts will allow us to graph the plane.

EXAMPLE 6 Graphing planes whose Cartesian equations involve three variables, $D \neq 0$

Sketch the plane defined by the equation $\pi_1 : 2x + 3y - z = 4$.

Solution

To sketch the plane, we calculate the coordinates of the points where the plane intersects each of the three coordinate axes (that is, we determine the three intercepts for the plane). This is accomplished by setting 2 of the 3 variables equal to zero and solving for the remaining variable. The x-, y-, and z-intercepts are 2, $\frac{4}{3}$, and -4, respectively. These three points form a triangle that forms part of the required plane.



EXAMPLE 7 Reasoning about direction vectors of planes

Determine two direction vectors for the planes $\pi_1 : 3x + 4y = 12$ and $\pi_2 : x - y - 5z = 0$.

Solution

The plane $\pi_1 : 3x + 4y = 12$ crosses the x-axis at the point $(4, 0, 0)$ and the y-axis at the point $(0, 3, 0)$. One direction vector is thus $\overrightarrow{m_1} = (4 - 0, 0 - 3, 0 - 0) = (4, -3, 0)$. Since the plane can be written as $3x + 4y + 0z = 12$, this implies that it does not intersect the z-axis, and therefore has $\overrightarrow{m_2} = (0, 0, 1)$ as a second direction vector.

The plane $\pi_2 : x - y - 5z = 0$ passes through $O(0, 0, 0)$ and cuts the xz-plane along the line $x - 5z = 0$. Convenient choices for x and z are 5 and 1, respectively. This means that $A(5, 0, 1)$ is on π_2. Similarly, the given plane cuts the xy-plane along the line $x - y = 0$. Convenient values for x and y are 1 and 1. This means that $B(1, 1, 0)$ is on π_2.

Possible direction vectors for π_2 are $\overrightarrow{m_1} = (5 - 0, 0 - 0, 1 - 0) = (5, 0, 1)$ and $\overrightarrow{m_2} = (1, 1, 0)$.

IN SUMMARY

Key Idea

- A sketch of a plane in R^3 can be created by using a combination of points and lines that help to define the plane.

Need to Know

- To sketch the graph of a plane, consider each of the following cases as it relates to the Cartesian equation $Ax + By + Cz + D = 0$:

  Case 1: The equation contains one variable.

  a. Two of the coefficients (two of A, B, or C) equal zero, and D equals zero.

     These are the three coordinate planes—xy-plane, xz-plane, and yz-plane.

  b. Two of the coefficients (two of A, B, or C) equal zero, and D does not equal zero.

     These are parallel to the three coordinate planes.

  Case 2: The equation contains two variables.

  a. One of the coefficients (one of A, B, or C) equals zero, and D equals zero.

     Find a point with missing variable set equal to 0. Join this point to (0, 0, 0), and draw a plane containing the missing variable axis and this point.

  b. One of the coefficients (one of A, B, or C) equals zero, and D does not equal zero.

     Find the two intercepts, and draw a plane parallel to the missing variable axis.

  Case 3: The equation contains three variables.

  a. None of the coefficients (none of A, B, or C) equals zero, and D equals zero.

     Determine two points in addition to (0, 0, 0), and draw the plane through these points.

  b. None of the coefficients (none of A, B, or C) equals zero, and D does not equal zero.

     Find the three intercepts, and draw a plane through these three points.

Exercise 8.6

PART A

C 1. Describe each of the following planes in words:

   a. $x = -2$ b. $y = 3$ c. $z = 4$

2. For the three planes given in question 1, what are coordinates of their point of intersection?

3. On which of the planes $\pi_1: x = 5$ or $\pi_2: y = 6$ could the point $P(5, -3, -3)$ lie? Explain.

PART B

A 4. Given that $x^2 - 1 = (x - 1)(x + 1)$, sketch the two graphs associated with $x^2 - 1 = 0$ in R^2 and R^3.

5. a. State the x-, y-, and z-intercepts for each of the following three planes:
   i. $\pi_1 : 2x + 3y = 18$
   ii. $\pi_2 : 3x - 4y + 5z = 120$
   iii. $\pi_3 : 13y - z = 39$
   b. State two direction vectors for each plane.

6. a. For the plane with equation $\pi : 2x - y + 5z = 0$, determine
   i. the coordinates of three points on this plane
   ii. the equation of the line where this plane intersects the xy-plane
   b. Sketch this plane.

7. Name the three planes that the equation $xyz = 0$ represents in R^3.

K 8. For each of the following equations, sketch the corresponding plane:
   a. $\pi_1 : 4x - y = 0$
   b. $\pi_2 : 2x + y - z = 4$
   c. $\pi_3 : z = 4$
   d. $\pi_4 : y - z = 4$

9. a. Write the expression $xy + 2y = 0$ in factored form.
   b. Sketch the lines corresponding to this expression in R^2.
   c. Sketch the planes corresponding to this expression in R^3.

10. For each given equation, sketch the corresponding plane.
   a. $2x + 2y + z - 4 = 0$
   b. $3x - 4z = 12$
   c. $5y - 15 = 0$

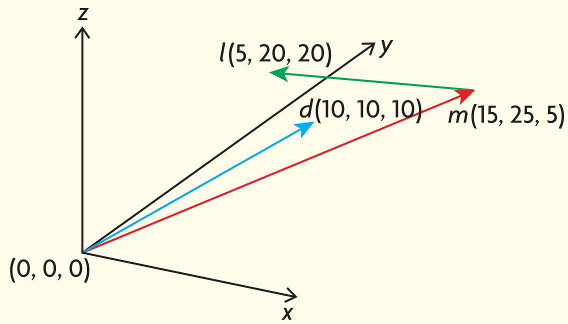
PART C

T 11. It is sometimes useful to be able to write an equation of a plane in terms of its intercepts. If *a, b,* and *c* represent the x-, y-, and z-intercepts, respectively, then the resulting equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
   a. Determine the equation of the plane that has x-, y-, and z-intercepts of 3, 4, and 6, respectively.
   b. Determine the equation of the plane that has x- and z-intercepts of 5 and –7, respectively, and is parallel to the y-axis.
   c. Determine the equation of the plane that has no x- or y-intercept, but has a z-intercept of 8.

CAREER LINK WRAP-UP Investigate and Apply

CHAPTER 8: COMPUTER PROGRAMMING WITH VECTORS

A computer programmer is designing a 3-D space game. She wants to have an asteroid fly past a spaceship along the path of vector m, collide with another asteroid, and be deflected along vector path $\overrightarrow{ml}$. The spaceship is treated as the origin and is travelling along vector d.



a. Determine the vector and parametric equations for the line determined by vector $\overrightarrow{ml}$ in its current position.

b. Determine the vector and parametric equations for the line determined by vector $\vec{d}$ in its current position.

c. By using the previous parts, can you determine if the asteroid and spaceship could possibly collide as they travel along their respective trajectories? Explain in detail all that would have to take place for this collision to occur (if, indeed, a collision is even possible).