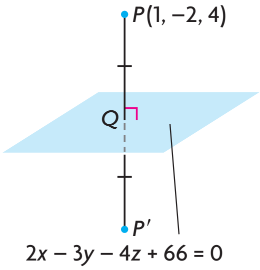
Cumulative Review of Vectors

1. For the vectors $\vec{a} = (2, -1, -2)$ and $\vec{b} = (3, -4, 12)$, determine the following:
   a. the angle between the two vectors
   b. the scalar and vector projections of $\vec{a}$ on $\vec{b}$
   c. the scalar and vector projections of $\vec{b}$ on $\vec{a}$
2. a. Determine the line of intersection between $\pi_1: 4x + 2y + 6z - 14 = 0$ and $\pi_2: x - y + z - 5 = 0$.
   b. Determine the angle between the two planes.
3. If $\vec{x}$ and $\vec{y}$ are unit vectors, and the angle between them is 60°, determine the value of each of the following:
   a. $|\vec{x} \cdot \vec{y}|$
   b. $|2\vec{x} \cdot 3\vec{y}|$
   c. $|(2\vec{x} - \vec{y}) \cdot (\vec{x} + 3\vec{y})|$
4. Expand and simplify each of the following, where $\vec{i}$, $\vec{j}$, and $\vec{k}$ represent the standard basis vectors in R^3:
   a. $2(\vec{i} - 2\vec{j} + 3\vec{k}) - 4(2\vec{i} + 4\vec{j} + 5\vec{k}) - (\vec{i} - \vec{j})$
   b. $-2(3\vec{i} - 4\vec{j} - 5\vec{k}) \cdot (2\vec{i} + 3\vec{k}) + 2\vec{i} \cdot (3\vec{j} - 2\vec{k})$
5. Determine the angle that the vector $\vec{a} = (4, -2, -3)$ makes with the positive x-axis, y-axis, and z-axis.
6. If $\vec{a} = (1, -2, 3)$, $\vec{b} = (-1, 1, 2)$, and $\vec{c} = (3, -4, -1)$, determine each of the following:
   a. $\vec{a} \times \vec{b}$
   b. $2\vec{a} \times 3\vec{b}$
   c. the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$
   d. $\vec{c} \cdot (\vec{b} \times \vec{a})$
7. Determine the coordinates of the unit vector that is perpendicular to $\vec{a} = (1, -1, 1)$ and $\vec{b} = (2, -2, 3)$.
8. a. Determine vector and parametric equations for the line that contains $A(2, -3, 1)$ and $B(1, 2, 3)$.
   b. Verify that $C(4, -13, -3)$ is on the line that contains A and B.
9. Show that the lines $L_1: \vec{r} = (2, 0, 9) + t(-1, 5, 2)$, $t \in \mathbf{R}$, and $L_2: x - 3 = \frac{y + 5}{-5} = \frac{z - 10}{-2}$ are parallel and distinct.
10. Determine vector and parametric equations for the line that passes through $(0, 0, 4)$ and is parallel to the line with parametric equations $x = 1$, $y = 2 + t$, and $z = -3 + t$, $t \in \mathbf{R}$.
11. Determine the value of c such that the plane with equation $2x + 3y + cz - 8 = 0$ is parallel to the line with equation $\frac{x - 1}{2} = \frac{y - 2}{3} = z + 1$.

12. Determine the intersection of the line $\frac{x-2}{3} = y + 5 = \frac{z-3}{5}$ with the plane $5x + y - 2z + 2 = 0$.

13. Sketch the following planes, and give two direction vectors for each.

    a. $x + 2y + 2z - 6 = 0$ b. $2x - 3y = 0$ c. $3x - 2y + z = 0$



14. If $P(1, -2, 4)$ is reflected in the plane with equation $2x - 3y - 4z + 66 = 0$, determine the coordinates of its image point, P'. (Note that the plane $2x - 3y - 4z + 66 = 0$ is the right bisector of the line joining $P(1, -2, 4)$ with its image.)

15. Determine the equation of the line that passes through the point $A(1, 0, 2)$ and intersects the line $\vec{r} = (-2, 3, 4) + s(1, 1, 2)$, $s \in \mathbf{R}$, at a right angle.

16. a. Determine the equation of the plane that passes through the points $A(1, 2, 3)$, $B(-2, 0, 0)$, and $C(1, 4, 0)$.

    b. Determine the distance from $O(0, 0, 0)$ to this plane.

17. Determine a Cartesian equation for each of the following planes:

    a. the plane through the point $A(-1, 2, 5)$ with $\vec{n} = (3, -5, 4)$

    b. the plane through the point $K(4, 1, 2)$ and perpendicular to the line joining the points $(2, 1, 8)$ and $(1, 2, -4)$

    c. the plane through the point $(3, -1, 3)$ and perpendicular to the z-axis

    d. the plane through the points $(3, 1, -2)$ and $(1, 3, -1)$ and parallel to the y-axis

18. An airplane heads due north with a velocity of 400 km/h and encounters a wind of 100 km/h from the northeast. Determine the resultant velocity of the airplane.

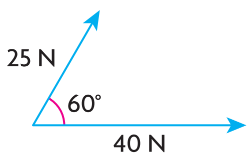
19. a. Determine a vector equation for the plane with Cartesian equation $3x - 2y + z - 6 = 0$, and verify that your vector equation is correct.

    b. Using coordinate axes you construct yourself, sketch this plane.

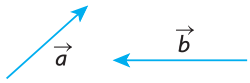
20. a. A line with equation $\vec{r} = (1, 0, -2) + s(2, -1, 2)$, $s \in \mathbf{R}$, intersects the plane $x + 2y + z = 2$ at an angle of θ degrees. Determine this angle to the nearest degree.

    b. Show that the planes π_1: $2x - 3y + z - 1 = 0$ and π_2: $4x - 3y - 17z = 0$ are perpendicular.

    c. Show that the planes π_3: $2x - 3y + 2z - 1 = 0$ and π_4: $2x - 3y + 2z - 3 = 0$ are parallel but not coincident.



21. Two forces, 25 N and 40 N, have an angle of 60° between them. Determine the resultant and equilibrant of these two vectors.



22. You are given the vectors $\vec{a}$ and $\vec{b}$, as shown at the left.
    a. Sketch $\vec{a} - \vec{b}$.
    b. Sketch $2\vec{a} + \frac{1}{2}\vec{b}$.
23. If $\vec{a} = (6, 2, -3)$, determine the following:
    a. the coordinates of a unit vector in the same direction as $\vec{a}$
    b. the coordinates of a unit vector in the opposite direction to $\vec{a}$
24. A parallelogram $OBCD$ has one vertex at $O(0, 0)$ and two of its remaining three vertices at $B(-1, 7)$ and $D(9, 2)$.
    a. Determine a vector that is equivalent to each of the two diagonals.
    b. Determine the angle between these diagonals.
    c. Determine the angle between $\overrightarrow{OB}$ and $\overrightarrow{OD}$.
25. Solve the following systems of equations:
    a. ① $x - y + z = 2$
       ② $-x + y + 2z = 1$
       ③ $x - y + 4z = 5$
    b. ① $-2x - 3y + z = -11$
       ② $x + 2y + z = 2$
       ③ $-x - y + 3z = -12$
    c. ① $2x - y + z = -1$
       ② $4x - 2y + 2z = -2$
       ③ $2x + y - z = 5$
    d. ① $x - y - 3z = 1$
       ② $2x - 2y - 6z = 2$
       ③ $-4x + 4y + 12z = -4$
26. State whether each of the following pairs of planes intersect. If the planes do intersect, determine the equation of their line of intersection.
    a. $x - y + z - 1 = 0$
       $x + 2y - 2z + 2 = 0$
    b. $x - 4y + 7z = 28$
       $2x - 8y + 14z = 60$
    c. $x - y + z - 2 = 0$
       $2x + y + z - 4 = 0$
27. Determine the angle between the line with symmetric equations $x = -y$, $z = 4$ and the plane $2x - 2z = 5$.
28. a. If $\vec{a}$ and $\vec{b}$ are unit vectors, and the angle between them is 60°, calculate $(6\vec{a} + \vec{b}) \cdot (\vec{a} - 2\vec{b})$.
    b. Calculate the dot product of $4\vec{x} - \vec{y}$ and $2\vec{x} + 3\vec{y}$ if $|\vec{x}| = 3$, $|\vec{y}| = 4$, and the angle between $\vec{x}$ and $\vec{y}$ is 60°.

29. A line that passes through the origin is perpendicular to a plane π and intersects the plane at $(-1, 3, 1)$. Determine an equation for this line and the cartesian equation of the plane.

30. The point $P(-1, 0, 1)$ is reflected in the plane π: $y - z = 0$ and has P' as its image. Determine the coordinates of the point P'.

31. A river is 2 km wide and flows at 4 km/h. A motorboat that has a speed of 10 km/h in still water heads out from one bank, which is perpendicular to the current. A marina lies directly across the river, on the opposite bank.
    a. How far downstream from the marina will the motorboat touch the other bank?
    b. How long will it take for the motorboat to reach the other bank?

32. a. Determine the equation of the line passing through $A(2, -1, 3)$ and $B(6, 3, 4)$.
    b. Does the line you found lie on the plane with equation $x - 2y + 4z - 16 = 0$? Justify your answer.

33. A sailboat is acted upon by a water current and the wind. The velocity of the wind is 16 km/h from the west, and the velocity of the current is 12 km/h from the south. Find the resultant of these two velocities.

34. A crate has a mass of 400 kg and is sitting on an inclined plane that makes an angle of 30° with the level ground. Determine the components of the *weight* of the mass, perpendicular and parallel to the plane. (Assume that a 1 kg mass exerts a force of 9.8 N.)

35. State whether each of the following is true or false. Justify your answer.
    a. Any two non-parallel lines in R^2 must always intersect at a point.
    b. Any two non-parallel planes in R^3 must always intersect on a line.
    c. The line with equation $x = y = z$ will always intersect the plane with equation $x - 2y + 2z = k$, regardless of the value of k.
    d. The lines $\frac{x}{2} = y - 1 = \frac{z+1}{2}$ and $\frac{x-1}{-4} = \frac{y-1}{-2} = \frac{z+1}{-2}$ are parallel.

36. Consider the lines L_1: $x = 2, \frac{y-2}{3} = z$ and L_2: $x = y + k = \frac{z+14}{k}$.
    a. Explain why these lines can never be parallel, regardless of the value of k.
    b. Determine the value of k that makes these two lines intersect at a single point, and find the actual point of intersection.