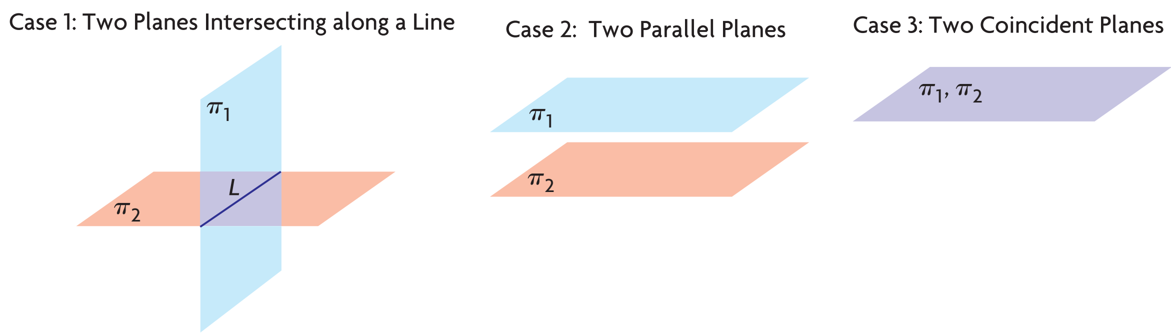
Section 9.3—The Intersection of Two Planes

In the previous section, we introduced elementary operations and their use in the solution of systems of equations. In this section, we will again examine systems of equations but will focus specifically on dealing with the intersections of two planes. Algebraically, these are typically represented by a system of two equations in three unknowns.

In our discussion on the intersections of two planes, there are three different cases to be considered, each of which is illustrated below.

Possible Intersections for Two Planes



Case 1: Two planes can intersect along a line. The corresponding system of equations will therefore have an infinite number of solutions.

Case 2: Two planes can be parallel and non-coincident. The corresponding system of equations will have no solutions.

Case 3: Two planes can be coincident and will have an infinite number of solutions.

Solutions for a System of Equations Representing Two Planes

The system of equations corresponding to the intersection of two planes will have either zero solutions or an infinite number of solutions.

It is not possible for two planes to intersect at a single point.

EXAMPLE 1 Reasoning about the nature of the intersection between two planes (Case 2)

Determine the solution to the system of equations $x - y + z = 4$ and $x - y + z = 5$. Discuss how these planes are related to each other.

Solution

Since the two planes have the same normals, $\vec{n_1} = \vec{n_2} = (1, -1, 1)$, this implies that the planes are parallel. Since the equations have different constants on the right side, the equations represent parallel and non-coincident planes. This indicates that there are no solutions to this system because the planes do not intersect.

The corresponding system of equations is

① $x - y + z = 4$

② $x - y + z = 5$

Using elementary operations, the following equivalent system of equations is obtained:

① $\quad x - y + z = 4$

③ $0x + 0y + 0z = 1 \qquad -1 \times$ ① + ②

Since there are no values that satisfy equation ③, there are no solutions to this system, confirming our earlier conclusion.

EXAMPLE 2 Reasoning about the nature of the intersection between two planes (Case 3)

Determine the solution to the following system of equations:

① $\quad x + 2y - 3z = -1$

② $4x + 8y - 12z = -4$

Solution

Since equation ② can be written as $4(x + 2y - 3z) = 4(-1)$, the two equations represent coincident planes. This means that there are an infinite number of values that satisfy the system of equations. The solution to the system of equations can be written using parameters in equation ①. If we let $y = s$ and $z = t$, then $x = -2s + 3t - 1$.

The solution to the system is $x = -2s + 3t - 1$, $y = s$, $z = t$, $s, t \in \mathbf{R}$. This is the equation of a plane, expressed in parametric form. Every point that lies on the plane is a solution to the given system of equations.

If we had solved the system using elementary operations, we would have arrived at the following equivalent system:

① $x + 2y - 3z = -1$

③ $0x + 0y + 0z = 0 \qquad -4 \times ① + ②$

There are an infinite number of ordered triples (x, y, z) that satisfy both equations ① and ③, confirming our earlier conclusion.

The normals of two planes give us important information about their intersection.

Intersection of Two Planes and their Normals

If the planes π_1 and π_2 have $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ as their respective normals, we know the following:

1. If $\overrightarrow{n_1} = k\overrightarrow{n_2}$ for some scalar, k, the planes are coincident or they are parallel and non-coincident. If they are coincident, there are an infinite number of points of intersection. If they are parallel and non-coincident, there are no points of intersection.
2. If $\overrightarrow{n_1} \neq k\overrightarrow{n_2}$, the two planes intersect in a line. This results in an infinite number of points of intersection.

EXAMPLE 3 Reasoning about the nature of the intersection between two planes (Case 1)

Determine the solution to the following system of equations:

① $x - y + z = 3$

② $2x + 2y - 2z = 3$

Solution

When solving a system involving two planes, it is useful to start by determining the normals for the two planes. The first plane has normal $\overrightarrow{n_1} = (1, -1, 1)$, and the second plane has $\overrightarrow{n_2} = (2, 2, -2)$. Since these vectors are not scalar multiples of each other, the normals are not parallel, which implies that the two planes intersect. Since the planes intersect and do not coincide, they intersect along a line.

We will use elementary operations to solve the system.

① $x - y + z = 3$

③ $0x + 4y - 4z = -3 \qquad -2 \times ① + ②$

To determine the equation of the line of intersection, a parameter must be introduced. From equation ③, which is written as $4y - 4z = -3$, we start by letting $z = s$. Substituting $z = s$ gives

$$\begin{aligned} 4y - 4s &= -3 \\ 4y &= 4s - 3 \\ y &= s - \frac{3}{4} \end{aligned}$$

Substituting $z = s$ and $y = s - \frac{3}{4}$ into equation ①, we obtain

$$\begin{aligned} x - \left(s - \frac{3}{4}\right) + s &= 3 \\ x &= \frac{9}{4} \end{aligned}$$

Therefore, the line of intersection expressed in parametric form is $x = \frac{9}{4}, y = s - \frac{3}{4}, z = s, s \in \mathbf{R}$.

Check:
To check, we'll substitute into each of the two original equations.

Substituting into equation ①,

$$x - y + z = \frac{9}{4} - \left(s - \frac{3}{4}\right) + s = \frac{9}{4} + \frac{3}{4} - s + s = 3$$

Substituting into equation ②,

$$2x + 2y - 2z = 2\left(\frac{9}{4}\right) + 2\left(s - \frac{3}{4}\right) - 2s = \frac{9}{2} - \frac{3}{2} = 3$$

This confirms our conclusion.

EXAMPLE 4 **Selecting the most efficient strategy to determine the intersection between two planes**

Determine the solution to the following system of equations:

① $2x - y + 3z = -2$

② $x - 3z = 1$

Solution

As in the first example, we note that the first plane has normal $\overrightarrow{n_1} = (2, -1, 3)$ and the second $\overrightarrow{n_2} = (1, 0, -3)$. These normals are not scalar multiples of each other, implying that the two planes have a line of intersection.

To find the line of intersection, it is not necessary to use elementary operations to reduce one of the equations. Since the second equation is missing a y-term, the best approach is to write the second equation using a parameter for z. If $z = s$, then $x = 3s + 1$. Now it is a matter of substituting these parametric values into the first equation and determining y in terms of s. Substituting gives

$$\begin{aligned} 2(3s + 1) - y + 3(s) &= -2 \\ 6s + 2 - y + 3s &= -2 \\ 9s + 4 &= y \end{aligned}$$

The line of intersection is given by the parametric equations $x = 3s + 1$, $y = 9s + 4$, and $z = s$, $s \in \mathbf{R}$.

Check:
Substituting into equation ①,
$2(3s + 1) - (9s + 4) + 3s = 6s + 2 - 9s - 4 + 3s = -2$
Substituting into equation ②, $(3s + 1) - 3s = 1$

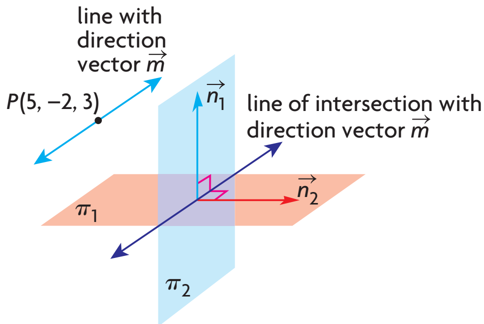
In the next example, we will demonstrate how a problem involving the intersection of two planes can be solved in more than one way.

EXAMPLE 5 Selecting a strategy to solve a problem involving two planes

Determine an equation of a line that passes through the point $P(5, -2, 3)$ and is parallel to the line of intersection of the planes $\pi_1: x + 2y - z = 6$ and $\pi_2: y + 2z = 1$.

Solution

Method 1:
Since the required line is parallel to the line of intersection of the planes, then the direction vectors for both of these lines must be parallel. Since the line of intersection is contained in both planes, its direction vector must then be perpendicular to the normals of each plane.



If $\vec{m}$ represents the direction vector of the required line, and it is perpendicular to $\vec{n_1} = (1, 2, -1)$ and $\vec{n_2} = (0, 1, 2)$, then we can choose $\vec{m} = \vec{n_1} \times \vec{n_2}$.

$$\begin{aligned} \text{Thus, } \vec{m} &= (1, 2, -1) \times (0, 1, 2) \\ &= (2(2) - (-1)(1), -1(0) - 1(2), 1(1) - 2(0)) \\ &= (5, -2, 1) \end{aligned}$$

Thus, the required line that passes through $P(5, -2, 3)$ and has direction vector $\vec{m} = (5, -2, 1)$ has parametric equations $x = 5 + 5t$, $y = -2 - 2t$, and $z = 3 + t$, $t \in \mathbf{R}$.

Method 2:
We start by finding the equation of the line of intersection between the two planes. In equation ②, if $z = t$, then $y = -2t + 1$ by substitution. Substituting these values into equation ① gives

$$\begin{aligned} x + 2(-2t + 1) - t &= 6 \\ x - 4t + 2 - t &= 6 \\ x &= 5t + 4 \end{aligned}$$

The line of intersection has $x = 5t + 4$, $y = -2t + 1$, and $z = t$ as its parametric equations $t \in \mathbf{R}$. Since the direction vector for this line is $(5, -2, 1)$, we can choose the direction vector for the required line to also be $(5, -2, 1)$.

The equation for the required line is $x = 5 + 5t$, $y = -2 - 2t$, $z = 3 + t$, $t \in \mathbf{R}$.

IN SUMMARY

Key Ideas

- A system of two (linear) equations in three unknowns geometrically represents two planes in R^3. These planes may intersect at zero points or an infinite number of points, depending on how the planes are related to each other.

  Case 1: Two planes can intersect along a line and will therefore have an infinite number of points of intersection.

  Case 2: Two planes can be parallel and non-coincident. In this case, there are no points of intersection.

  Case 3: Two planes can be coincident and will have an infinite number of points of intersection.

Need to Know

- If the normals of two planes are known, examining how these are related to each other provides information about how the two planes are related.

  If planes π_1 and π_2 have $\vec{n_1}$ and $\vec{n_2}$ as their respective normals, we know the following:

  1. If $\vec{n_1} = k\vec{n_2}$ for some scalar k, the planes are either coincident or they are parallel and non-coincident. If they are coincident, there are an infinite number of points of intersection, and if they are parallel and non-coincident, there are no points of intersection.
  2. If $\vec{n_1} \neq k\vec{n_2}$ for some scalar k, the two planes intersect in a line. This results in an infinite number of points of intersection.

Exercise 9.3

PART A

1. A system of two equations in three unknowns has been manipulated, and, after correctly using elementary operations, a student arrives at the following equivalent system of equations:

   ① $x - y + z = 1$
   ③ $0x + 0y + 0z = 3$

   a. Explain what this equivalent system means.
   b. Give an example of a system of equations that might lead to this solution.

2. A system of two equations in three unknowns has been manipulated, and, after correctly using elementary operations, a student arrives at the following equivalent system of equations:

   ① $2x - y + 2z = 1$
   ③ $0x + 0y + 0z = 0$

   a. Write a solution to this system of equations, and explain what your solution means.
   b. Give an example of a system of equations that leads to your solution in part a.

C 3. A system of two equations in three unknowns has been manipulated, and, after correctly using elementary operations, a student arrives at the following equivalent system of equations:

   ① $x - y + z = -1$
   ③ $0x + 0y + 2z = -4$

   a. Write a solution to this system of equations, and explain what your solution means.
   b. Give an example of a system of equations that leads to your solution in part a.

PART B

K 4. Consider the following system of equations:

   ① $2x + y + 6z = p$
   ② $x + my + 3z = q$

   a. Determine values of m, p, and q such that the two planes are coincident. Are these values unique? Explain.
   b. Determine values of m, p, and q such that the two planes are parallel and not coincident. Are these values unique? Explain.
   c. A value of m such that the two planes intersect at right angles. Is this value unique? Explain.
   d. Determine values of m, p, and q such that the two planes intersect at right angles. Are these values unique? Explain.

5. Consider the following system of equations:

① $x + 2y - 3z = 0$

② $y + 3z = 0$

a. Solve this system of equations by letting $z = s$.

b. Solve this system of equations by letting $y = t$.

c. Show that the solution you found in part a. is the same as the solution you found in part b.

6. The following systems of equations involve two planes. State whether the planes intersect, and, if they do intersect, specify if their intersection is a line or a plane.

a. ① $x + y + z = 1$
② $2x + 2y + 2z = 2$

b. ① $2x - y + z + 1 = 0$
② $2x - y + z + 2 = 0$

c. ① $x - y + 2z = 2$
② $x + y + 2z = -2$

d. ① $x + y + 2z = 4$
② $x - y = 6$

e. ① $2x - y + 2z = 2$
② $-x + 2y + z = 1$

f. ① $x - y + 2z = 0$
② $z = 4$

7. Determine the solution to each system of equations in question 6.

8. A system of equations is given as follows:

① $x + y + 2z = 1$

② $kx + 2y + 4z = k$

a. For what value of k does the system have an infinite number of solutions? Determine the solution to the system for this value of k.

b. Is there any value of k for which the system does not have a solution? Explain.

9. Determine the vector equation of the line that passes through $A(-2, 3, 6)$ and is parallel to the line of intersection of the planes π_1: $2x - y + z = 0$ and π_2: $y + 4z = 0$.

A 10. For the planes $2x - y + 2z = 0$ and $2x + y + 6z = 4$, show that their line of intersection lies on the plane with equation $5x + 3y + 16z - 11 = 0$.

T 11. The line of intersection of the planes π_1: $2x + y - 3z = 3$ and π_2: $x - 2y + z = -1$ is L.

a. Determine parametric equations for L.

b. If L meets the xy-plane at point A and the z-axis at point B, determine the length of line segment AB.

PART C

12. Determine the Cartesian equation of the plane that is parallel to the line with equation $x = -2y = 3z$ and that contains the line of intersection of the planes with equations $x - y + z = 1$ and $2y - z = 0$.