12. Determine the equation of the parabola that passes through the points $A(-1, -7)$, $B(2, 20)$, and $C(-3, -5)$, and has an axis of symmetry parallel to the y-axis. (A parabola whose axis of symmetry is parallel to the y-axis has an equation of the form $y = ax^2 + bx + c$.)

PART C

13. Solve for p, q, and r in the following system of equations:

① $pq - 2\sqrt{q} + 3rq = 8$

② $2pq - \sqrt{q} + 2rq = 7$

③ $-pq + \sqrt{q} + 2rq = 4$

14. A system of equations has the following augmented matrix:

$$\left[\begin{array}{ccc|c} a & 1 & 1 & a \\ 1 & a & 1 & a \\ 1 & 1 & a & a \end{array}\right]$$

Determine the values of the parameter a if the corresponding system of equations has

a. no solutions

b. an infinite number of solutions

c. exactly one solution

Gauss-Jordan Method for Solving Systems of Equations

In the previous section, we introduced Gaussian elimination as a method for solving systems of equations. This method uses elementary row operations on an augmented matrix so it can be written in row-echelon form. We will now introduce the concept of Gauss-Jordan elimination as a method for solving systems of equations. This new method uses elementary operations in the same way as before, but the augmented matrix is written in what is called **reduced row-echelon form.** The writing of a matrix in reduced row-echelon form allows us to avoid using the method of back substitution and, instead, to determine the solution(s) to a system of equations directly from the matrix. We begin by first defining what is meant by a reduced row-echelon matrix.

Reduced Row-Echelon Form of a Matrix

A matrix is in reduced row-echelon form if

1. it is in row-echelon form
2. the first nonzero number in every row is a 1 (this is known as a leading 1 for that row)
3. any *column* containing a leading 1 has all of its other column entries equal to zero

For example, a system of three equations and three unknowns might appear as follows in reduced row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

EXAMPLE 1 Reasoning about reduced row-echelon form

Explain why each of the following augmented matrices is in reduced row-echelon form:

a. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{array}\right]$ b. $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & -4 & 0 \end{array}\right]$ c. $\left[\begin{array}{ccc|c} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Solution

We start by noting that each of these augmented matrices is in row-echelon form, so the first condition is met. The leading nonzero entry in each row is a 1, and each of the other column entries for a leading 1 is also 0. This means that each of these three matrices is in reduced row-echelon form.

EXAMPLE 2 Connecting the solution to a system of equations to a matrix in reduced row-echelon form

The following two augmented matrices are in row-echelon form. Determine the solution to the corresponding system of equations by writing each of these matrices in reduced row-echelon form.

a. $\left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -7 \end{array}\right]$ b. $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right]$

Solution

a. This matrix is not in reduced row-echelon form because, in the second column, there is a -2 above the 1. This -2 can be eliminated by multiplying row 2 by 2 and adding it to row 1, which gives the following:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -7 \end{array}\right] \quad 2\text{ (row 2)} + \text{row 1}$$

The solution to the system of equations is $x = 11$, $y = 4$, and $z = -7$, which implies that the original system of equations represents three planes intersecting at the point $(11, 4, -7)$.

b. This matrix is in row-echelon form, but it is not in reduced row-echelon form because, in the third column, there should be a 0 instead of the -2 in row 2, and instead of the 4 in row 1. This matrix can be put in the required form by using elementary row operations.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -18 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 5 \end{array}\right] \begin{array}{l} -4\text{ (row 3)} + \text{row 1} \\ 2\text{ (row 3)} + \text{row 2} \\ \end{array}$$

The solution to this system is $x = -18$, $y = 11$, and $z = 5$, which means that the original system of equations represents three planes intersecting at the point $(-18, 11, 5)$.

In the following example, we use the interchanging of rows along with the other elementary operations.

EXAMPLE 3

Solving a system of equations by putting the augmented matrix in reduced row-echelon form

Write the following matrix in reduced row-echelon form. Then determine the solution to the corresponding system of equations.

$$\left[\begin{array}{ccc|c} 0 & 0 & -1 & 4 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right]$$

Solution

We start by writing the given matrix in row-echelon form. This involves interchanging the three rows.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 4 \end{array}\right]$$

In this matrix, the difficulty is now with the third column: $\begin{array}{r} 2 \\ 0 \\ -1 \end{array}$. There should be a 1 where the -1 is located and a 0 where the 2 is located. Elementary operations must be used to put the matrix in reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 4 \end{array}\right] \begin{array}{l} 2\text{ (row 3)} + \text{row 1} \\ \\ \end{array}$$

All that is needed now is to have a leading 1 in each row.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -4 \end{array}\right] \begin{array}{l} \\ \\ -1\text{ (row 3)} \end{array}$$

The solution is $x = 8$, $y = -1$, and $z = -4$, meaning that we have three planes intersecting at the point $(8, -1, -4)$.

In the next example, we will first put the matrix into row-echelon form and then write it in reduced row-echelon form.

EXAMPLE 4 **Using Gauss-Jordan elimination to solve a system of equations**

Solve the following system of equations using Gauss-Jordan elimination:

① $x - y + 2z = -9$
② $-x - y + 2z = -7$
③ $x + 2y - z = 6$

Solution

The given system of equations is first written in augmented matrix form.

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & -9 \\ -1 & -1 & 2 & -7 \\ 1 & 2 & -1 & 6 \end{array}\right]$$

Step 1: Write the given augmented matrix in row-echelon form.

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & -9 \\ 0 & -2 & 4 & -16 \\ 0 & 3 & -3 & 15 \end{array}\right] \begin{array}{l} \\ \text{row 1} + \text{row 2} \\ -1\,(\text{row 1}) + \text{row 3} \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & -9 \\ 0 & 1 & -2 & 8 \\ 0 & 3 & -3 & 15 \end{array}\right] \quad -\frac{1}{2}(\text{row 2})$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & -9 \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 3 & -9 \end{array}\right] \begin{array}{l} \\ \\ -3\,(\text{row 2}) + \text{row 3} \end{array}$$

The original matrix is now in row-echelon form.

Step 2: Write the matrix in reduced row-echelon form.
First change the leading 3 in row 3 to a 1.

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & -9 \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 1 & -3 \end{array}\right] \quad \frac{1}{3}(\text{row 3})$$

Use elementary row operations to obtain 0 in the third column for all entries but the third row.

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right] \begin{array}{l} -2\,(\text{row 3}) + \text{row 1} \\ 2\,(\text{row 3}) + \text{row 2} \\ \\ \end{array}$$

Finally, convert the entry in the second column of the first row to a 0.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right] \quad \text{row 2 + row 3}$$

The solution to this system of equations is $x = -1$, $y = 2$, and $z = -3$. The three given planes intersect at a point with coordinates $(-1, 2, -3)$.

Check:
As with previous calculations, this result should be checked by substitution in each of the three original equations.

The previous example leads to the following generalization for finding the intersection of three planes at a point using the Gauss-Jordan method of elimination:

Finding the Point of Intersection for Three Planes with the Gauss-Jordan Method

When three planes intersect at a point, the resulting coefficient matrix can always be put in the form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which is known as the **identity matrix**.

Solving equations using the Gauss-Jordan method takes about the same amount of effort as Gaussian elimination when solving small systems of equations. The main advantage of the Gauss-Jordan method is its usefulness in higher-level applications and the understanding it provides regarding the general theory of matrices. To solve smaller systems of equations, such as our examples, either method (Gaussian elimination or Gauss-Jordan elimination) can be used.

Exercises

PART A

1. Using elementary operations, write each of the following matrices in reduced row-echelon form:

a. $\left[\begin{array}{cc|c} -1 & 3 & 1 \\ 0 & -1 & 2 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -1 & 0 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} 1 & 1 & 4 & 2 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

d. $\left[\begin{array}{ccc|c} 0 & 0 & -1 & 4 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right]$

2. After you have written each of the matrices in question 1 in reduced row-echelon form, determine the solution to the related system of equations.

PART B

3. Solve each system of equations using the Gauss-Jordan method of elimination.

a. ① $x - y + z = 0$
② $x + 2y - z = 8$
③ $2x - 2y + z = -11$

b. ① $3x - 2y + z = 6$
② $x - 3y - 2z = -26$
③ $-x + y + z = 9$

c. ① $2x + 2y + 5z = -14$
② $-x + z = -5$
③ $y - z = 6$

d. ① $x - y - 3z = 3$
② $2x + 2y + z = -1$
③ $-x - y + z = -1$

e. ① $\frac{1}{2}x + 9y - z = 1$
② $x - 6y + z = -6$
③ $2x + 3y - z = -7$

f. ① $2x + 3y + 6z = 3$
② $x - y - z = 0$
③ $4x + 3y - 6z = 2$

4. Using either Gaussian elimination or the Gauss-Jordan method of elimination, solve the following systems of equations:

a. ① $2x + y - z = -6$
② $x - y + 2z = 9$
③ $-x + y + z = 9$

b. ① $x - y + z = -30$
② $-2x + y + 6z = 90$
③ $2x - y - z = -20$

5. a. Determine the value of k for which the following system will have an infinite number of solutions:

① $x + y + z = -1$
② $x - y + z = 2$
③ $3x - y + 3z = k$

b. For what value(s) of k will this system have no solutions?

c. Explain why it is not possible for this system to have a unique solution.

PART C

6. The following system of equations is called a homogeneous system. This term is used to describe a system of equations in which the number to the right of the equal sign in every equation equals 0.

① $2x - y + z = 0$
② $x + y + z = 0$
③ $5x - y + 3z = 0$

a. Explain why every homogeneous system of equations has at least one solution.

b. Write the related augmented matrix in reduced row-echelon form, and explain the meaning of this result.

7. Solve the following system of equations using the Gauss-Jordan method of elimination:

① $\frac{1}{x} - \frac{2}{y} + \frac{6}{z} = \frac{5}{6}$
② $\frac{2}{x} - \frac{3}{y} + \frac{12}{z} = 2$
③ $\frac{3}{x} + \frac{6}{y} + \frac{2}{z} = \frac{23}{6}$