Vector Appendix

Gaussian Elimination

In Chapter 9, we developed a method for solving systems of linear equations that uses elementary operations to eliminate unknowns. We will now introduce a method for solving equations that is known as Gaussian elimination. This method uses matrices and elementary operations to deal with larger systems of equations more easily.

Consider the following system of three equations in three unknowns:

① $x + 2y + 2z = 9$
② $x + y = 1$
③ $2x + 3y - z = 1$

To use Gaussian elimination to solve a system of equations, first write the system in matrix form—an abbreviated form that omits the variables and uses only the coefficients of each equation. Each row in the matrix represents an equation from the original system. For example, the second equation in the original system is represented by the second row in the matrix. The coefficients of the unknowns are entered in columns on the left side of the matrix, with a vertical line separating the coefficients from the numbers on the right side. When a system of equations is written in this form, the associated matrix is called its **augmented matrix**. The matrix below is the augmented matrix representing the original system of equations. Note that the 0 in row 2 of the augmented matrix represents the coefficient of z in the second equation, if the equation had been written as $1x + 1y + 0z = 1$. We have also included a second matrix, called the **coefficient matrix**. We make sure that in the coefficient matrix, each column corresponds to a single variable. For instance, in this case the first column corresponds to the coefficients for x. This matrix represents the coefficients of the unknowns in each of the equations.

Coefficient Matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix}$$

Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & -1 & 1 \end{array}\right]$$

The benefit of using matrices (the plural of *matrix*) is that it allows us to introduce a method for solving systems of equations that is systematic, methodical, and useful for dealing with larger systems of equations.

For solving systems of equations using matrices, we introduce operations that are similar to the previously introduced elementary operations. When dealing with matrices, however, we call them **elementary row operations**. Notice the term *row operations*—they are only applied to the rows. Just as with elementary operations, elementary row operations are always used to reduce matrices into simpler form by eliminating unknowns. In Example 1, notice that the previously introduced elementary operations have been modified for use with matrices.

Elementary Row Operations for Matrices (Systems of Equations)

1. Multiply a row (equation) by a nonzero constant.
2. Interchange any pair of rows (equations).
3. Add a multiple of one row (equation) to a second row (equation) to replace the second row (equation).

EXAMPLE 1 Using elementary row operations to solve a system of equations

Solve the following system of equations using elementary row operations:

① $x + 2y + 2z = 9$
② $x + y = 1$
③ $2x + 3y - z = 1$

Solution

1: Start by writing the system of equations in its augmented matrix form.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & -1 & 1 \end{array}\right]$$

2: Multiply row 1 by -1, and add the result to row 2. This leaves the first row unchanged, and the second row is replaced with the result.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 2 & 3 & -1 & 1 \end{array}\right] \quad -1\text{ (row 1)} + \text{row 2}$$

When solving a system of equations using elementary row operations, normally we try to make the first entry in the second row 0, because our ultimate objective is to have an augmented matrix in **row-echelon form.** In this form, the second row of the coefficient matrix can only have nonzero entries in the y- and z-columns, and the third row can only have a nonzero entry in the z-column (so that the coefficient matrix looks like an *upper triangle* of nonzero entries). Notice that we indicate the operations we use to reduce the matrix.

3: Multiply row 1 by -2, and add the result to row 3.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & -1 & -5 & -17 \end{array}\right] \quad -2\text{ (row 1)} + \text{row 3}$$

We have now produced a new row 3, with 0 as the coefficient of x.

4: Multiply row 2 by -1, and add the result to row 3.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & 0 & -3 & -9 \end{array}\right] \quad -1\text{ (row 2)} + \text{row 3}$$

This final operation is equivalent to subtracting row 2 from row 3, with the final result being a new row 3.

The new matrix corresponds to the following equivalent system of equations:

① $x + 2y + 2z = 9$
④ $-y - 2z = -8$
⑤ $-3z = -9$

This system can now be solved using back substitution. From equation ⑤, $z = 3$. Substituting into equation ④, $-y - 2(3) = -8$, or $y = 2$. If we substitute $y = 2$ and $z = 3$ into equation ①, we obtain $x + 2(2) + 2(3) = 9$, or $x = -1$.

We conclude that $x = -1$, $y = 2$, and $z = 3$ is the solution to the system of equations. So the original system represents three planes intersecting at the point $(-1, 2, 3)$.

Check:
Substituting into equation ①, $x + 2y + 2z = -1 + 2(2) + 2(3) = 9$.
Substituting into equation ②, $x + y = -1 + 2 = 1$.
Substituting into equation ③, $2x + 3y - z = 2(-1) + 3(2) - 3 = 1$.

When solving a system of equations, either with or without matrices, it is always a good idea to verify the final result.

Important Points about Elementary Row Operations

1. Elementary row operations are used to produce equivalent matrices, but they can be applied in different orders, provided that they are applied properly. There are many ways to get to the final answer.
2. Steps can be combined, and every step does *not* have to be written out in words. In the following examples, we will demonstrate ways of abbreviating steps and reducing the amount of written work.
3. When using elementary row operations, steps should be documented to show how a new matrix was determined. This allows for easy checking in case a mistake is made.

When using elementary row operations to solve systems of equations, the objective is to have the final matrix written in what is described as row-echelon form. In Example 1, we accomplished this but did not define row-echelon form. If a matrix is written in row-echelon form, it must have the following properties:

Properties of a Matrix in Row-Echelon Form

1. All rows that consist entirely of zeros must be written at the bottom of the matrix.
2. In any two successive rows that do not consist entirely of zeros, the first nonzero number in the lower row must occur farther to the right than the first nonzero number in the row directly above.

EXAMPLE 2 Reasoning about the row-echelon form of a matrix

Determine whether the following matrices are in row-echelon form. If they are not in this form, use an elementary row operation to put them in this form.

a. $\left[\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} 2 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$

d. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{array}\right]$

Solution

a. This augmented matrix is not in row-echelon form. It can be changed to the correct form by multiplying row 1 by -1 and adding the product to row 2. The resulting matrix will be in row-echelon form.

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right] \quad -1\text{ (row 1)} + \text{row 2}$$

b. This matrix is in row-echelon form. The first nonzero number in row 3 is 1, which occurs to the right of the first nonzero number, -1, in row 2. Similarly the -1 in row 2 occurs to the right of the first 1 in row 1.

c. This matrix is not in row-echelon form. To put it in row-echelon form, interchange the second and third rows.

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{Interchanging rows 2 and 3}$$

d. This matrix is in row-echelon form. The first nonzero number in row 3 is 8, which is to the right of -2 in row 2, and -2 occurs to the right of 1 in row 1.

In the next example, by using elementary row operations, we will solve a system in which the three planes intersect along a line.

EXAMPLE 3 Using elementary row operations to solve a system of equations

Solve the following system of equations using matrices:

① $2x + y + 2z = 2$
② $x + y + z = 2$
③ $x + z = 0$

Solution

To solve this system, first write it in augmented matrix form, but interchange the order of the equations so that 1 will be the coefficient of x in the first equation (row 2 becomes row 1, row 3 becomes row 2, and row 1 becomes row 3). There is more than one way to write the augmented matrix, which is a reminder that there is more than one way to solve this system using matrices.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 2 \end{array}\right]$$

This matrix results from interchanging the rows in the original equations.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & -2 \\ 0 & -1 & 0 & -2 \end{array}\right]$$

-1 (row 1) $+$ row 2
-2 (row 1) $+$ row 3

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

-1 (row 2) $+$ row 3

This system is now in row-echelon form, which means that we can solve it using the method of back substitution. Row 2 corresponds to $-y = -2$, or $y = 2$. Since row 3 consists only of zeros, this corresponds to $0x + 0y + 0z = 0$, which implies a parametric solution. If $z = s$ and $y = 2$, these values can be substituted into the first equation to obtain x in terms of the parameter.

Since the first equation is $x + y + z = 2$, substituting gives $x + 2 + s = 2$. So, $x = -s$ and the solution to the system is $x = -s$, $y = 2$, and $z = s$.

This means that the three planes intersect along a line with parametric equations $x = -s$, $y = 2$, and $z = s$, or, written as a vector equation, $\vec{r} = (0, 2, 0) + s(-1, 0, 1)$, $s \in \mathbf{R}$.

Check:
Substituting into equation ②, $x + y + z = -s + 2 + s = 2$.
Substituting into equation ③, $x + z = -s + s = 0$.
Substituting into equation ①, $2x + y + 2z = 2(-s) + 2 + 2(s) = 2$.

Exercise

PART A

1. Write an augmented matrix for each system of equations.

a. ① $x + 2y - z = -1$
② $-x + 3y - 2z = -1$
③ $3y - 2z = -3$

b. ① $2x - z = 1$
② $2y - z = 16$
③ $-3x + y = 10$

c. ① $2x - y - z = -2$
② $x - y + 4z = -1$
③ $-x - y = 13$

2. Determine two different row-echelon forms for the following augmented matrix:

$$\left[\begin{array}{rr|r} 2 & 3 & 0 \\ 3 & -1 & 1 \end{array}\right]$$

3. Reduce the following augmented matrix to row-echelon form. Make sure that there are no fractions in the final matrix.

$$\left[\begin{array}{rrr|r} 2 & 1 & 6 & 0 \\ 0 & -2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right]$$

4. a. Write the following augmented matrix in row-echelon form. Make sure that every number in this matrix is an integer.

$$\left[\begin{array}{rrr|r} -1 & 0 & 1 & 2 \\ 0 & -1 & 2 & 0 \\ \frac{1}{2} & -\frac{3}{4} & -2 & \frac{1}{3} \end{array}\right]$$

b. Solve the system of equations corresponding to the matrix you derived in part a.

5. Write the system of equations that corresponds to each augmented matrix.

a. $\left[\begin{array}{rr|r} 1 & -2 & -1 \\ 2 & -3 & 1 \\ 2 & -1 & 0 \end{array}\right]$

b. $\left[\begin{array}{rrr|r} -2 & 0 & -1 & 0 \\ 1 & -2 & 0 & 4 \\ 0 & 1 & 2 & -3 \end{array}\right]$

c. $\left[\begin{array}{rrr|r} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right]$

6. The following matrices represent the final row-echelon form matrix in the solution to a system of equations. Write the solution to each system, if it exists.

a. $\left[\begin{array}{rr|r} -2 & 1 & 6 \\ 0 & -5 & 15 \end{array}\right]$

b. $\left[\begin{array}{rrr|r} 2 & -1 & 1 & 11 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 6 & -36 \end{array}\right]$

c. $\left[\begin{array}{rrr|r} -1 & 3 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -13 \end{array}\right]$

d. $\left[\begin{array}{rrr|r} 4 & -1 & -1 & 0 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & -1 & 5 \end{array}\right]$

e. $\left[\begin{array}{rrr|r} 1 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

f. $\left[\begin{array}{rrr|r} -1 & 0 & 0 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -1 & 2 \end{array}\right]$

7. A student performs elementary row operations on an augmented matrix and comes up with the following matrix:

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

a. Explain why this matrix is in row-echelon form.

b. Explain why there is no solution to the corresponding system of equations.

c. Give an example of an augmented matrix consisting of all nonzero numbers that might have produced this matrix.

PART B

8. Determine whether the following matrices are in row-echelon form. If they are not, use elementary row operations to put them in row-echelon form.

a. $\left[\begin{array}{ccc|c} -1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 3 & 1 & -4 & 2 \\ 0 & 0 & 3 & 6 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -6 \end{array}\right]$

d. $\left[\begin{array}{ccc|c} 1 & -4 & 1 & 0 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

9. a. Write the solution to the system of equations corresponding to each of the augmented matrices in question 8. if a solution exists.

b. Give a geometric interpretation of your result.

10. Solve each system of equations using matrices, and interpret your result geometrically.

a. ① $-x + y + z = 9$
② $x - 2y + z = 15$
③ $2x - y - z = -12$

b. ① $x + y + z = 0$
② $2x + 3y + z = 0$
③ $-3x - 2y - 4z = 0$

c. ① $x - y + 3z = -1$
② $5x + y - 3z = -5$
③ $2x + y - 3z = -2$

d. ① $x + 3y + 4z = 4$
② $-x + 3y + 8z = -4$
③ $x - 3y - 4z = -4$

e. ① $2x + y + z = 1$
② $4x + 2y + 2z = 2$
③ $-2x + y + z = 3$

f. ① $x - y = -500$
② $2y - z = 3500$
③ $x - z = 2000$

11. In the following system of equations, a, b, and c are written as linear combinations of x, y, and z:

$a = x + 2y - z$

$b = x - y + 2z$

$c = 3x + 3y + z$

Express each of x, y, and z as a linear combination of a, b, and c.