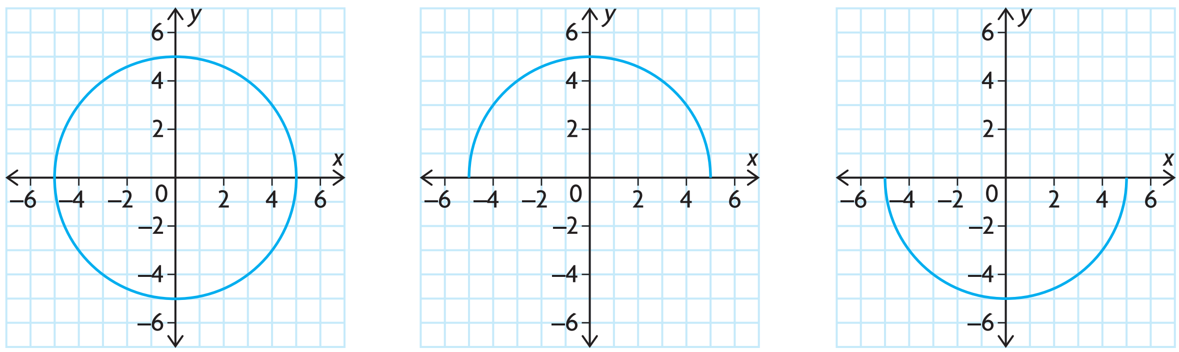
Calculus Appendix

Implicit Differentiation

In Chapters 1 to 5, most functions were written in the form $y = f(x)$, in which y was defined explicitly as a function of x, such as $y = x^3 - 4x$ and $y = \frac{7}{x^2 + 1}$. In these equations, y is isolated on one side and is expressed explicitly as a function of x.

Functions can also be defined implicitly by relations, such as the circle $x^2 + y^2 = 25$. In this case, the dependent variable, y, is not isolated or explicitly defined in terms of the independent variable, x. Since there are x-values that correspond to two y-values, y is not a function of x on the entire circle. Solving for y gives $y = \pm\sqrt{25 - x^2}$, where $y = \sqrt{25 - x^2}$ represents the upper semicircle and $y = -\sqrt{25 - x^2}$ represents the lower semicircle. The given relation defines two different functions of x.



$x^2 + y^2 = 25$ $\qquad$ $y = \sqrt{25 - x^2}$ $\qquad$ $y = -\sqrt{25 - x^2}$

Consider the problem of determining the slope of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$. Since this point lies on the lower semicircle, we could differentiate the function $y = -\sqrt{25 - x^2}$ and substitute $x = 3$. An alternative, which avoids having to solve for y explicitly in terms of x, is to use the method of **implicit differentiation**. Example 1 illustrates this method.

EXAMPLE 1 Selecting a strategy to differentiate an implicit relation

a. If $x^2 + y^2 = 25$, determine $\frac{dy}{dx}$.

b. Determine the slope of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution

a. Differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

To determine $\frac{d}{dx}(y^2)$, use the chain rule, since y is a function of x.

$$\frac{d}{dx}(y^2) = \frac{d(y^2)}{dy} \times \frac{dy}{dx}$$

$$= 2y\frac{dy}{dx}$$

So, $\frac{d}{dx}(x^2) + \frac{d(y^2)}{dy} \times \frac{dy}{dx} = \frac{d}{dx}(25)$ (Substitute)

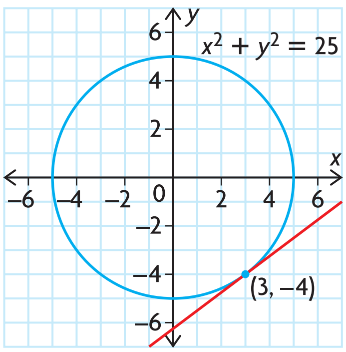
$$2x + 2y\frac{dy}{dx} = 0$$ $\left(\text{Solve for } \frac{dy}{dx}\right)$

$$\frac{dy}{dx} = -\frac{x}{y}$$

b. The derivative in part a. depends on both x and y. With the derivative in this form, we need to substitute values for both variables.

At the point $(3, -4)$, $x = 3$ and $y = -4$.

The slope of the tangent line to $x^2 + y^2 = 25$ at $(3, -4)$ is $\frac{dy}{dx} = -\left(\frac{3}{-4}\right) = \frac{3}{4}$.



In Example 1, the derivative could be determined either by using implicit differentiation or by solving for y in terms of x and using one of the methods introduced earlier in the text. There are many situations in which solving for y in terms of x is very difficult and, in some cases, impossible. In such cases, implicit differentiation is the only algebraic method available to us.

EXAMPLE 2 Using implicit differentiation to determine the derivative

Determine $\frac{dy}{dx}$ for $2xy - y^3 = 4$.

Solution

Differentiate both sides of the equation with respect to x as follows:

$$\frac{d}{dx}(2xy) - \frac{d}{dx}(y^3) = \frac{d}{dx}(4)$$

Use the product rule to differentiate the first term and the chain rule to differentiate the second term.

$$\left[\left(\frac{d}{dx}(2x)\right)y + 2x\frac{dy}{dx}\right] - \frac{d(y^3)}{dy} \times \frac{dy}{dx} = \frac{d}{dx}(4)$$

$$2y + 2x\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0 \qquad \text{(Rearrange and factor)}$$

$$(2x - 3y^2)\frac{dy}{dx} = -2y \qquad \left(\text{Solve for } \frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = -\frac{2y}{2x - 3y^2}$$

Procedure for Implicit Differentiation

If an equation defines y implicitly as a differentiable function of x, determine $\frac{dy}{dx}$ as follows:

1: Differentiate both sides of the equation with respect to x. Remember to use the chain rule when differentiating terms containing y.

2: Solve for $\frac{dy}{dx}$.

Note that implicit differentiation leads to a derivative expression that usually includes terms with both x and y. The derivative is defined at a specific point on the original function if, after substituting the x and y coordinates of the point, the value of the denominator is nonzero.

Exercise

PART A

1. State the chain rule. Outline a procedure for implicit differentiation.
2. Determine $\frac{dy}{dx}$ for each of the following in terms of x and y, using implicit differentiation:
   a. $x^2 + y^2 = 36$
   b. $15y^2 = 2x^3$
   c. $3xy^2 + y^3 = 8$
   d. $9x^2 - 16y^2 = -144$
   e. $\frac{x^2}{16} + \frac{3y^2}{13} = 1$
   f. $x^2 + y^2 + 5y = 10$
3. For each relation, determine the equation of the tangent at the given point.
   a. $x^2 + y^2 = 13, (2, -3)$
   b. $x^2 + 4y^2 = 100, (-8, 3)$
   c. $\frac{x^2}{25} - \frac{y^2}{36} = -1, (5\sqrt{3}, -12)$
   d. $\frac{x^2}{81} - \frac{5y^2}{162} = 1, (-11, -4)$

PART B

4. At what point is the tangent to the curve $x + y^2 = 1$ parallel to the line $x + 2y = 0$?
5. The equation $5x^2 - 6xy + 5y^2 = 16$ represents an ellipse.
   a. Determine $\frac{dy}{dx}$ at $(1, -1)$.
   b. Determine two points on the ellipse at which the tangent is horizontal.
6. Determine the slope of the tangent to the ellipse $5x^2 + y^2 = 21$ at the point $A(-2, -1)$.
7. Determine the equation of the normal to the curve $x^3 + y^3 - 3xy = 17$ at the point (2, 3).
8. Determine the equation of the normal to $y^2 = \frac{x^3}{2 - x}$ at the point $(1, -1)$.
9. Determine $\frac{dy}{dx}$.
   a. $(x + y)^3 = 12x$
   b. $\sqrt{x + y} - 2x = 1$
10. The equation $4x^2y - 3y = x^3$ implicitly defines y as a function of x.
    a. Use implicit differentiation to determine $\frac{dy}{dx}$.
    b. Write y as an explicit function of x, and compute $\frac{dy}{dx}$ directly.
    c. Show that your results for parts a. and b. are equivalent.
11. Graph each relation using graphing technology. For each graph, determine the number of tangents that exist when $x = 1$.
    a. $y = \sqrt{3 - x}$
    b. $y = -\sqrt{5 - x}$
    c. $y = x^7 - x$
    d. $x^3 + 4x^2 + (x - 4)y^2 = 0$ (This curve is known as the strophoid.)

PART C

12. Show that $\frac{dy}{dx} = \frac{y}{x}$ for the relation $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = 10, x, y \neq 0$.
13. Determine the equations of the lines that are tangent to the ellipse $x^2 + 4y^2 = 16$ and also pass through the point (4, 6).
14. The angle between two intersecting curves is defined as the angle between their tangents at the point of intersection. If this angle is 90°, the two curves are said to be orthogonal at this point.
    Prove that the curves defined by $x^2 - y^2 = k$ and $xy = p$ intersect orthogonally for all values of the constants k and p. Illustrate your proof with a sketch.
15. Let l be any tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{k}$, where k is a constant. Show that the sum of the intercepts of l is k.
16. Two circles of radius $3\sqrt{2}$ are tangent to the graph $y^2 = 4x$ at the point (1, 2). Determine the equations of these two circles.