Section 9.4—The Intersection of Three Planes

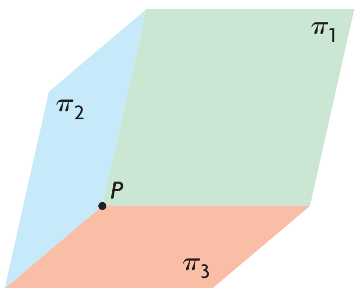
In the previous section, we discussed the intersection of two planes. In this section, we will extend these ideas and consider the intersection of three planes. Algebraically, the three planes are typically represented by a system of three linear equations in three unknowns.

First we will consider consistent systems. Later in the section, we will consider inconsistent systems.

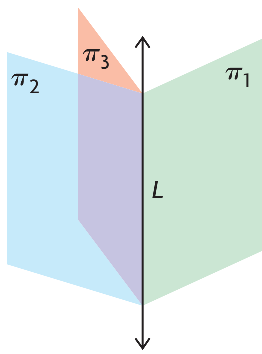
Consistent Systems for Three Equations Representing Three Planes

There are four cases that should be considered for the intersection of three planes. These four cases, which all result in one or more points of intersection between all three planes, are shown below.

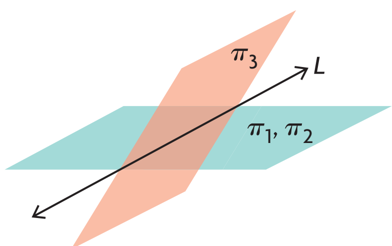
Case 1: P is the point of intersection of three planes.



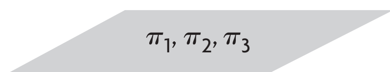
Case 2a: L is the line of intersection of three planes.



Case 2b: L is the line of intersection of two coincident planes and a third plane not parallel to the coincident planes.



Case 3: The plane of intersection of three coincident planes is the plane itself.



Possible Intersections for Three Planes

A description is given below for each situation represented in the diagram on the previous page.

Case 1: There is just one solution to the corresponding system of equations. This is a single point. The coordinates of the point of intersection will satisfy each of the three equations.

This case can be visualized by looking at the ceiling in a rectangular room. The point where the plane of the ceiling meets two walls represents the point of intersection of three planes. Although planes are not usually at right angles to each other and they extend infinitely far in all directions, this gives some idea of how planes can intersect at a point.

Case 2: There are an infinite number of solutions to the related system of equations. Geometrically, this corresponds to a line, and the solution is given in terms of one parameter. There are two sub-cases to consider.

Case 2a: The three planes intersect along a line and are mutually non-coincident.

Case 2b: Two planes are coincident, and the third plane cuts through these two planes intersecting along a line.

Case 3: Three planes are coincident, and there are an infinite number of solutions to the related system of equations. The number of solutions corresponds to the infinite number of points on a plane, and the solution is given in terms of two parameters. In this case, there are three coincident planes that have identical equations or can be reduced to three equivalent equations.

In the following examples, we use elementary operations to determine the solution of three equations in three unknowns.

EXAMPLE 1 **Using elementary operations to solve a system of three equations in three unknowns**

Determine the intersection of the three planes with the equations $x - y + z = -2$, $2x - y - 2z = -9$, and $3x + y - z = -2$.

Solution

For the intersection of the three planes, we must find the solution to the following system of equations:

① $x - y + z = -2$

② $2x - y - 2z = -9$

③ $3x + y - z = -2$

1: Create two new equations, ④ and ⑤, each containing an x-term with a coefficient of 0.

To determine the required intersection, we use elementary operations and solve the system of equations as shown earlier.

$$
\begin{array}{lll}
① & x - y + z = -2 & \\
④ & 0x + y - 4z = -5 & -2 \times ① + ② \\
⑤ & 0x + 4y - 4z = 4 & -3 \times ① + ③
\end{array}
$$

2: We create equation ⑥ by eliminating y from equation ⑤.

$$
\begin{array}{lll}
① & x - y + z = -2 & \\
④ & 0x + y - 4z = -5 & \\
⑥ & 0x + 0y + 12z = 24 & -4 \times ④ + ⑤
\end{array}
$$

This equivalent system can now be solved by first solving equation ⑥ for z.

Thus, $12z = 24$

$z = 2$

If we use the method of back substitution, we can substitute into equation ④ and solve for y.

Substituting into equation ④,

$y - 4(2) = -5$

$y = 3$

If we now substitute $y = 3$ and $z = 2$ into equation ①, we obtain the value of x.

$x - 3 + 2 = -2$

$x = -1$

Thus, the three planes intersect at the point with coordinates $(-1, 3, 2)$.

Check:
Substituting into equation ①, $x - y + z = -1 - 3 + 2 = -2$.

Substituting into equation ②, $2x - y - 2z = 2(-1) - 3 - 2(2) = -9$.

Substituting into equation ③, $3x + y - z = 3(-1) + 3 - 2 = -2$.

Checking each of the equations confirms the solution.

Two of the other possibilities involving consistent systems are demonstrated in the next two examples.

EXAMPLE 2 Selecting a strategy to determine the intersection of three planes

Determine the solution to the following system of equations:

① $2x - y + z = 1$

② $3x - 5y + 4z = 3$

③ $3x + 2y - z = 0$

Solution

In this situation, there is not a best way to solve the system. Because the coefficients of x for two of the equations are equal, however, the computation might be easier if we arrange them as follows (although, in situations like this, it is often a matter of individual preference).

① $3x + 2y - z = 0$ **(Interchange equations ① and ③)**

② $3x - 5y + 4z = 3$

③ $2x - y + z = 1$

Again, we try to create a zero for the coefficient of x in two of the equations.

Applying elementary operations gives the following system of equations:

① $3x + 2y - z = 0$

④ $0x - 7y + 5z = 3$ $\quad -1 \times$ ① + ②

⑤ $0x - \frac{7}{3}y + \frac{5}{3}z = 1$ $\quad -\frac{2}{3} \times$ ① + ③

Before proceeding with further computations, we should observe that equations ④ and ⑤ are scalar multiples of each other and that, if equation ⑤ is multiplied by 3, there will be two identical equations.

① $3x + 2y - z = 0$

④ $0x - 7y + 5z = 3$

⑥ $0x - 7y + 5z = 3$ $\quad 3 \times$ ⑤

By using elementary operations again, we create the following equivalent system:

① $3x + 2y - z = 0$

④ $0x - 7y + 5z = 3$

⑦ $0x + 0y + 0z = 0$ $\quad -1 \times$ ④ + ⑥

Equation ⑦, in conjunction with equations ① and ④, indicates that this system has an infinite number of solutions. To solve this system, we let $z = t$ and solve for y in equation ④.

Thus, $-7y = -5t + 3$. Dividing by -7, we get $y = \frac{5}{7}t - \frac{3}{7}$.

We determine the parametric equation for x by substituting in equation ①.

Substituting $z = t$ and $y = \frac{5}{7}t - \frac{3}{7}$ into equation ① gives

$$3x + 2\left(\frac{5}{7}t - \frac{3}{7}\right) - t = 0$$

$$3x + \frac{3}{7}t - \frac{6}{7} = 0$$

$$x = -\frac{1}{7}t + \frac{2}{7}$$

Therefore, the solution to this system is $x = -\frac{1}{7}t + \frac{2}{7}$, $y = \frac{5}{7}t - \frac{3}{7}$, and $z = t$.

To help simplify the verification, we will remove the fractions from the *direction numbers* of this line by multiplying them by 7. (Recall that we cannot multiply the points by 7.)

In simplified form, the solution to the system of equations is
$x = -t + \frac{2}{7}$, $y = 5t - \frac{3}{7}$, and $z = 7t$, $t \in \mathbf{R}$.

Check:
Substituting into equation ①,
$$3\left(-t + \frac{2}{7}\right) + 2\left(5t - \frac{3}{7}\right) - 7t = -3t + \frac{6}{7} + 10t - \frac{6}{7} - 7t = 0$$

Substituting into equation ②,
$$3\left(-t + \frac{2}{7}\right) - 5\left(5t - \frac{3}{7}\right) + 4(7t) = -3t + \frac{6}{7} - 25t + \frac{15}{7} + 28t = 3$$

Substituting into equation ③,
$$2\left(-t + \frac{2}{7}\right) - \left(5t - \frac{3}{7}\right) + 7t = -2t + \frac{4}{7} - 5t + \frac{3}{7} + 7t = 1$$

The solution to the system of equations is a line with parametric equations
$x = -t + \frac{2}{7}$, $y = 5t - \frac{3}{7}$, and $z = 7t$, $t \in \mathbf{R}$. This is a line that has direction vector $\vec{m} = (-1, 5, 7)$ and passes through the point $\left(\frac{2}{7}, -\frac{3}{7}, 0\right)$.

It is useful to note that the normals for these three planes are $\vec{n_1} = (3, 2, -1)$, $\vec{n_2} = (3, -5, 4)$, and $\vec{n_3} = (2, -1, 1)$. Because none of these normals are collinear, this situation corresponds to Case 2a.

EXAMPLE 3 More on solving a consistent system of equations

Determine the solution to the following system of equations:

① $2x + y + z = 1$

② $4x - y - z = 5$

③ $8x - 2y - 2z = 10$

Solution

Again, using elementary operations,

① $2x + y + z = 1$

④ $0x - 3y - 3z = 3 \qquad -2 \times ① + ②$

⑤ $0x - 6y - 6z = 6 \qquad -4 \times ① + ③$

Continuing, we obtain

① $2x + y + z = 1$

④ $0x - 3y - 3z = 3$

⑥ $0x + 0y + 0z = 0 \qquad -2 \times ④ + ⑤$

Equation ⑥ indicates that this system has an infinite number of solutions.

We can solve this system by using a parameter for either y or z.

Substituting $y = s$ in equation ④ gives $-3s - 3z = 3$ or $z = -s - 1$.

Substituting into equation ①, $2x + s + (-s - 1) = 1$ or $x = 1$.

Therefore, the solution to this system is $x = 1$, $y = s$, and $z = -s - 1$, $s \in \mathbf{R}$.

Check:
Substituting into equation ①, $2(1) + s + (-s - 1) = 2 + s - s - 1 = 1$.

Substituting into equation ②, $4(1) - s - (-s - 1) = 4 - s + s + 1 = 5$.

There is no need to check in our third equation since $2 \times$ ② = ③. Equation ③ represents the same plane as equation ②.

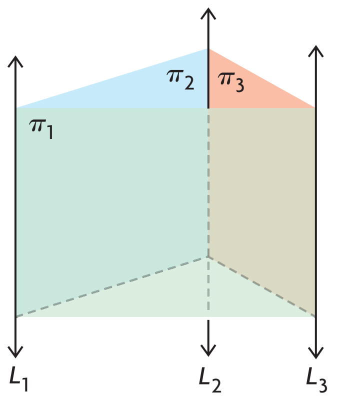
It is worth noting that the normals of the second and third planes, $\vec{n_2} = (4, -1, -1)$ and $\vec{n_3} = (8, -2, -2)$, are scalar multiples of each other, and that the constants on the right-hand side are related by the same factor. This indicates that the two equations represent the same plane. Since neither of these normals and the first plane's normal $\vec{n_1} = (2, 1, 1)$ are scalar multiples of each other, the first plane must intersect the two coincident planes along a line passing through the point $(1, 0, -1)$ with direction vector $\vec{m} = (0, 1, -1)$. This corresponds to Case 2b.

Inconsistent Systems for Three Equations Representing Three Planes

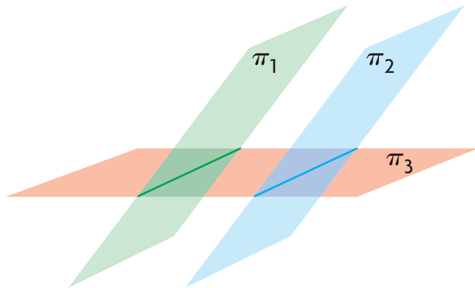
There are four cases to consider for inconsistent systems of equations that represent three planes.

The four cases, which all result in no points of intersection between all three planes, are shown below.

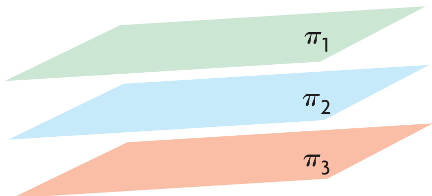
Case 1: A triangular prism is formed by three parallel lines.



Case 2: Two parallel planes intersect a third plane.



Case 3: Three planes are parallel; none are coincident.



Case 4: Two planes are coincident; a third plane is parallel to and non-coincident with the first two planes.

Non-intersections for Three Planes

A description of each case above is given below.

Case 1: Three planes (π_1, π_2, and π_3) form a triangular prism as shown. This means that, if you consider any two of the three planes, they intersect in a line and each of these three lines is parallel to the others. In the diagram, the lines L_1, L_2, and L_3 represent the lines of intersection between the three pairs of planes, and these lines have direction vectors that are identical to, or scalar multiples of, each other.

Even though the planes intersect in a pair-wise fashion, there is no common intersection between all three of the planes.

As well, the normals of the three planes are not scalar multiples of each other, and the system is inconsistent. The only geometric possibility is that the planes form a triangular prism. This idea is discussed in Example 4.

Case 2: We consider two parallel planes, each intersecting a third plane. Each of the parallel planes has a line of intersection with the third plane, but there is no intersection between all three planes.

Cases 3 and 4: In these two cases, which again implies that all three planes do not have any points of intersection.

EXAMPLE 4 **Selecting a strategy to solve an inconsistent system of equations**

Determine the solution to the following system of equations:

① $x - y + z = 1$

② $x + y + 2z = 2$

③ $x - 5y - z = 1$

Solution

Applying elementary operations to this system, the following system is obtained:

① $x - y + z = 1$

④ $0x + 2y + z = 1$ $\quad -1 \times$ ① + ②

⑤ $0x - 4y - 2z = 0$ $\quad -1 \times$ ① + ③

At this point, it can be observed that there is an inconsistency between equations ④ and ⑤. If equation ④ is multiplied by -2, it becomes $0x - 4y - 2z = -2$, which is inconsistent with equation ⑤ $(0x - 4y - 2z = 0)$. This implies that there is no solution to the system of equations. It is instructive, however, to continue using elementary operations and observe the results.

① $x - y + z = 1$

④ $0x + 2y + z = 1$

⑥ $0x + 0y + 0z = 2$ $\quad 2 \times$ ④ + ⑤

Equation ⑥, $0x + 0y + 0z = 2$, tells us there is no solution to the system, because there are no values of x, y, and z that satisfy this equation. The system is inconsistent.

If we use the normals for these three equations, we can calculate direction vectors for each pair of intersections. The normals for the three planes are $\overrightarrow{n_1} = (1, -1, 1)$, $\overrightarrow{n_2} = (1, 1, 2)$, and $\overrightarrow{n_3} = (1, -5, -1)$.
Let $\overrightarrow{m_1}$ be a direction vector for the line of intersection between π_1 and π_2.
Let $\overrightarrow{m_2}$ be a direction vector for the line of intersection between π_1 and π_3.
Let $\overrightarrow{m_3}$ be a direction vector for the line of intersection between π_2 and π_3.

Therefore, we can choose

$$\begin{aligned}\overrightarrow{m_1} &= \overrightarrow{n_1} \times \overrightarrow{n_2}\\ &= (1, -1, 1) \times (1, 1, 2)\\ &= (-1(2) - 1(1), 1(1) - 1(2), 1(1) - (-1)(1))\\ &= (-3, -1, 2)\end{aligned}$$

$$\begin{aligned}\overrightarrow{m_2} &= \overrightarrow{n_1} \times \overrightarrow{n_3}\\ &= (1, -1, 1) \times (1, -5, -1)\\ &= (-1(-1) - 1(-5), 1(1) - 1(-1), 1(-5) - (-1)(1))\\ &= -2(-3, -1, 2)\end{aligned}$$

$$\begin{aligned}\overrightarrow{m_3} &= \overrightarrow{n_2} \times \overrightarrow{n_3}\\ &= (1, 1, 2) \times (1, -5, -1)\\ &= (1(-1) - 2(-5), 2(1) - 1(-1), 1(-5) - 1(1))\\ &= -3(-3, -1, 2)\end{aligned}$$

We can see from our calculations that the system of equations corresponds to Case 1 for systems of inconsistent equations (triangular prism).

This conclusion could have been anticipated without doing any calculations. We have shown that the system of equations is inconsistent, and, because the normals are not scalar multiples of each other, we can reach the same conclusion.

In the following example, we deal with another inconsistent system.

EXAMPLE 5 Reasoning about an inconsistent system of equations

Determine the solution to the following system of equations:

① $x - y + 2z = -1$

② $x - y + 2z = 3$

③ $x - 3y + z = 0$

Solution

Using elementary operations,

① $x - y + 2z = -1$

④ $0x + 0y + 0z = 4$ $\quad -1 \times$ ① + ②

③ $x - 3y + z = 0$

It is only necessary to use elementary operations once, and we obtain equation ④. As before, we create an equivalent system of equations that does not have a solution, implying that the original system has no solution.

It should be noted that it is not necessary to use elementary operations in this example. Because equations ① and ② are the equations of non-coincident parallel planes, no intersection is possible. This corresponds to Case 2 for systems of inconsistent equations, since the third plane is not parallel to the first two.

EXAMPLE 6 Identifying coincident and parallel planes in an inconsistent system

Solve the following system of equations:

① $x + y + z = 5$

② $x + y + z = 4$

③ $x + y + z = 5$

Solution

It is clear, from observation, that this system of equations is inconsistent. Equations ① and ③ represent the same plane, and equation ② represents a plane that is parallel to, but different from, the other plane. This corresponds to Case 4 for systems of inconsistent equations, so there are no solutions.

IN SUMMARY

Key Idea

- A system of three (linear) equations in three unknowns geometrically represents three planes in R^3. These planes may intersect at zero, one, or an infinite number of points, depending on how the planes are related to each other.

Need to Know

- Consistent Systems for Three Equations Representing Three Planes

  Case 1 (one solution): There is a single point.

  Case 2 (infinite number of solutions): The solution uses one parameter.

  Case 2a: The three planes intersect along a line.

  Case 2b: Two planes are coincident, and the third plane cuts through these two planes.

  Case 3 (infinite number of solutions): The solution uses two parameters. There are three planes that have identical equations (after reducing the equations) that coincide with one another.

- Inconsistent Systems for Three Equations Representing Three Planes (No Intersection)

  Case 1: Three planes (π_1, π_2, and π_3) form a triangular prism.

  Case 2: Two non-coincident parallel planes each intersect a third plane.

  Case 3: The three planes are parallel and non-coincident.

  Case 4: Two planes are coincident and parallel to the third plane.

Exercise 9.4

PART A

1. A student is manipulating a system of equations and obtains the following equivalent system:

   ① $x - 3y + z = 2$

   ② $0x + y - z = -1$

   ③ $0x + 0y + 3z = -12$

   a. Determine the solution to this system of equations.

   b. How would your solution be interpreted geometrically?

2. When manipulating a system of equations, a student obtains the following equivalent system:

   ① $x - y + z = 4$

   ② $0x + 0y + 0z = 0$

   ③ $0x + 0y + 0z = 0$

   a. Give a system of equations that would produce this equivalent system.

   b. How would you interpret the solution to this system geometrically?

   c. Write the solution to this system using parameters for x and y.

   d. Write the solution to this system using parameters for y and z.

3. When manipulating a system of equations, a student obtains the following equivalent system:

   ① $2x - y + 3z = -2$

   ② $x - y + 4z = 3$

   ③ $0x + 0y + 0z = 1$

   a. Give *two* systems of equations that could have produced this result.

   b. What does this equivalent system tell you about possible solutions for the original system of equations?

4. When manipulating a system of equations, a student obtains the following equivalent system:

   ① $x + 2y - z = 4$

   ② $x + 0y - 2z = 0$

   ③ $2x + 0y + 0z = -6$

   a. Without using any further elementary operations, determine the solution to this system.

   b. How can the solution to this system be interpreted geometrically?

PART B

5. a. Without solving the following system, how can you deduce that these three planes must intersect in a line?

   ① $2x - y + z = 1$

   ② $x + y - z = -1$

   ③ $-3x - 3y + 3z = 3$

   b. Find the solution to the given system using elementary operations.

C 6. Explain why there is no solution to the following system of equations:

   ① $2x + 3y - 4z = -5$

   ② $x - y + 3z = -201$

   ③ $5x - 5y + 15z = -1004$

7. Avery is solving a system of equations using elementary operations and derives, as one of the equations, $0x + 0y + 0z = 0$.

   a. Is it true that this equation will always have a solution? Explain.

   b. Construct your own system of equations in which the equation $0x + 0y + 0z = 0$ appears, but for which there is no solution to the constructed system of equations.

K 8. Solve the following systems of equations using elementary operations. Interpret your results geometrically.

   a. ① $2x + y - z = -3$

   ② $x - y + 2z = 0$

   ③ $3x + 2y - z = -5$

   b. ① $\frac{x}{3} - \frac{y}{4} + z = \frac{7}{8}$

   ② $2x + 2y - 3z = -20$

   ③ $x - 2y + 3z = 2$

   c. ① $x - y = -199$

   ② $x + z = -200$

   ③ $y - z = 201$

   d. ① $x - y - z = -1$

   ② $y - 2 = 0$

   ③ $x + 1 = 5$

9. Solve each system of equations using elementary operations. Interpret your results geometrically.

a. ① $x - 2y + z = 3$
② $2x + 3y - z = -9$
③ $5x - 3y + 2z = 0$

b. ① $x - 2y + z = 3$
② $x + y + z = 2$
③ $x - 3y + z = -6$

c. ① $x - y + z = -2$
② $x + y + z = 2$
③ $x - 3y + z = -6$

10. Determine the solution to each system.

a. ① $x - y + z = 2$
② $2x - 2y + 2z = 4$
③ $x + y - z = -2$

b. ① $2x - y + 3z = 0$
② $4x - 2y + 6z = 0$
③ $-2x + y - 3z = 0$

A

11. a. Use elementary operations to show that the following system does not have a solution:
① $x + y + z = 1$
② $x - 2y + z = 0$
③ $x - y + z = 0$

b. Calculate the direction vectors for the lines of intersection between each pair of planes, as shown in Example 4.

c. Explain, in your own words, why the planes represented in this system of equations must correspond to a triangular prism.

d. Explain how the same conclusion could have been reached without doing the calculations in part b.

12. Each of the following systems does not have a solution. Explain why.

a. ① $x - y + 3z = 3$
② $x - y + 3z = 6$
③ $3x - 5z = 0$

b. ① $5x - 2y + 3z = 1$
② $5x - 2y + 3z = -1$
③ $5x - 2y + 3z = 13$

c. ① $x - y + z = 9$
② $2x - 2y + 2z = 18$
③ $2x - 2y + 2z = 17$

d. ① $3x - 2y + z = 4$
② $9x - 6y + 3z = 12$
③ $6x - 4y + 2z = 5$

13. Determine the solution to each system of equations, if a solution exists.

a. ① $2x - y - z = 10$
② $x + y + 0z = 7$
③ $0x + y - z = 8$

b. ① $2x - y + z = -3$
② $x + y - 2z = 1$
③ $5x + 2y - 5z = 0$

c. ① $x + y - z = 0$
② $2x - y + z = 0$
③ $4x - 5y + 5z = 0$

d. ① $x - 10y + 13z = -4$
② $2x - 20y + 26z = -8$
③ $x - 10y + 13z = -8$

e. ① $x - y + z = -2$
② $x + y + z = 2$
③ $3x + y + 3z = 2$

f. ① $x + y + z = 0$
② $x - 2y + 3z = 0$
③ $2x - y + 3z = 0$

PART C

14. The following system of equations represents three planes that intersect in a line:

① $2x + y + z = 4$
② $x - y + z = p$
③ $4x + qy + z = 2$

a. Determine p and q.
b. Determine an equation in parametric form for the line of intersection.

T 15. Consider the following system of equations:

① $4x + 3y + 3z = -8$
② $2x + y + z = -4$
③ $3x - 2y + (m^2 - 6)z = m - 4$

Determine the value(s) of m for which this system of equations will have

a. no solution
b. one solution
c. an infinite number of solutions

16. Determine the solution to the following system of equations:

① $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} = 0$
② $\frac{2}{a} + \frac{3}{b} + \frac{2}{c} = \frac{13}{6}$
③ $\frac{4}{a} - \frac{2}{b} + \frac{3}{c} = \frac{5}{2}$