The Derivatives of General Logarithmic Functions

In the previous section, we learned how to determine the derivative of the natural logarithmic function (base e). But what is the derivative of $y = \log_2 x$? The base of this function is 2, not e.

To differentiate the general logarithmic function $y = \log_a x$, $a > 0$, $a \neq 1$, we can use the properties of logarithms so that we can use the base e.

Let $y = \log_a x$.

Then $a^y = x$.

Take the logarithm of both sides using the base e.

$$\ln a^y = \ln x$$

$$y \ln a = \ln x$$

$$y = \frac{\ln x}{\ln a}$$

Differentiating both sides with respect to x, we obtain

$$\frac{dy}{dx} = \frac{d\left(\frac{\ln x}{\ln a}\right)}{dx}$$

$$= \frac{1}{\ln a} \times \frac{d(\ln x)}{dx} \quad (\ln a \text{ is a constant.})$$

$$= \frac{1}{\ln a} \times \frac{1}{x}$$

$$= \frac{1}{x \ln a}$$

The Derivative of the Logarithmic Function $y = \log_a x$

If $y = \log_a x$, $a > 0$, $a \neq 1$, then $\frac{dy}{dx} = \frac{1}{x \ln a}$.

EXAMPLE 1 **Solving a tangent problem involving a logarithmic function**

Determine the equation of the tangent to $y = \log_2 x$ at $(8, 3)$.

Solution

The slope of the tangent is given by the derivative $\frac{dy}{dx}$, where $y = \log_2 x$.

$$\frac{dy}{dx} = \frac{1}{x \ln 2}$$

At $x = 8$, $\frac{dy}{dx} = \frac{1}{8 \ln 2}$.

The equation of the tangent is

$$y - 3 = \frac{1}{8 \ln 2}(x - 8)$$

$$y = \frac{1}{8 \ln 2}x + 3 - \frac{1}{\ln 2}$$

We can determine the derivatives of other logarithmic functions using the rule $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$, along with other derivative rules.

EXAMPLE 2 **Selecting a strategy to differentiate a composite logarithmic function**

Determine the derivative of $y = \log_4(2x + 3)^5$.

Solution

We can rewrite the logarithmic function as follows:

$$y = \log_4(2x + 3)^5$$

$$y = 5 \log_4(2x + 3) \quad \text{(Property of logarithms)}$$

$$\frac{dy}{dx} = \frac{d}{dx}[5 \log_4(2x + 3)]$$

$$= 5\frac{d}{dx}[\log_4(2x + 3)]$$

$$= 5\frac{d[\log_4(2x + 3)]}{d[(2x + 3)]}\frac{d(2x + 3)}{dx} \quad \text{(Chain rule)}$$

$$= 5\left(\frac{1}{(2x + 3) \ln 4}\right)(2) \quad \text{(Simplify)}$$

$$= \frac{10}{(2x + 3) \ln 4}$$

The Derivative of a Composite Function Involving $y = \log_a x$

If $y = \log_a f(x)$, $a > 0, a \neq 1$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x) \ln a}$.

Exercise

PART A

1. Determine $\frac{dy}{dx}$ for each function.
   a. $y = \log_5 x$
   b. $y = \log_3 x$
   c. $y = 2 \log_4 x$
   d. $y = -3 \log_7 x$
   e. $y = -(\log x)$
   f. $y = 3 \log_6 x$
2. Determine the derivative of each function.
   a. $y = \log_3(x + 2)$
   b. $y = \log_8(2x)$
   c. $y = -3 \log_3(2x + 3)$
   d. $y = \log_{10}(5 - 2x)$
   e. $y = \log_8(2x + 6)$
   f. $y = \log_7(x^2 + x + 1)$

PART B

3. a. If $f(t) = \log_2\left(\frac{t + 1}{2t + 7}\right)$, evaluate $f'(3)$.
   b. If $h(t) = \log_3(\log_2(t))$, determine $h'(8)$.
4. Differentiate.
   a. $y = \log_{10}\left(\frac{1 + x}{1 - x}\right)$
   b. $y = \log_2 \sqrt{x^2 + 3x}$
   c. $y = 2 \log_3(5^x) - \log_3(4^x)$
   d. $y = 3^x \log_3 x$
   e. $y = 2x \log_4 x$
   f. $y = \frac{\log_5(3x^2)}{\sqrt{x + 1}}$
5. Determine the equation of the tangent to the curve $y = x \log x$ at $x = 10$. Graph the function and the tangent.
6. Explain why the derivative of $y = \log_a kx$, $k > 0$, is $\frac{dy}{dx} = \frac{1}{x \ln a}$ for any constant k.
7. Determine the equation of the tangent to the curve $y = 10^{2x-9} \log_{10}(x^2 - 3x)$ at $x = 5$.
8. A particle's distance, in metres, from a fixed point at time, t, in seconds is given by $s(t) = t \log_6(t + 1)$, $t \geq 0$. Is the distance increasing or decreasing at $t = 15$? How do you know?

PART C

9. a. Determine the equation of the tangent to $y = \log_3 x$ at the point (9, 2).
   b. Graph the function and include any asymptotes.
   c. Will this tangent line intersect any asymptotes? Explain.
10. Determine the domain, critical numbers, and intervals of increase and decrease of $f(x) = \ln(x^2 - 4)$.
11. Do the graphs of either of these functions have points of inflection? Justify your answers with supporting calculations.
    a. $y = x \ln x$
    b. $y = 3 - 2 \log x$
12. Determine whether the slope of the graph of $y = 3^x$ at the point $(0, 1)$ is greater than the slope of the graph of $y = \log_3 x$ at the point $(1, 0)$. Include graphs with your solution.