Section 9.2—Systems of Equations

To solve problems in real-life situations, we often need to solve systems of linear equations. Thus far, we have seen systems of linear equations in a variety of different contexts dealing with lines and planes. The following is a typical example of a system of two equations in two unknowns:

① $2x + y = -9$

② $x + 2y = -6$

Each of the equations in this system is a linear equation. A linear equation is an equation of the form $a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = b$, where $a_1, a_2, a_3, \ldots, a_n$ and b are real numbers with the variables $x_1, x_2, x_3, \ldots, x_n$ being the unknowns. Typical examples of linear equations are $y = 2x - 3$, $x + 4y = 9$, and $x + 3y - 2z - 2 = 0$. All the variables in each of these equations are raised to the first power only (degree of one). Linear equations do not include any products or powers of variables, and there are no trigonometric, logarithmic, or exponential functions making up part of the equation. Typical examples of nonlinear equations are $x - 3y^2 = 3$, $2x - xyz = 4$, and $y = \sin 2x$.

A system of linear equations is a set of one or more linear equations. When we solve a system of linear equations, we are trying to find values that will simultaneously satisfy the unknowns in each of the equations. In the following example, we consider a system of two equations in two unknowns and possible solutions for this system.

EXAMPLE 1

Reasoning about the solutions to a system of two equations in two unknowns

The number of solutions to the following system of equations depends on the value(s) of a and b. Determine values of a and b for which this system has no solutions, an infinite number of solutions, and one solution.

① $x + 4y = a$

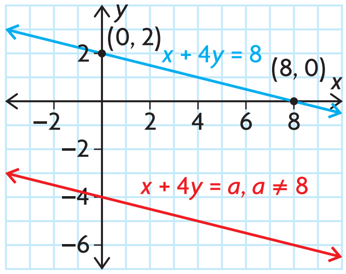
② $x + by = 8$

Solution

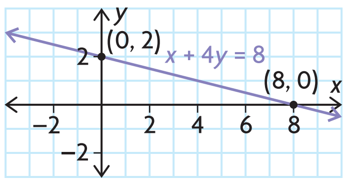
Each of the equations in this system represents a line in R^2. For these two lines, there are three cases to consider, each depending on the values of a and b.

Case 1: These equations represent two parallel and non-coincident lines. If these lines are parallel, they must have the same slope, implying that $b = 4$.

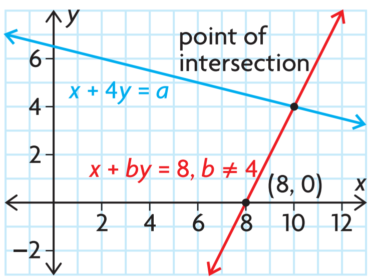
This means that the second equation is $x + 4y = 8$, and the slope of each line is $-\frac{1}{4}$. If $a \neq 8$, this implies that the two lines are parallel and have different equations. Since the lines would be parallel and not intersect, there is no solution to this system when $b = 4$ and $a \neq 8$.



Case 2: These equations represent two parallel and coincident lines. This means that the two equations must be equivalent. If $a = 8$ and $b = 4$, then both equations are identical and this system would be reduced to finding values of x and y that satisfy the equation $x + 4y = 8$. Since there are an infinite number of points that satisfy this equation, the original system will have an infinite number of solutions.



Case 3: These two equations represent two intersecting, non-coincident lines. The third possibility for these two lines is that they intersect at a single point in R^2. These lines will intersect at a single point if they are not parallel—that is, if $b \neq 4$. In this case, the solution is the point of intersection of these lines.



This system of linear equations is typical in that it can only have zero, one, or an infinite number of solutions. In general, it is not possible for any system of linear equations to have a finite number of solutions greater than one.

Number of Solutions to a Linear System of Equations

A linear system of equations can have zero, one, or an infinite number of solutions.

In Example 2, the idea of equivalent systems is introduced as a way of understanding how to solve a system of equations. Equivalent systems of equations are defined in the following way:

Definition of Equivalent Systems

Two systems of equations are defined as equivalent if every solution to one system is also a solution to the second system of equations, and vice versa.

The idea of equivalent systems is important because, when solving a system of equations, what we are attempting to do is create a system of equations that is easier to solve than the previous system. To construct an equivalent system of equations, the new system is obtained in a series of steps using a set of well-defined operations. These operations are referred to as **elementary operations**.

Elementary Operations Used to Create Equivalent Systems

1. Multiply an equation by a nonzero constant.
2. Interchange any pair of equations.
3. Add a nonzero multiple of one equation to a second equation to replace the second equation.

In previous courses, when we solved systems of equations, we often multiplied two equations by different constants and then added or subtracted to eliminate variables. Although these kinds of operations can be used algebraically to solve systems, elementary operations are used because of their applicability in higher-level mathematics.

The use of elementary operations to create equivalent systems is illustrated in the following example.

EXAMPLE 2 Using elementary operations to solve a system of two equations in two unknowns

Solve the following system of equations:

① $2x + y = -9$

② $x + 2y = -6$

Solution

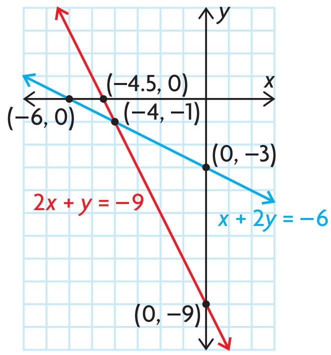
1: Interchange equations ① and ②.

① $x + 2y = -6$

② $2x + y = -9$

The equations have been interchanged to make the coefficient of x in the first equation equal to 1. This is always a good strategy when solving systems of linear equations.

This original system of equations is illustrated in the following diagram.



After Step 1

2: Multiply equation ① by -2, and then add equation ② to eliminate the variable x from the second equation to create equation ③. Note that the coefficient of the x-term in the new equation is 0.

① $x + 2y = -6$

③ $0x - 3y = 3$ $\quad -2 \times$ ① + ②

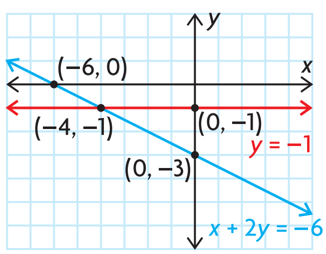
When solving a system of equations, the elementary operations that are used are specified beside the newly created equation.

3: Multiply each side of equation ③ by $-\frac{1}{3}$ to obtain a new equation that is labelled equation ④.

① $x + 2y = -6$

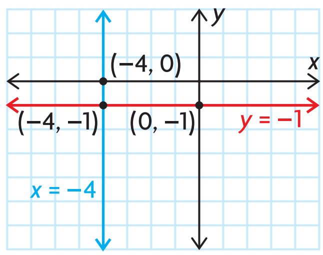
④ $0x + y = -1$ $\quad -\frac{1}{3} \times$ ③

This new system of equations is illustrated in the diagram below.



After Step 3

The new system of equations that we produced is easier to solve than the original system. If we substitute $y = -1$ into equation ①, we obtain $x + 2(-1) = -6$ or $x = -4$.
The solution to this system of equations is $x = -4$, $y = -1$, which is shown in the graph below.



Final system

The equivalence of the systems of equations was illustrated geometrically at different points in the calculations. As each elementary operation is applied, we create an equivalent system such that the two lines always have the point $(-4, -1)$ in common. When we create the various equivalent systems, the solution to each set of equations remains the same. This is what we mean when we use elementary operations to create equivalent systems.

The solution to the original system of equations is $x = -4$ and $y = -1$. This means that these two values of x and y must satisfy each of the given equations. It is easy to verify that these values satisfy each of the given equations. For the first equation, $2(-4) + (-1) = -9$. For the second equation, $-4 + 2(-1) = -6$.

A solution to a system of equations must satisfy each equation in the system for it to be a solution to the overall system. This is demonstrated in the following example.

EXAMPLE 3 Reasoning about the solution to a system of two equations in three unknowns

Determine whether $x = -3$, $y = 5$, and $z = 6$ is a solution to the following system:

① $2x + 3y - 5z = -21$

② $x - 6y + 6z = 8$

Solution

For the given values to be a solution to this system of equations, they must satisfy both equations.

Substituting into the first equation,

$$2(-3) + 3(5) - 5(6) = -6 + 15 - 30 = -21$$

Substituting into the second equation,

$-3 - 6(5) + 6(6) = -3 - 30 + 36 = 3 \neq 8$

Since the values of x, y, and z do not satisfy both equations, they are not a solution to this system.

If a system of equations has no solutions, it is said to be **inconsistent**. If a system has at least one solution, it is said to be **consistent**.

Consistent and Inconsistent Systems of Equations

A system of equations is consistent if it has either one solution or an infinite number of solutions. A system is inconsistent if it has no solutions.

In the next example, we show how to use elementary operations to solve a system of three equations in three unknowns.

EXAMPLE 4 Using elementary operations to solve a system of three equations in three unknowns

Solve the following system of equations for x, y, and z using elementary operations:

① $x - y + z = 1$

② $2x + y - z = 11$

③ $3x + y + 2z = 12$

Solution

1: Use equation ① to eliminate x from equations ② and ③.

① $x - y + z = 1$

④ $0x + 3y - 3z = 9$ $\quad -2 \times$ ① + ②

⑤ $0x + 4y - z = 9$ $\quad -3 \times$ ① + ③

2: Use equations ④ and ⑤ to eliminate y from equation ⑤, and then scale equation ④.

① $x - y + z = 1$

⑥ $0x + y - z = 3$ $\quad \frac{1}{3} \times$ ④

⑦ $0x + 0y + 3z = -3$ $\quad -\frac{4}{3} \times$ ④ + ⑤

We can now solve this system by using a method known as **back substitution**. Start by solving for z in equation ⑦, and use this value to solve for y in equation ⑥. From there, we use the values for y and z to solve for x in equation ①.

From equation ⑦, $3z = -3$

$$z = -1$$

If we then substitute into equation ⑥,

$$y - (-1) = 3$$

$$y = 2$$

If $y = 2$ and $z = -1$, these values can now be substituted into equation ① to obtain $x - 2 + (-1) = 1$, or $x = 4$.

Therefore, the solution to this system is $(4, 2, -1)$.

Check:
These values should be substituted into each of the original equations and checked to see that they satisfy each equation.

To solve this system of equations, we used elementary operations and ended up with a triangle of zeros in the lower left part:

$$ax + by + cz = d$$
$$0x + ey + fz = g$$
$$0x + 0y + hz = i$$

The use of elementary operations to create the lower triangle of zeros is our objective when solving systems of equations. Large systems of equations are solved using computers and elementary operations to eliminate unknowns. This is by far the most efficient and cost-effective method for their solution.

In the following example, we consider a system of equations with different possibilities for its solution.

EXAMPLE 5 Connecting the value of a parameter to the nature of the intersection between two lines in R^2

Consider the following system of equations:

① $x + ky = 4$
② $kx + 4y = 8$

Determine the value(s) of k for which this system of equations has

a. no solutions
b. one solution
c. an infinite number of solutions

Solution

Original System of Equations:

① $x + ky = 4$
② $kx + 4y = 8$

1: Multiply equation ① by $-k$, and add it to equation ② to eliminate x from equation ②.

① $x + ky = 4$

③ $0x - k^2y + 4y = -4k + 8, -k \times ① + ②$

Actual Solution to Problem:
To solve the problem, it is only necessary to deal with the equation $0x - k^2y + 4y = -4k + 8$ to determine the necessary conditions on k.

$$-k^2y + 4y = -4k + 8 \quad \text{(Factor)}$$
$$y(-k^2 + 4) = -4(k - 2) \quad \text{(Multiply by } -1\text{)}$$
$$y(k^2 - 4) = 4(k - 2)$$
$$(k - 2)(k + 2)y = 4(k - 2)$$

There are three different cases to consider.

Case 1: $k = 2$
If $k = 2$, this results in the equation $(2 - 2)(2 + 2)y = 4(2 - 2)$, or $0y = 0$.

Since this equation is true for all real values of y, we will have an infinite number of solutions. Substituting $k = 2$ into the original system of equations gives

① $x + 2y = 4$

② $2(x + 2y) = 2(4)$

This system can then be reduced to just a single equation, $x + 2y = 4$, which, as we have seen, has an infinite number of solutions.

Case 2: $k = -2$
If $k = -2$, this equation becomes $(-2 - 2)(-2 + 2)y = 4(-2 - 2)$, or $0y = -16$.

There are no solutions to this equation. Substituting $k = -2$ into the original system of equations gives

① $x - 2y = 4$

② $-2(x - 2y) = -2(-4)$

This system can be reduced to the two equations, $x - 2y = 4$ and $x - 2y = -4$, which are two parallel lines that do not intersect. Thus, there are no solutions.

Case 3: $k \neq \pm 2$
If $k \neq \pm 2$, we get an equation of the form $ay = b$, $a \neq 0$. This equation will always have a unique solution for y, which implies that the original system of equations will have exactly one solution, provided that $k \neq \pm 2$.

IN SUMMARY

Key Idea

- A system of two (linear) equations in two unknowns geometrically represents two lines in R^2. These lines may intersect at zero, one, or an infinite number of points, depending on how the lines are related to each other.

Need to Know

- Elementary operations can be used to solve a system of equations. The operations are defined as follows:
  1. Multiply an equation by a nonzero constant.
  2. Interchange any pair of equations.
  3. Add a multiple of one equation to a second equation to replace the second equation.

  As each elementary operation is applied, we create an equivalent system, which gets progressively easier to solve.
- The solution to a system of equations consists of the values of the variables that satisfy all the equations in the system simultaneously.
- A system of equations is consistent if it has either one solution or an infinite number of solutions. The system is inconsistent if it has no solutions.

Exercise 9.2

PART A

1. Given that k is a nonzero constant, which of the following are linear equations?

   a. $kx - \frac{1}{k}y = 3$

   b. $2 \sin x = kx$

   c. $2^k x + 3y - z = 0$

   d. $\frac{1}{x} - y = 3$

2. a. Create a system of three equations in three unknowns that has $x = -3$, $y = 4$, and $z = -8$ as its solution.

   b. Solve this system of equations using elementary operations.

3. Determine whether $x = -7$, $y = 5$, and $z = \frac{3}{4}$ is a solution to the following systems:

   a. ① $x - 3y + 4z = -19$
   ② $x - 8z = -13$
   ③ $x + 2y = 3$

   b. ① $3x - 2y + 16z = -19$
   ② $3x - 2y = -23$
   ③ $8x - y + 4z = -58$

PART B

4. Solve each system of equations, and state whether the systems given in parts a. and b. are equivalent or not. Explain.

   a. ① $x = -2$
   ② $3y = -9$

   b. ① $3x + 5y = -21$
   ② $\frac{1}{6}x - \frac{1}{2}y = \frac{7}{6}$

K 5. Solve each of the following systems using elementary operations:

   a. ① $2x - y = 11$
   ② $x + 5y = 11$

   b. ① $2x + 5y = 19$
   ② $3x + 4y = 11$

   c. ① $-x + 2y = 10$
   ② $3x + 5y = 3$

C 6. Solve the following systems of equations, and explain the nature of each intersection:

   a. ① $2x + y = 3$
   ② $2x + y = 4$

   b. ① $7x - 3y = 9$
   ② $35x - 15y = 45$

7. Write a solution to each equation using parameters.

   a. $2x - y = 3$

   b. $x - 2y + z = 0$

8. a. Determine a linear equation that has $x = t$, $y = -2t - 11$, $t \in \mathbf{R}$, as its general solution.

   b. Show that $x = 3t + 3$, $y = -6t - 17$, $t \in \mathbf{R}$, is also a general solution to the linear equation found in part a.

9. Determine the value(s) of the constant k for which the following system of equations has

   a. no solutions
   b. one solution
   c. infinitely many solutions

   ① $x + y = 6$
   ② $2x + 2y = k$

10. For the equation $2x + 4y = 11$, determine

   a. the number of solutions
   b. a generalized parametric solution
   c. an explanation as to why it will *not* have any integer solutions

11. a. Solve the following system of equations for x and y:

   ① $x + 3y = a$
   ② $2x + 3y = b$

   b. Explain why this system of equations will always be consistent, irrespective of the values of a and b.

12. Solve each system of equations using elementary operations.

a. ① $x + y + z = 0$
② $x - y = 1$
③ $y - z = -5$

b. ① $2x - 3y + z = 6$
② $x + y + 2z = 31$
③ $x - 2y - z = -17$

c. ① $x + y = 10$
② $y + z = -2$
③ $x + z = -4$

d. ① $\frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 14$
② $\frac{x}{4} + \frac{y}{5} + \frac{z}{3} = -21$
③ $\frac{x}{5} + \frac{y}{3} + \frac{z}{4} = 7$

e. ① $2x - y = 0$
② $2y - z = 7$
③ $2z - x = 0$

f. ① $x + y + 2z = 13$
② $2y - 3z = -12$
③ $x - y + 4z = 19$

A 13. A system of equations is given by the lines L_1: $ax + by = p$, L_2: $dx + ey = q$, and L_3: $gx + hy = r$. Sketch the lines under the following conditions:

a. when the system of equations represented by these lines has no solutions
b. when the system of equations represented by these lines has exactly one solution
c. when the system of equations represented by these lines has an infinite number of solutions

T 14. Determine the solution to the following system of equations:

① $x + y + z = a$
② $x + y = b$
③ $y + z = c$

PART C

15. Consider the following system of equations:

① $x + 2y = -1$
② $2x + k^2y = k$

Determine the values of k for which this system of equations has

a. no solutions
b. an infinite number of solutions
c. a unique solution