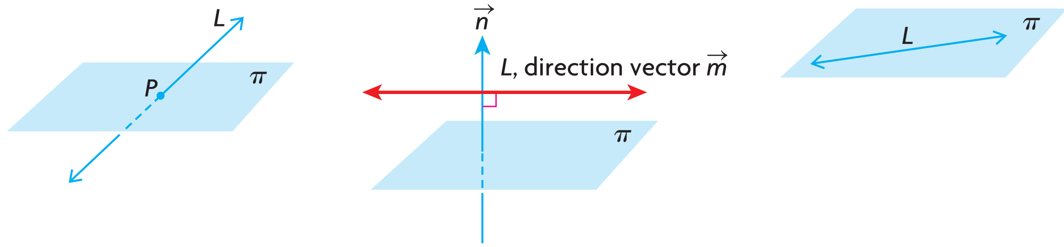
Section 9.1—The Intersection of a Line with a Plane and the Intersection of Two Lines

We start by considering the intersection of a line with a plane.

Intersection between a Line and a Plane

Before considering mathematical techniques for the solution to this problem, we consider the three cases for the intersection of a line with a plane.

Case 1: L intersects π at a point. Case 2: L is parallel to, but not on, π. Case 3: L is on π.



Case 1: The line L intersects the plane π at exactly one point, P.
Case 2: The line L does *not* intersect the plane so it is parallel to the plane. There are no points of intersection.
Case 3: The line L lies on the plane π. Every point on L intersects the plane. There are an infinite number of points of intersection.

For the intersection of a line with a plane, there are three different possibilities, which correspond to zero, one, or an infinite number of intersection points. It is not possible to have a finite number of intersection points, other than zero or one. These three possible intersections are considered in the following examples.

EXAMPLE 1 **Selecting a strategy to determine the point of intersection between a line and a plane**

Determine points of intersection between the line $L: \vec{r} = (3, 1, 2) + s(1, -4, -8), s \in \mathbf{R}$, and the plane $\pi: 4x + 2y - z - 8 = 0$, if any exist.

Solution

To determine the required point of intersection, first convert the line from its vector form to its corresponding parametric form. The parametric form is $x = 3 + s, y = 1 - 4s, z = 2 - 8s$. Using parametric equations allows for direct substitution into π.

$$4(3 + s) + 2(1 - 4s) - (2 - 8s) - 8 = 0 \quad \text{(Use substitution)}$$
$$12 + 4s + 2 - 8s - 2 + 8s - 8 = 0 \quad \text{(Isolate } s\text{)}$$
$$4s = -4$$
$$s = -1$$

This means that the point where L meets π corresponds to a single point on the line with a parameter value of $s = -1$. To obtain the coordinates of the required point, $s = -1$ is substituted into the parametric equations of L. The point of intersection is

$$x = 3 + (-1) = 2$$
$$y = 1 - 4(-1) = 5$$
$$z = 2 - 8(-1) = 10$$

Check (by substitution):
The point lies on the plane because
$4(2) + 2(5) - 10 - 8 = 8 + 10 - 10 - 8 = 0$.

The point that satisfies the equation of the plane and the line is $(2, 5, 10)$. Now we consider the situation in which the line does not intersect the plane.

EXAMPLE 2 Connecting the algebraic representation to the situation with no points of intersection

Determine points of intersection between the line
$L: x = 2 + t, y = 2 + 2t, z = 9 + 8t, t \in \mathbf{R}$,
and the plane $\pi: 2x - 5y + z - 6 = 0$, if any exist.

Solution

Method 1:
Because the line L is already in parametric form, we substitute the parametric equations into the equation for π.

$$2(2 + t) - 5(2 + 2t) + (9 + 8t) - 6 = 0 \quad \text{(Use substitution)}$$
$$4 + 2t - 10 - 10t + 9 + 8t - 6 = 0 \quad \text{(Isolate } t\text{)}$$
$$0t = -3$$

Since there is no value of t that, when multiplied by zero, gives -3, there is no solution to this equation. Because there is no solution to this equation, there is no point of intersection. Thus, L and π do not intersect. L is a line that lies on a plane that is parallel to π.

Method 2:
It is also possible to show that the given line and plane do not intersect by first considering $\vec{n} = (2, -5, 1)$, which is the normal for the plane, and $\vec{m} = (1, 2, 8)$, which is the direction vector for the line, and calculating their dot product. If the dot product is zero, this implies that the line is either on the plane or parallel to the plane.

$$\begin{aligned} \vec{n} \times \vec{m} &= (2, -5, 1) \cdot (1, 2, 8) \\ &= 2(1) - 5(2) + 1(8) \\ &= 0 \end{aligned}$$

(Definition of dot product)

We can prove that the line does not lie on the plane by showing that the point $(2, 2, 9)$, which we know is on the line, is not on the plane.

Substituting $(2, 2, 9)$ into the equation of the plane, we get
$2(2) - 5(2) + 9 - 6 = -3 \neq 0$.

Since the point does not satisfy the equation of the plane, the point is not on the plane. The line and the plane are parallel and do not intersect.

Next, we examine the intersection of a line and a plane where the line lies on the plane.

EXAMPLE 3 Connecting the algebraic representation to the situation with infinite points of intersection

Determine points of intersection of the line $L: \vec{r} = (3, -2, 1) + s(14, -5, -3)$, $s \in \mathbf{R}$, and the plane $x + y + 3z - 4 = 0$, if any exist.

Solution

Method 1:
As before, we convert the equation of the line to its parametric form. Doing so, we obtain the equations $x = 3 + 14s$, $y = -2 - 5s$, and $z = 1 - 3s$.

$$\begin{aligned} (3 + 14s) + (-2 - 5s) + 3(1 - 3s) - 4 &= 0 \\ 3 + 14s - 2 - 5s + 3 - 9s - 4 &= 0 \\ 0s &= 0 \end{aligned}$$

(Use substitution)
(Isolate s)

Since any real value of s will satisfy this equation, there are an infinite number of solutions to this equation, each corresponding to a real value of s. Since any value will work for s, every point on L will be a point on the plane. Therefore, the given line lies on the plane.

Method 2:
Again, this result can be achieved by following the same procedure as in the previous example. If $\vec{n} = (1, 1, 3)$ and $\vec{m} = (14, -5, -3)$, then $\vec{n} \times \vec{m} = 1(14) + 1(-5) + 3(-3) = 0$, implying that the line and plane are parallel. We substitute the coordinates $(3, -2, 1)$, which is a point on the line, into the equation of the plane and find that $3 + (-2) + 3(1) - 4 = 0$. So this point lies on the plane as well. Since the line and plane are parallel, and $(3, -2, 1)$ lies on the plane, the entire line lies on the plane.

Next, we consider the intersection of a line with a plane parallel to a coordinate plane.

EXAMPLE 4 **Reasoning about the intersection between a line and the *yz*-plane**

Determine points where $L: x = 2 - s, y = -1 + 3s, z = 4 - 2s, s \in \mathbf{R}$, and $\pi : x = -3$ intersect, if any exist.

Solution

At the point of intersection, the x-values for the line and the plane will be equal.

Equating the two gives $2 - s = -3$, or $s = 5$. The y- and z-values for the point of intersection can now be found by substituting $s = 5$ into the other two parametric equations. Thus, $y = -1 + 3s = -1 + 3(5) = 14$ and $z = 4 - 2s = 4 - 2(5) = -6$. The point of intersection between L and π is $(-3, 14, -6)$.

Intersection between Two Lines

Thus far, we have discussed the possible intersections between a line and a plane. Next, we consider the possible intersection between two lines.

There are four cases to consider for the intersection of two lines in R^3.

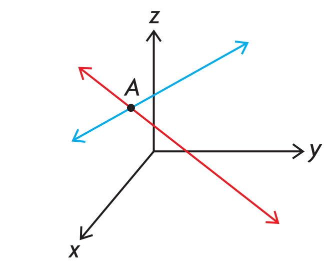
Intersecting Lines
Case 1: The lines are not parallel and intersect at a single point.
Case 2: The lines are coincident, meaning that the two given lines are identical. There are an infinite number of points of intersection.

Non-intersecting Lines
Case 3: The two lines are parallel, and there is no intersection.
Case 4: The two lines are not parallel, and there is no intersection. The lines in this case are called **skew lines**. (Skew lines do not exist in R^2, only in R^3.)

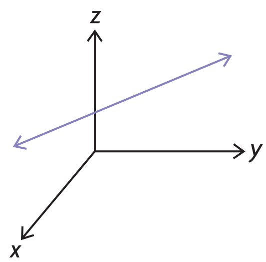
These four cases are shown in the diagram below.

Intersecting Lines

Case 1: Intersecting Lines at a Point

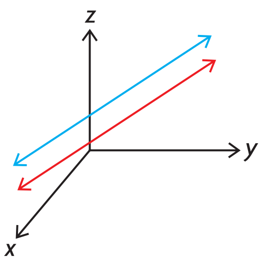


Case 2: Coincident Lines

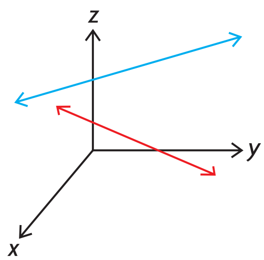


Non-intersecting Lines

Case 3: Parallel Lines



Case 4: Skew Lines



EXAMPLE 5 Selecting a strategy to determine the intersection of two lines in R^3

For $L_1 : \vec{r} = (-3, 1, 4) + s(-1, 1, 4), s \in \mathbf{R}$, and $L_2 : \vec{r} = (1, 4, 6) + t(-6, -1, 6), t \in \mathbf{R}$, determine points of intersection, if any exist.

Solution

Before calculating the coordinates of points of intersection between the two lines, we note that these lines are not parallel to each other because their direction vectors are not scalar multiples of each other—that is, $(-1, 1, 4) \neq k(-6, -1, 6)$. This indicates that these lines either intersect each other exactly once or are skew lines. If these lines intersect, there must be a single point that is on both lines. To use this idea, the vector equations for L_1 and L_2 must be converted to parametric form.

L_1	L_2
$x = -3 - s$	$x = 1 - 6t$
$y = 1 + s$	$y = 4 - t$
$z = 4 + 4s$	$z = 6 + 6t$

We can now select any two of the three equations from each line and equate them. Comparing the x and y components gives $-3 - s = 1 - 6t$ and $1 + s = 4 - t$. Rearranging and simplifying gives

① $s - 6t = -4$

② $s + t = 3$

Subtracting ② from ① yields the following:

$-7t = -7$

$t = 1$

Substituting $t = 1$ into equation ①,

$s - 6(1) = -4$

$s = 2$

We find $s = 2$ and $t = 1$. These two values can now be substituted into the parametric equations to find the corresponding values of x, y, and z.

L_1	L_2
$x = -3 - 2 = -5$	$x = 1 - 6(1) = -5$
$y = 1 + 2 = 3$	$y = 4 - 1 = 3$
$z = 4 + 4(2) = 12$	$z = 6 + 6(1) = 12$

Since we found that substituting $s = 2$ and $t = 1$ into the corresponding parametric equations gives the same values of x, y, and z, the point of intersection is $(-5, 3, 12)$.

It is important to understand that when finding the points of intersection between any pair of lines, the parametric values must be substituted back into the original equations to check that a consistent result is obtained. In other words, $s = 2$ and $t = 1$ must give the same point for each line. In this case, there were consistent values, and so we can be certain that the point of intersection is $(-5, 3, 12)$.

In the next example, we will demonstrate the importance of checking for consistency to find the possible point of intersection.

EXAMPLE 6 Connecting the solution to a system of equations to the case of skew lines

For $L_1: x = -1 + s, y = 3 + 4s, z = 6 + 5s, s \in \mathbf{R}$, and $L_2: x = 4 - t, y = 17 + 2t, z = 30 - 5t, t \in \mathbf{R}$, determine points of intersection, if any exist.

Solution

We use the same approach as in the previous example. In this example, we'll start by equating corresponding y- and z-coordinates.

L_1	L_2
$x = -1 + s$	$x = 4 - t$
$y = 3 + 4s$	$y = 17 + 2t$
$z = 6 + 5s$	$z = 30 - 5t$

Comparing y- and z-values, we get $3 + 4s = 17 + 2t$ and $6 + 5s = 30 - 5t$. Rearranging and simplifying gives

① $4s - 2t = 14$

② $5s + 5t = 24$

③ $10s - 5t = 35$ $\quad \frac{5}{2} \times$ ①

$15s = 59$ $\quad$ ② + ③

$$s = \frac{59}{15}$$

If $s = \frac{59}{15}$ is substituted into either equation ① or equation ②, we obtain the value of t.

Substituting into equation ①,

$$4\left(\frac{59}{15}\right) - 2t = 14$$

$$\frac{236}{15} - \frac{210}{15} = 2t$$

$$t = \frac{13}{15}$$

We found that $s = \frac{59}{15}$ and $t = \frac{13}{15}$. These two values can now be substituted back into the parametric equations to find the values of x, y, and z.

L_1	L_2
$x = -1 + \frac{59}{15} = \frac{44}{15}$	$x = 4 - \frac{13}{15} = \frac{47}{15}$
$y = 3 + 4\left(\frac{59}{15}\right) = \frac{281}{15}$	$y = 17 + 2\left(\frac{13}{15}\right) = \frac{281}{15}$
$z = 6 + 5\left(\frac{59}{15}\right) = \frac{77}{3}$	$z = 30 - 5\left(\frac{13}{15}\right) = \frac{77}{3}$

For these lines to intersect at a point, we must obtain equal values for each coordinate. From observation, we can see that the x-coordinates are different, which implies that these lines do not intersect. Since the two given lines do not intersect and have different direction vectors, they must be skew lines.

IN SUMMARY

Key Ideas

- Line and plane intersections can occur in three different ways.

  Case 1: The line L intersects the plane π at exactly one point, P.

  Case 2: The line L does *not* intersect the plane and is parallel to the plane π. In this case, there are no points of intersection and solving the system of equations results in an equation that has no solution ($0 \times$ variable $=$ a nonzero number).

  Case 3: The line L lies on the plane π. In this case, there are an infinite number of points of intersection between the line and the plane, and solving the system of equations results in an equation with an infinite number of solutions ($0 \times$ variable $= 0$).

- Line and line intersections can occur in four different ways.

  Case 1: The lines intersect at a single point.

  Case 2: The two lines are parallel, and there is no intersection.

  Case 3: The two lines are not parallel and do not intersect. The lines in this case are called *skew lines*.

  Case 4: The two lines are parallel and coincident. They are the same line.

Exercise 9.1

PART A

1. Tiffany is given the parametric equations for a line L and the Cartesian equation for a plane π and is trying to determine their point of intersection. She makes a substitution and gets $(1 + 5s) - 2(2 + s) - 3(-3 + s) - 6 = 0$.

   a. Give a possible equation for both the line and the plane.

   b. Finish the calculation, and describe the nature of the intersection between the line and the plane.

2. a. If a line and a plane intersect, in how many different ways can this occur? Describe each case.

   b. It is only possible to have zero, one, or an infinite number of intersections between a line and a plane. Explain why it is not possible to have a finite number of intersections, other than zero or one, between a line and a plane.

C 3. A line has the equation $\vec{r} = s(1, 0, 0)$, $s \in \mathbf{R}$, and a plane has the equation $y = 1$.

   a. Describe the line.

   b. Describe the plane.

c. Sketch the line and the plane.

d. Describe the nature of the intersection between the line and the plane.

PART B

4. For each of the following, show that the line lies on the plane with the given equation. Explain how the equation that results implies this conclusion.

   a. $L: x = -2 + t, y = 1 - t, z = 2 + 3t, t \in \mathbf{R}; \pi: x + 4y + z - 4 = 0$

   b. $L: \vec{r} = (1, 5, 6) + t(1, -2, -2), t \in \mathbf{R}; \pi: 2x - 3y + 4z - 11 = 0$

5. For each of the following, show that the given line and plane do not intersect. Explain how the equation that results implies there is no intersection.

   a. $L: \vec{r} = (-1, 1, 0) + s(-1, 2, 2), s \in \mathbf{R}; \pi: 2x - 2y + 3z - 1 = 0$

   b. $L: x = 1 + 2t, y = -2 + 5t, z = 1 + 4t, t \in \mathbf{R};$
   $\pi: 2x - 4y + 4z - 13 = 0$

6. Verify your results for question 5 by showing that the direction vector of the line and the normal for the plane meet at right angles, and the given point on the line does not lie on the plane.

7. For the following, determine points of intersection between the given line and plane, if any exist:

   a. $L: \vec{r} = (-1, 3, 4) + p(6, 1, -2), p \in \mathbf{R}; \pi: x + 2y - z + 29 = 0$

   b. $L: \dfrac{x - 1}{4} = \dfrac{y + 2}{-1} = z - 3; \pi: 2x + 7y + z + 15 = 0$

8. Determine points of intersection between the following pairs of lines, if any exist:

   a. $L_1: \vec{r} = (3, 1, 5) + s(4, -1, 2), s \in \mathbf{R};$
   $L_2: x = 4 + 13t, y = 1 - 5t, z = 5t, t \in \mathbf{R}$

   b. $L_3: \vec{r} = (3, 7, 2) + m(1, -6, 0), m \in \mathbf{R};$
   $L_4: \vec{r} = (-3, 2, 8) + s(7, -1, -6), s \in \mathbf{R}$

K

9. Determine which of the following pairs of lines are skew lines:

   a. $\vec{r} = (-2, 3, 4) + p(6, -2, 3), p \in \mathbf{R};$
   $\vec{r} = (-2, 3, -4) + q(6, -2, 11), q \in \mathbf{R}$

   b. $\vec{r} = (4, 1, 6) + t(1, 0, 4), t \in \mathbf{R}; \vec{r} = (2, 1, -8) + s(1, 0, 5), s \in \mathbf{R}$

   c. $\vec{r} = (2, 2, 1) + m(1, 1, 1), m \in \mathbf{R};$
   $\vec{r} = (-2, 2, 1) + p(3, -1, -1), p \in \mathbf{R}$

   d. $\vec{r} = (9, 1, 2) + m(5, 0, 4), m \in \mathbf{R}; \vec{r} = (8, 2, 3) + s(4, 1, -2), s \in \mathbf{R}$

10. The line with the equation $\vec{r} = (-3, 2, 1) + s(3, -2, 7), s \in \mathbf{R}$, intersects the z-axis at the point $Q(0, 0, q)$. Determine the value of q.

11. a. Show that the lines $L_1: \vec{r} = (-2, 3, 4) + s(7, -2, 2)$, $s \in \mathbf{R}$, and $L_2: \vec{r} = (-30, 11, -4) + t(7, -2, 2)$, $t \in \mathbf{R}$, are coincident by writing each line in parametric form and comparing components

b. Show that the point $(-2, 3, 4)$ lies on L_2. How does this show that the lines are coincident?

12. The lines $\vec{r} = (-3, 8, 1) + s(1, -1, 1)$, $s \in \mathbf{R}$, and $\vec{r} = (1, 4, 2) + t(-3, k, 8)$, $t \in \mathbf{R}$, intersect at a point.

a. Determine the value of k.

b. What are the coordinates of the point of intersection?

A 13. The line $\vec{r} = (-8, -6, -1) + s(2, 2, 1)$, $s \in \mathbf{R}$, intersects the xz- and yz-coordinate planes at the points A and B, respectively. Determine the length of line segment AB.

14. The lines $\vec{r} = (2, 1, 1) + p(4, 0, -1)$, $p \in \mathbf{R}$, and $\vec{r} = (3, -1, 1) + q(9, -2, -2)$, $q \in \mathbf{R}$, intersect at the point A.

a. Determine the coordinates of point A.

b. What is the distance from point A to the xy-plane?

T 15. The lines $\vec{r} = (-1, 3, 2) + s(5, -2, 10)$, $s \in \mathbf{R}$, and $\vec{r} = (4, -1, 1) + t(0, 2, 11)$, $t \in \mathbf{R}$, intersect at point A.

a. Determine the coordinates of point A.

b. Determine the vector equation for the line that is perpendicular to the two given lines and passes through point A.

16. a. Sketch the lines $L_1: \vec{r} = p(0, 1, 0)$, $p \in \mathbf{R}$, and $L_2: \vec{r} = q(0, 1, 1)$, $q \in \mathbf{R}$.

b. At what point do these lines intersect?

c. Verify your conclusion for part b. algebraically.

PART C

17. a. Show that the lines $\frac{x}{1} = \frac{y - 7}{-8} = \frac{z - 1}{2}$ and $\frac{x - 4}{3} = \frac{z - 1}{-2}$, $y = -1$, lie on the plane with equation $2x + y + 3z - 10 = 0$.

b. Determine the point of intersection of these two lines.

18. A line passing through point $P(-4, 0, -3)$ intersects the two lines with equations $L_1: \vec{r} = (1, 1, -1) + s(1, 1, 0)$, $s \in \mathbf{R}$, and $L_2: \vec{r} = (0, 1, 3) + t(-2, 1, 3)$, $t \in \mathbf{R}$. Determine a vector equation for this line.