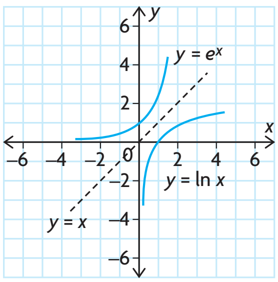
The Natural Logarithm and its Derivative



The logarithmic function is the inverse of the exponential function. For the particular exponential function $y = e^x$, the inverse is $x = e^y$ or $y = \log_e x$, a logarithmic function where $e \doteq 2.718$. This logarithmic function is referred to as the "natural" logarithmic function and is usually written as $y = \ln x$.

The functions $y = e^x$ and $y = \ln x$ are inverses of each other. This means that the graphs of the functions are reflections of each other in the line $y = x$, as shown.

What is the derivative of the natural logarithmic function?

For $y = \ln x$, the definition of the derivative yields $\frac{dy}{dx} = \lim_{h \to 0} \frac{\ln(x + h) - \ln(x)}{h}$.

We can determine the derivative of the natural logarithmic function using the derivative of the exponential function that we developed earlier.

Given $y = \ln x$, we can rewrite this as $e^y = x$. Differentiating both sides of the equation with respect to x, and using implicit differentiation on the left side, yields

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

$$= \frac{1}{x}$$

The Derivative of the Natural Logarithmic Function

The derivative of the natural logarithmic function $y = \ln x$ is $\frac{dy}{dx} = \frac{1}{x}$, $x > 0$.

This derivative makes sense when we consider the graph of $y = \ln x$. The function is defined only for $x > 0$, and the slopes are all positive. We see that, as $x \to \infty$, $\frac{dy}{dx} \to 0$. As x increases, the slope of the tangent decreases.

We can apply this new derivative, along with the product, quotient, and chain rules to determine derivatives of fairly complicated functions.

EXAMPLE 1 Selecting a strategy to determine the derivative of a function involving a natural logarithm

Determine $\frac{dy}{dx}$ for the following functions:

a. $y = \ln(5x)$ b. $y = \frac{\ln x}{x^3}$ c. $y = \ln(x^2 + e^x)$

Solution

a. $y = \ln(5x)$

Using the chain rule,

$$\frac{dy}{dx} = \frac{1}{5x}(5) = \frac{1}{x}$$

Using properties of logarithms,

$$y = \ln(5x) = \ln(5) + \ln(x)$$

$$\frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$$

b. $y = \frac{\ln x}{x^3}$

Using the quotient and power rules,

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\ln(x))\left(x^3 - \ln(x)\frac{d}{dx}(x^3)\right)}{(x^3)^2}$$

$$= \frac{\frac{1}{x}x^3 - \ln(x) \times 3x^2}{x^6}$$ (Simplify)

$$= \frac{x^2 - 3x^2\ln(x)}{x^6}$$ (Divide by x^2)

$$= \frac{1 - 3\ln(x)}{x^4}$$

c. $y = \ln(x^2 + e^x)$

Using the chain rule,

$$\frac{dy}{dx} = \frac{1}{(x^2 + e^x)}\frac{d}{dx}(x^2 + e^x)$$

$$= \frac{2x + e^x}{(x^2 + e^x)}$$

The Derivative of a Composite Natural Logarithmic Function

If $f(x) = \ln(g(x))$, then $f'(x) = \frac{1}{g(x)}g'(x)$, by the chain rule.

EXAMPLE 2 Selecting a strategy to solve a tangent problem

Determine the equation of the line that is tangent to $y = \frac{\ln x^2}{3x}$ at the point where $x = 1$.

Solution

$\ln 1 = 0$, so $y = 0$ when $x = 1$, and the point of contact of the tangent is $(1, 0)$.

The slope of the tangent is given by $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{3x\left(\frac{1}{x^2}\right)2x - 3\ln x^2}{9x^2}$$ (Quotient rule)

$$= \frac{6 - 3\ln x^2}{9x^2}$$

When $x = 1$, $\frac{dy}{dx} = \frac{2}{3}$.

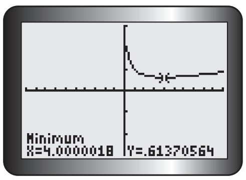
The equation of the tangent is $y - 0 = \frac{2}{3}(x - 1)$, or $2x - 3y - 2 = 0$.

EXAMPLE 3 Determining where the minimum value of a function occurs

a. For the function $f(x) = \sqrt{x} - \ln x$, $x > 0$, use your graphing calculator to determine the x-value that minimizes the value of the function.

b. Use calculus methods to determine the exact x-value where the minimum is attained.

Solution



a. The graph of $f(x) = \sqrt{x} - \ln x$ is shown.

Use the minimum value operation of your calculator to determine the minimum value of $f(x)$. The minimum value occurs at $x = 4$.

b. $f(x) = \sqrt{x} - \ln x$

To minimize $f(x)$, set the derivative equal to zero.

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x}$$

$$\frac{1}{2\sqrt{x}} - \frac{1}{x} = 0$$

$$\frac{1}{2\sqrt{x}} = \frac{1}{x}$$

$$x = 2\sqrt{x}$$

$$x^2 = 4x$$

$x(x - 4) = 0$

$x = 4$ or $x = 0$

But $x = 0$ is not in the domain of the function, so $x = 4$.

Therefore, the minimum value of $f(x)$ occurs at $x = 4$.

We now look back at the derivative of the natural logarithmic function using the definition.

For the function $f(x) = \ln(x)$,

$$f'(x) = \lim_{h \to 0} \frac{\ln(x + h) - \ln(x)}{h}$$

and, specifically,

$$f'(1) = \lim_{h \to 0} \frac{\ln(1 + h) - \ln(1)}{h}$$

$$= \lim_{h \to 0} \frac{\ln(1 + h)}{h}$$

$$= \lim_{h \to 0} \ln(1 + h)^{\frac{1}{h}}, \text{ since } \frac{1}{h}\ln(1 + h) = \ln(1 + h)^{\frac{1}{h}}$$

However, we know that $f'(x) = \frac{1}{x}, f'(1) = 1$.

We conclude that $\lim_{h \to 0} \ln(1 + h)^{\frac{1}{h}} = 1$.

Since the natural logarithmic function is a continuous and one-to-one function (meaning that, for each function value, there is exactly one value of the independent variable that produces this function value), we can rewrite this as $\ln[\lim_{h \to 0}(1 + h)^{\frac{1}{h}}] = 1$.

Since $\ln e = 1$, $\ln[\lim_{h \to 0}(1 + h)^{\frac{1}{h}}] = \ln e$.

Therefore, $\lim_{h \to 0}(1 + h)^{\frac{1}{h}} = e$.

We now have a way to approximate the value of e using the above limit.

h	0.1	0.01	0.001	0.0001
$(1 + h)^{\frac{1}{h}}$	2.593 742 46	2.704 813 829	2.716 923 932	2.718 145 927

From the table, it appears that $e \doteq 2.718$ is a good approximation as h approaches zero.

Exercise

PART A

1. Distinguish between natural logarithms and common logarithms.
2. At the end of this section, we found that we could approximate the value of e (Euler's constant) using $e = \lim_{h\to 0}(1 + h)^{\frac{1}{h}}$. By substituting $h = \frac{1}{n}$, we can express e as $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$. Justify this definition by evaluating the limit for increasing values of n.
3. Determine the derivative for each of the following:
   a. $y = \ln(5x + 8)$
   b. $y = \ln(x^2 + 1)$
   c. $s = 5 \ln t^3$
   d. $y = \ln\sqrt{x + 1}$
   e. $s = \ln(t^3 - 2t^2 + 5)$
   f. $w = \ln\sqrt{z^2 + 3z}$
4. Differentiate each of the following:
   a. $f(x) = x \ln x$
   b. $y = e^{\ln x}$
   c. $v = e^t \ln t$
   d. $g(z) = \ln(e^{-z} + ze^{-z})$
   e. $s = \frac{e^t}{\ln t}$
   f. $h(u) = e^{\sqrt{u}}\ln\sqrt{u}$
5. a. If $g(x) = e^{2x-1}\ln(2x - 1)$, evaluate $g'(1)$.
   b. If $f(t) = \ln\left(\frac{t - 1}{3t + 5}\right)$, evaluate $f'(5)$.
   c. Check your calculations for parts a. and b. using either a calculator or a computer.
6. For each of the following functions, solve the equation $f'(x) = 0$:
   a. $f(x) = \ln(x^2 + 1)$
   b. $f(x) = (\ln x + 2x)^{\frac{1}{3}}$
   c. $f(x) = (x^2 + 1)^{-1}\ln(x^2 + 1)$
7. a. Determine the equation of the tangent to the curve defined by $f(x) = \frac{\ln\sqrt[3]{x}}{x}$ at the point where $x = 1$.
   b. Use technology to graph the function in part a. and then draw the tangent at the point where $x = 1$.
   c. Compare the equation you obtained in part a. with the equation you obtained in part b.

PART B

8. Determine the equation of the tangent to the curve defined by $y = \ln x - 1$ that is parallel to the straight line with equation $3x - 6y - 1 = 0$.
9. a. If $f(x) = (x \ln x)^2$, determine all the points at which the graph of $f(x)$ has a horizontal tangent line.
   b. Use graphing technology to check your work in part a.
   c. Comment on the efficiency of the two solutions.
10. Determine the equation of the tangent to the curve defined by $y = \ln(1 + e^{-x})$ at the point where $x = 0$.
11. The velocity, in kilometres per hour, of a car as it begins to slow down is given by the equation $v(t) = 90 - 30\ln(3t + 1)$, where t is in seconds.
    a. What is the velocity of the car as the driver begins to brake?
    b. What is the acceleration of the car?
    c. What is the acceleration at $t = 2$?
    d. How long does the car take to stop?

PART C

12. Use the definition of the derivative to evaluate $\lim_{h\to 0}\frac{\ln(2 + h) - \ln(2)}{h}$.
13. Consider $f(x) = \ln(\ln x)$.
    a. Determine $f'(x)$.
    b. State the domains of $f(x)$ and $f'(x)$.