Related Rates

Oil that is spilled from a tanker spreads in a circle. The area of the circle increases at a constant rate of 6 km^2/h. How fast is the radius of the spill increasing when the area is 9π km^2? Knowing the rate of increase of the radius is important for planning the containment operation.

In this section, you will encounter some interesting problems that will help you understand the applications of derivatives and how they can be used to describe and predict the phenomena of change. In many practical applications, several quantities vary in relation to one another. The rates at which they vary are also related to one another. With calculus, we can describe and calculate such rates.

EXAMPLE 1 Solving a related rate problem involving a circular model

When a raindrop falls into a still puddle, it creates a circular ripple that spreads out from the point where the raindrop hit. The radius of the circle grows at a rate of 3 cm/s.

a. Determine the rate of increase of the circumference of the circle with respect to time.

b. Determine the rate of increase of the area of the circle when its area is 81π cm^2.

Solution

The radius, r, and the circumference of a circle, C, are related by the formula $C = 2\pi r$.

The radius, r, and the area of a circle, A, are related by the formula $A = \pi r^2$.

We are given $\frac{dr}{dt} = 3$ at any time t.

a. To determine $\frac{dC}{dt}$ at any time, it is necessary to differentiate the equation $C = 2\pi r$ with respect to t, using the chain rule.

$$\frac{dC}{dt} = \frac{dC}{dr}\frac{dr}{dt}$$

$$\frac{dC}{dt} = 2\pi\frac{dr}{dt}$$

At time t, since $\frac{dr}{dt} = 3$,

$$\frac{dC}{dt} = 2\pi(3)$$

$$= 6\pi$$

Therefore, the circumference is increasing at a constant rate of 6π cm/s.

b. To determine $\frac{dA}{dt}$, differentiate $A = \pi r^2$ with respect to t, using the chain rule.

$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

We know that $\frac{dr}{dt} = 3$, so we need to determine r.

Since $A = 81\pi$ and $A = \pi r^2$,

$$\pi r^2 = 81\pi$$

$$r^2 = 81$$

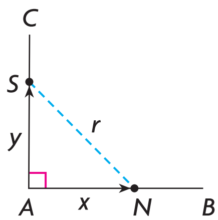
$$r = 9,\ r > 0, \text{ and } \frac{dA}{dt} = 2\pi(9)(3)$$

$$= 54\pi$$

The area of the circle is increasing at a rate of 54π cm^2/s at the given instant.

Many related-rate problems involve right triangles and the Pythagorean theorem. In these problems, the lengths of the sides of the triangle vary with time. The lengths of the sides and the related rates can be represented quite simply on the Cartesian plane.

EXAMPLE 2 Solving a related rate problem involving a right triangle model



Natalie and Shannon start from point A and drive along perpendicular roads AB and AC, respectively, as shown. Natalie drives at a speed of 45 km/h, and Shannon travels at a speed of 40 km/h. If Shannon begins 1 h before Natalie, at what rate are their cars separating 3 h after Shannon leaves?

Solution

Let x represent the distance that Natalie's car has travelled along AB, and let y represent the distance that Shannon's car has travelled along AC.

Therefore, $\frac{dx}{dt} = 45$ and $\frac{dy}{dt} = 40$, where t is the time in hours. (Note that both of these rates of change are positive since both distances, x and y, are increasing with time.)

Let r represent the distance between the two cars at time t.

Therefore, $x^2 + y^2 = r^2$.

Differentiate both sides of the equation with respect to time.

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(r^2)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt} \quad \text{or} \quad x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}$$

Natalie has travelled for 2 h, or $2 \times 45 = 90$ km.
Shannon has travelled for 3 h, or $3 \times 40 = 120$ km.
The distance between the cars is

$$90^2 + 120^2 = r^2$$
$$r = 150$$

Thus, $x = 90$, $\frac{dx}{dt} = 45$, $y = 120$, $\frac{dy}{dt} = 40$, and $r = 150$

$$\text{So, } 90 \times 45 + 120 \times 40 = 150\frac{dr}{dt}$$ (Substitute)

$$4050 + 4800 = 150\frac{dr}{dt}$$ $\left(\text{Solve for } \frac{dr}{dt}\right)$

$$59 = \frac{dr}{dt}.$$

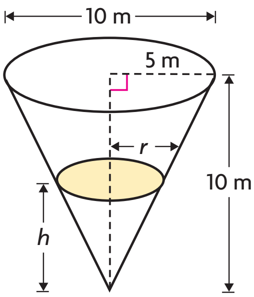
Therefore, the distance between Natalie's car and Shannon's car is increasing at a rate of 59 km/h, 3 h after Shannon leaves.

EXAMPLE 3 **Solving a related rate problem involving a conical model**

Water is pouring into an inverted right circular cone at a rate of π m^3/min. The height and the diameter of the base of the cone are both 10 m. How fast is the water level rising when the depth of the water is 8 m?

Solution

Let V represent the volume, r represent the radius, and h represent the height of the water in the cone at time t. The volume of the water in the cone, at any time, is $V = \frac{1}{3}\pi r^2 h$. Since we are given $\frac{dV}{dt}$ and we want to determine $\frac{dh}{dt}$ when $h = 8$, we solve for r in terms of h from the ratio determined from the similar triangles $\frac{r}{h} = \frac{5}{10}$ or $r = \frac{1}{2}h$. Therefore, we can simplify the volume formula so it involves only V and h.



Substituting into $V = \frac{1}{3}\pi r^2 h$, we get

$$V = \frac{1}{3}\pi\left(\frac{1}{4}h^2\right)h$$

$$V = \frac{1}{12}\pi h^3$$

Differentiating with respect to time, $\frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt}$.

At a specific time, when $h = 8$ and $\frac{dV}{dt} = \pi$,

$$\pi = \frac{1}{4}\pi(8)^2\frac{dh}{dt}$$

$$\frac{1}{16} = \frac{dh}{dt}$$

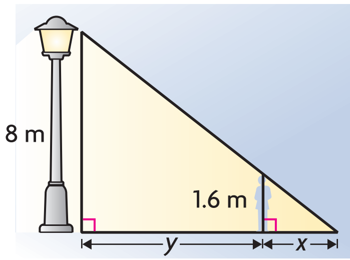
Therefore, at the moment when the depth of the water is 8 m, the level is rising at $\frac{1}{16}$ m/min.

EXAMPLE 4 **Solving a related rate problem involving similar triangle models**

A student who is 1.6 m tall walks directly away from a lamppost at a rate of 1.2 m/s. A light is situated 8 m above the ground on the lamppost. Show that the student's shadow is lengthening at a rate of 0.3 m/s when she is 20 m from the base of the lamppost.

Solution

Let x be the length of the student's shadow, and let y be her distance from the lamppost, in metres, as shown. Let t denote the time, in seconds.



We are given that $\frac{dy}{dt} = 1.2$ m/s, and we want to determine $\frac{dx}{dt}$ when $y = 20$ m.

To determine a relationship between x and y, use similar triangles.

$$\frac{x + y}{8} = \frac{x}{1.6}$$

$$1.6x + 1.6y = 8x$$

$$1.6y = 6.4x$$

Differentiating both sides with respect to t, $1.6\frac{dy}{dt} = 6.4\frac{dx}{dt}$.

When $y = 20$ and $\frac{dy}{dt} = 1.2$,

$$1.6(1.2) = 6.4\frac{dx}{dt}$$

$$\frac{dx}{dt} = 0.3$$

Therefore, the student's shadow is lengthening at 0.3 m/s. (Note that her shadow is lengthening at a constant rate, independent of her distance from the lamppost.)

Exercise

PART A

1. Express the following statements in symbols:

   a. The area, A, of a circle is increasing at a rate of 4 m^2/s.

   b. The surface area, S, of a sphere is decreasing at a rate of 3 m^2/min.

   c. After travelling for 15 min, the speed of a car is 70 km/h.

   d. The x- and y-coordinates of a point are changing at equal rates.

   e. The head of a short-distance radar dish is revolving at three revolutions per minute.

PART B

2. The function $T(x) = \frac{200}{1 + x^2}$ represents the temperature, in degrees Celsius, perceived by a person standing x metres from a fire.

   a. If the person moves away from the fire at 2 m/s, how fast is the perceived temperature changing when the person is 5 m away?

   b. Using a graphing calculator, determine the distance from the fire when the perceived temperature is changing the fastest.

   c. What other calculus techniques could be used to check the result?

3. The side of a square is increasing at a rate of 5 cm/s. At what rate is the area changing when the side is 10 cm long? At what rate is the perimeter changing when the side is 10 cm long?

4. Each edge of a cube is expanding at a rate of 4 cm/s.

   a. How fast is the volume changing when each edge is 5 cm?

   b. At what rate is the surface area changing when each edge is 7 cm?

5. The width of a rectangle increases at 2 cm/s, while the length decreases at 3 cm/s. How fast is the area of the rectangle changing when the width equals 20 cm and the length equals 50 cm?

6. The area of a circle is decreasing at the rate of 5 m^2/s when its radius is 3 m.

   a. At what rate is the radius decreasing at that moment?

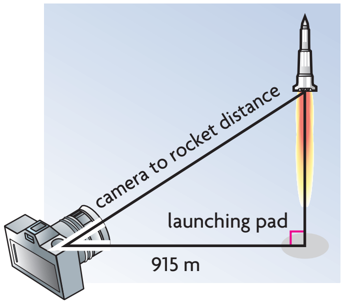
   b. At what rate is the diameter decreasing at that moment?

7. Oil that is spilled from a ruptured tanker spreads in a circle. The area of the circle increases at a constant rate of 6 km^2/h. How fast is the radius of the spill increasing when the area is 9π km^2?

8. The top of a 5 m wheeled ladder rests against a vertical wall. If the bottom of the ladder rolls away from the base of the wall at a rate of $\frac{1}{3}$ m/s, how fast is the top of the ladder sliding down the wall when it is 3 m above the base of the wall?

9. How fast must someone let out line if a kite is 30 m high, 40 m away horizontally, and continuing to move away horizontally at a rate of 10 m/min?

10. If the rocket shown below is rising vertically at 268 m/s when it is 1220 m up, how fast is the camera-to-rocket distance changing at that instant?



11. Two cyclists depart at the same time from a starting point along routes that make an angle of $\frac{\pi}{3}$ radians with each other. The first cyclist is travelling at 15 km/h, while the second cyclist is moving at 20 km/h. How fast are the two cyclists moving apart after 2 h?

12. A spherical balloon is being filled with helium at a rate of 8 cm^3/s. At what rate is its radius increasing at the following moments.

    a. when the radius is 12 cm

    b. when the volume is 1435 cm^3 (Your answer should be correct to the nearest hundredth.)

    c. when the balloon has been filling for 33.5 s

13. A cylindrical tank, with height 15 m and diameter 2 m, is being filled with gasoline at a rate of 500 L/min. At what rate is the fluid level in the tank rising? ($1 \text{ L} = 1000 \text{ cm}^3$) About how long will it take to fill the tank?

14. If $V = \pi r^2 h$, determine $\frac{dV}{dt}$ if r and h are both variables that depend on t. In your journal, write three problems that involve the rate of change of the volume of a cylinder such that

    a. r is a variable and h is a constant

    b. r is a constant and h is a variable

    c. r and h are both variables

15. The trunk of a tree is approximately cylindrical in shape and has a diameter of 1 m when the height of the tree is 15 m. If the radius is increasing at 0.003 m per year and the height is increasing at 0.4 m per year, determine the rate of increase of the volume of the trunk at this moment.

16. A conical paper cup, with radius 5 cm and height 15 cm, is leaking water at a rate of 2 cm^3/min. At what rate is the water level decreasing when the water is 3 cm deep?

17. The cross-section of a water trough is an equilateral triangle with a horizontal top edge. If the trough is 5 m long and 25 cm deep, and water is flowing in at a rate of 0.25 m^3/min, how fast is the water level rising when the water is 10 cm deep at the deepest point?

18. The shadow cast by a man standing 1 m from a lamppost is 1.2 m long. If the man is 1.8 m tall and walks away from the lamppost at a speed of 120 m/min, at what rate is the shadow lengthening after 5 s?

PART C

19. A railroad bridge is 20 m above, and at right angles to, a river. A person in a train travelling at 60 km/h passes over the centre of the bridge at the same instant that a person in a motorboat travelling at 20 km/h passes under the centre of the bridge. How fast are the two people separating 10 s later?

20. Liquid is being poured into the top of a funnel at a steady rate of 200 cm^3/s. The funnel is in the shape of an inverted right circular cone, with a radius equal to its height. It has a small hole in the bottom, where the liquid is flowing out at a rate of 20 cm^3/s.

    a. How fast is the height of the liquid changing when the liquid in the funnel is 15 cm deep?

    b. At the instant when the height of the liquid is 25 cm, the funnel becomes clogged at the bottom and no more liquid flows out. How fast does the height of the liquid change just after this occurs?

21. A ladder of length l, standing on level ground, is leaning against a vertical wall. The base of the ladder begins to slide away from the wall. Introduce a coordinate system so that the wall lies along the positive y-axis, the ground is on the positive x-axis, and the base of the wall is the origin.

    a. What is the equation of the path followed by the midpoint of the ladder?

    b. What is the equation of the path followed by any point on the ladder? (*Hint:* Let k be the distance from the top of the ladder to the point on the ladder.)