Logarithmic Differentiation

The derivatives of most functions involving exponential and logarithmic expressions can be determined by using the methods that we have developed. A function such as $y = x^x$ poses new problems, however. The power rule cannot be used because the exponent is not a constant. The method of determining the derivative of an exponential function also cannot be used because the base is not a constant. What can be done?

It is frequently possible, with functions presenting special difficulties, to simplify the function by using the properties of logarithms. In such cases we say that we are using **logarithmic differentiation**.

EXAMPLE 1 **Determining the derivative of a function using logarithmic differentiation**

Determine $\frac{dy}{dx}$ for the function $y = x^x$, $x > 0$.

Solution

Take the natural logarithms of each side, and rewrite.

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

Differentiate both sides with respect to x, using implicit differentiation on the left side and the product rule on the right side.

$$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \ln x$$

$$\frac{dy}{dx} = y(1 + \ln x)$$

$$= x^x(1 + \ln x)$$

This method of logarithmic differentiation also works well to help simplify a function with many factors and powers before the differentiation takes place.

We can use logarithmic differentiation to prove the power rule, $\frac{d}{dx}(x^n) = nx^{n-1}$, for all real values of n. (In a previous chapter, we proved this rule for positive integer values of n and we have been cheating a bit in using it for other values of n.)

Given the function $y = x^n$, for any real value of n where $x > 0$, how do we determine $\frac{dy}{dx}$?

To solve this, we take the natural logarithm of both sides of the expression and get $\ln y = \ln x^n = n \ln x$.

Differentiating both sides with respect to x, using implicit differentiation, and remembering that n is a constant, we get

$$\frac{1}{y}\frac{dy}{dx} = n\frac{1}{x}$$

$$\frac{dy}{dx} = ny\frac{1}{x}$$

$$= nx^n\frac{1}{x}$$

$$= nx^{n-1}$$

Therefore, $\frac{d}{dx}(x^n) = nx^{n-1}$ for any real value of n.

EXAMPLE 2 **Selecting a logarithmic differentiation strategy to determine a derivative**

For $y = (x^2 + 3)^x$, determine $\frac{dy}{dx}$.

Solution

Take the natural logarithm of both sides of the equation.

$$y = (x^2 + 3)^x$$

$$\ln y = \ln (x^2 + 3)^x$$

$$\ln y = x \ln (x^2 + 3)$$

Differentiate both sides of the equation with respect to x, using implicit differentiation on the left side and the product and chain rules on the right side.

$$\frac{1}{y}\frac{dy}{dx} = (1)\ln(x^2 + 3) + x\left(\frac{1}{x^2 + 3}(2x)\right)$$

$$\frac{dy}{dx} = y\left[\ln(x^2 + 3) + x\left(\frac{2x}{x^2 + 3}\right)\right]$$

$$= (x^2 + 3)^x\left[\ln (x^2 + 3) + \left(\frac{2x^2}{x^2 + 3}\right)\right]$$

You will recognize logarithmic differentiation as the method used in the previous section, and its use makes memorization of many formulas unnecessary. It also allows complicated functions to be handled much more easily.

EXAMPLE 3 Using logarithmic differentiation

Given $y = \frac{(x^4 + 1)\sqrt{x + 2}}{(2x^2 + 2x + 1)}$, determine $\frac{dy}{dx}$ at $x = -1$.

Solution

While it is possible to determine $\frac{dy}{dx}$ using a combination of the product, quotient, and chain rules, this process is awkward and time-consuming. Instead, before differentiating, we take the natural logarithm of both sides of the equation.

Since $y = \frac{(x^4 + 1)\sqrt{x + 2}}{(2x^2 + 2x + 1)}$,

$$\ln y = \ln\left[\frac{(x^4 + 1)\sqrt{x + 2}}{(2x^2 + 2x + 1)}\right]$$

$$\ln y = \ln(x^4 + 1) + \ln\sqrt{x + 2} - \ln(2x^2 + 2x + 1)$$

$$\ln y = \ln(x^4 + 1) + \frac{1}{2}\ln(x + 2) - \ln(2x^2 + 2x + 1)$$

The right side of this equation looks much simpler. We can now differentiate both sides with respect to x, using implicit differentiation on the left side.

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^4 + 1}(4x^3) + \frac{1}{2}\frac{1}{x + 2} - \frac{1}{2x^2 + 2x + 1}(4x + 2)$$

$$\frac{dy}{dx} = y\left[\frac{4x^3}{x^4 + 1} + \frac{1}{2(x + 2)} - \frac{4x + 2}{2x^2 + 2x + 1}\right]$$

$$= \frac{(x^4 + 1)\sqrt{x + 2}}{(2x^2 + 2x + 1)}\left[\frac{4x^3}{x^4 + 1} + \frac{1}{2(x + 2)} - \frac{4x + 2}{2x^2 + 2x + 1}\right]$$

While this derivative is a very complicated function, the process of determining the derivative is straightforward, using only the derivative of the natural logarithmic function and the chain rule.

We do not need to simplify this in order to determine the value of the derivative at $x = -1$.

For $x = -1$, $\frac{dy}{dx} = \frac{(1 + 1)\sqrt{1}}{(2 - 2 + 1)}\left[\frac{-4}{1 + 1} + \frac{1}{2(-1 + 2)} - \frac{-4 + 2}{2 - 2 + 1}\right]$

$$= 2\left[-2 + \frac{1}{2} + 2\right]$$

$$= 1$$

Exercise

PART A

1. Differentiate each of the following:

   a. $y = x^{\sqrt{10}} - 3$ c. $s = t^{\pi}$

   b. $f(x) = 5x^{3\sqrt{2}}$ d. $f(x) = x^{e} + e^{x}$

2. Use the method of logarithmic differentiation to determine the derivative for each of the following functions:

   a. $y = x^{\ln x}$ c. $y = x^{\sqrt{x}}$

   b. $y = \dfrac{(x + 1)(x - 3)^2}{(x + 2)^3}$ d. $s = \left(\dfrac{1}{t}\right)^{t}$

3. a. If $y = f(x) = x^{x}$, evaluate $f'(e)$.

   b. If $s = e^{t} + t^{e}$, determine $\dfrac{ds}{dt}$ when $t = 2$.

   c. If $f(x) = \dfrac{(x - 3)^2\sqrt[3]{x + 1}}{(x - 4)^5}$, determine $f'(7)$.

4. Determine the equation of the tangent to the curve defined by $y = x^{(x^2)}$ at the point where $x = 2$.

PART B

5. If $y = \dfrac{1}{(x + 1)(x + 2)(x + 3)}$, determine the slope of the tangent to the curve at the point where $x = 0$.

6. Determine the points on the curve defined by $y = x^{\frac{1}{x}}$, $x > 0$, where the slope of the tangent is zero.

7. If tangents to the curve defined by $y = x^2 + 4 \ln x$ are parallel to the line defined by $y - 6x + 3 = 0$, determine the points where the tangents touch the curve.

8. The tangent to the curve defined by $y = x^{\sqrt{x}}$ at the point $A(4, 16)$ is extended to cut the x-axis at B and the y-axis at C. Determine the area of $\triangle OBC$, where O is the origin.

PART C

9. Determine the slope of the line that is tangent to the curve defined by $y = \dfrac{e^{x}\sqrt{x^2 + 1}}{(x^2 + 2)^3}$ at the point $\left(0, \frac{1}{8}\right)$.

10. Determine $f'(x)$ if $f(x) = \left(\dfrac{x \sin x}{x^2 - 1}\right)^2$.

11. Differentiate $y = x^{\cos x}$, $x > 0$.

12. Determine the equation of the line that is tangent to the curve $y = x^{x}$ at the point $(1, 1)$.

13. The position of a particle that moves on a straight line is given by $s(t) = t^{\frac{1}{t}}$ for $t > 0$.

    a. Determine the velocity and acceleration functions.

    b. At what time, t, is the velocity zero? What is the acceleration at this time?

14. Make a conjecture about which number is larger: e^{π} or π^{e}. Verify your work with a calculator.