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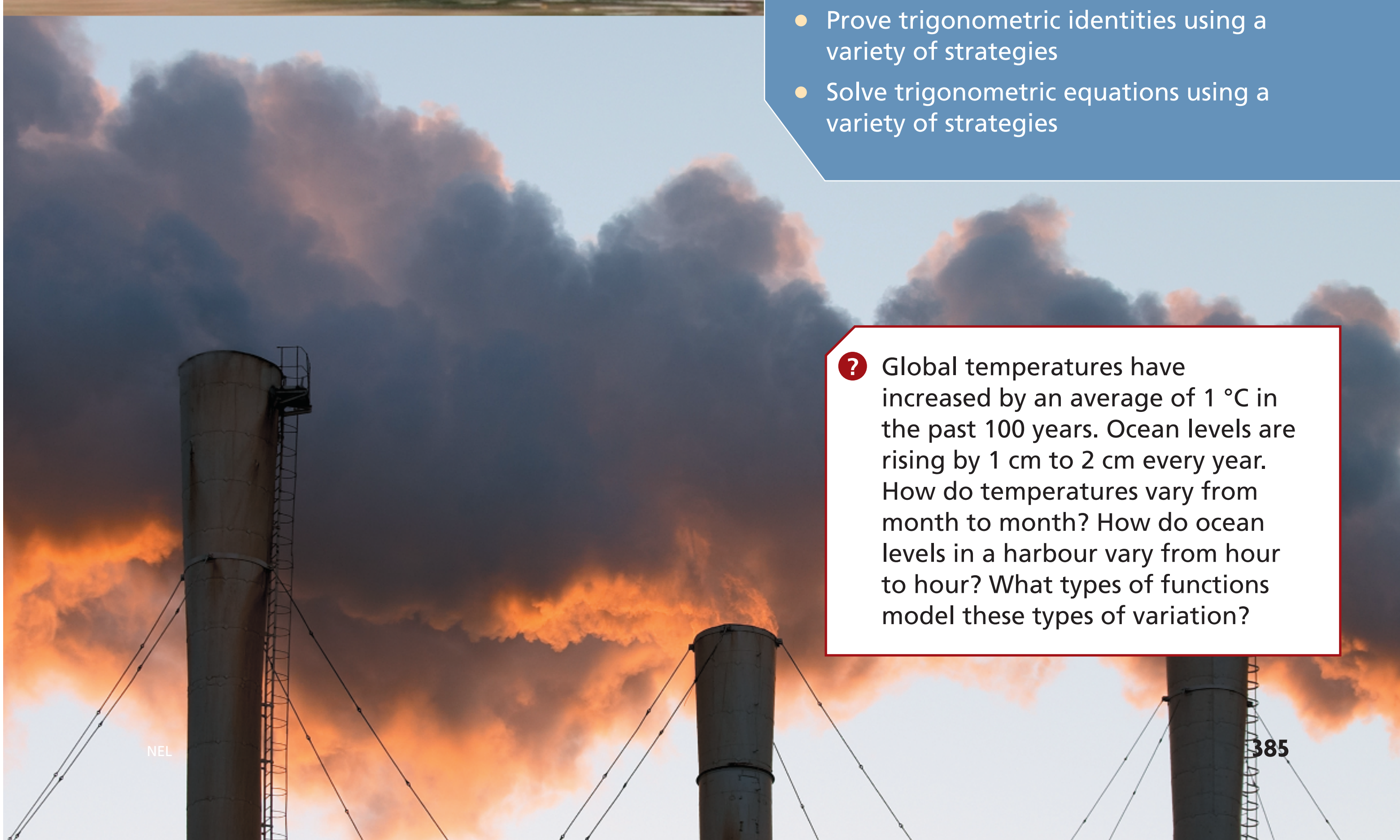
Chapter 7

Trigonometric Identities and Equations

GOALS

You will be able to

- Recognize equivalent trigonometric relationships
- Use compound angle formulas to determine the exact values of trigonometric ratios that involve sums, differences, and products of special angles
- Prove trigonometric identities using a variety of strategies
- Solve trigonometric equations using a variety of strategies



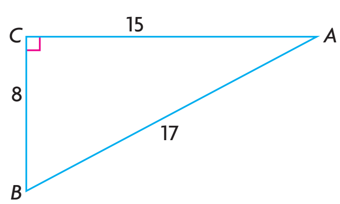
? Global temperatures have increased by an average of 1 °C in the past 100 years. Ocean levels are rising by 1 cm to 2 cm every year. How do temperatures vary from month to month? How do ocean levels in a harbour vary from hour to hour? What types of functions model these types of variation?

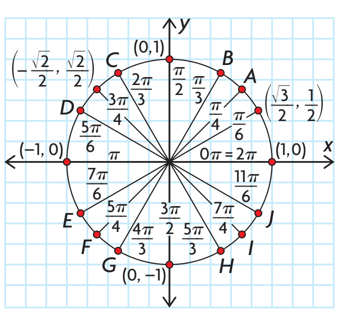
7 Getting Started

Study Aid

- For help, see the Review of Essential Skills found at the Nelson Advanced Functions website.

Question	Appendix/ Lesson
1	R-6
3	R-10
4, 5, 6	6.2
7	R-14
8	R-12





SKILLS AND CONCEPTS You Need

1. Solve each equation to two decimal places where necessary.

a) $3x - 7 = 5 - 9x$

b) $2(x + 3) - \frac{x}{4} = \frac{1}{2}$

c) $x^2 - 5x - 24 = 0$

d) $6x^2 + 11x = 10$

e) $x^2 + 2x - 1 = 0$

f) $3x^2 = 3x + 1$

2. Show that the line segment from $A(1, 0)$ to $B\left(2, \frac{1}{2}\right)$ is the same length as the line segment from $C\left(-\frac{1}{2}, 5\right)$ to $D(0, 6)$.

3. Given $\triangle ABC$ shown,

a) state the six trigonometric ratios for $\angle A$

b) determine the measure of $\angle A$ in **radians**, to one decimal place

c) determine the measure of $\angle B$ in **degrees**, to one decimal place

4. $P(-2, 2)$ lies on the terminal arm of an angle in **standard position**.

a) Sketch the **principal angle**, θ.

b) Determine the value of the **related acute angle** in radians.

c) Determine the value of θ in radians.

5. a) Determine the coordinates of each missing point on the unit circle shown.

b) Determine:

i) $\cos\left(\frac{3\pi}{4}\right)$ **ii)** $\sin\left(\frac{11\pi}{6}\right)$ **iii)** $\cos(\pi)$ **iv)** $\csc\left(\frac{\pi}{6}\right)$

6. Given $\tan x = -\frac{3}{4}$, where $0 \le x \le 2\pi$,

a) state the other five trigonometric ratios as fractions

b) determine the value(s) of x, to one decimal place

7. State whether each relationship is true or false.

a) $\tan\theta = \frac{\sin\theta}{\cos\theta}, \cos\theta \ne 0$

b) $\sin^2\theta + \cos^2\theta = 1$

c) $\sec\theta = \frac{1}{\sin\theta}, \sin\theta \ne 0$

d) $\cos^2\theta = \sin^2\theta - 1$

e) $1 + \tan^2\theta = \sec^2\theta$

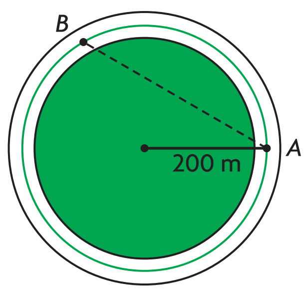
f) $\cot\theta = \frac{\cos\theta}{\sin\theta}, \sin\theta \ne 0$

8. Create a flow chart that shows how transformations can be used to sketch the graph of a sinusoidal function in the form $y = a\sin(k(x - d)) + c$.

APPLYING What You Know

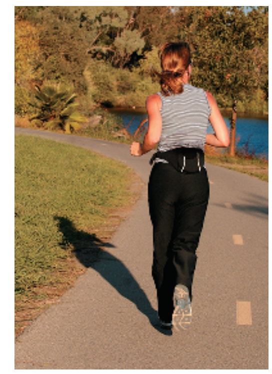
Going for a Run

Julie goes for a daily run in her local park. She parks her bike at point A and runs five times around the playing field, in a counterclockwise direction. The radius of the path that she runs is 200 m. This morning, she ran one-third of the way around the field, to point B, before realizing that she had left her heart-rate monitor on her bike. She ran in a straight line across the field, back to her bike, to get her monitor.



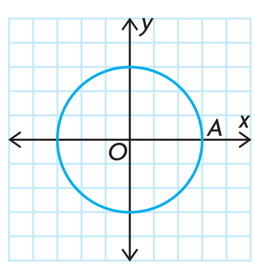
YOU WILL NEED

- graph paper



❓ How far did Julie run when she went across the field, back to her bike?

A. Draw a circle (centred at the origin) on graph paper, as shown, to represent the path that Julie runs. Write the coordinates of point A.



B. Mark point B one-third of the way around the circle from point A. What is the radian measure of $\angle AOB$? Write the coordinates $(r\cos\theta, r\sin\theta)$ of point B in terms of this angle.

C. Use the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, to calculate the distance from A to B.

D. What kind of triangle is $\triangle AOB$? What are the lengths of AO and BO?

E. Verify your answer in part C using the cosine law.

F. How far did Julie run when she went across the field, back to her bike, to get her heart-rate monitor?

7.1 Exploring Equivalent Trigonometric Functions

YOU WILL NEED

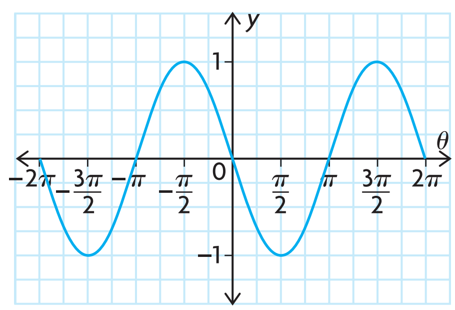
- graphing calculator or graphing software

GOAL

Identify equivalent trigonometric relationships.

EXPLORE the Math

What is a possible equation for the function shown?



Craig, Erin, Robin, and Sarah are comparing their answers to the question shown above.

Craig's function: $f(\theta) = -\sin \theta$

Erin's function: $g(\theta) = \sin (\theta + \pi)$

Robin's function: $h(\theta) = \sin (\theta - \pi)$

Sarah's function: $j(\theta) = \cos \left(\theta + \frac{\pi}{2}\right)$

Their teacher explains that they are all correct because they have written equivalent trigonometric functions.

? How can you verify that these equations are equivalent and identify other equivalent trigonometric expressions?

Tech Support

Scroll to the left of Y2, Y3, and Y4. Press Enter until the required graphing option appears.

A. Enter each student's function into Y1 to Y4 in the equation editor of a graphing calculator, using the settings shown. Use radian mode, and graph using the Zoom 7:Ztrig command. What do you notice?

B. Examine the table of values for each function. Are you convinced that the four functions are equivalent? Explain.

Creating equivalent expressions using the period of a function

C. Clear all functions from the calculator, and graph $f(\theta) = \sin \theta$. Using transformations, explain why $\sin (\theta + 2\pi) = \sin \theta$. Write a similar statement for $\cos \theta$ and another similar statement for $\tan \theta$.

D. Verify that your statements for part C are equivalent by graphing the corresponding pair of functions. Write similar statements for the reciprocal trigonometric functions, and verify them by graphing.

Creating equivalent expressions by classifying a function as odd or even

E. $f(\theta) = \cos\theta$ is an **even function** because its graph is symmetrical in the y-axis. Use transformations to explain why $\cos(-\theta) = \cos\theta$, and then verify by graphing.

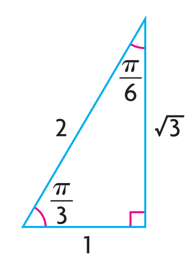
F. $f(\theta) = \sin\theta$ is an **odd function** because its graph has rotational symmetry about the origin. Use transformations to explain why $\sin(-\theta) = -\sin\theta$, and then verify by graphing.

G. Classify the tangent functions as even or odd. Based on your classification, write the corresponding pair of equivalent expressions.

Creating equivalent expressions using complementary angles

H. Determine the exact values of the six trigonometric ratios for each acute angle in the triangle shown. Record the values in a table like the one below. Describe any relationships that you see.

$\sin\left(\frac{\pi}{3}\right) =$	$\csc\left(\frac{\pi}{3}\right) =$	$\sin\left(\frac{\pi}{6}\right) =$	$\csc\left(\frac{\pi}{6}\right) =$
$\cos\left(\frac{\pi}{3}\right) =$	$\sec\left(\frac{\pi}{3}\right) =$	$\cos\left(\frac{\pi}{6}\right) =$	$\sec\left(\frac{\pi}{6}\right) =$
$\tan\left(\frac{\pi}{3}\right) =$	$\cot\left(\frac{\pi}{3}\right) =$	$\tan\left(\frac{\pi}{6}\right) =$	$\cot\left(\frac{\pi}{6}\right) =$



I. Repeat part H for a right triangle in which one acute angle is $\frac{\pi}{8}$ and the other acute angle is $\frac{3\pi}{8}$. Use a calculator to determine the approximate values of the six trigonometric ratios for each of these acute angles. Record the values in a table like the one above. How do the relationships in this table compare with the relationships in the table you completed for part H?

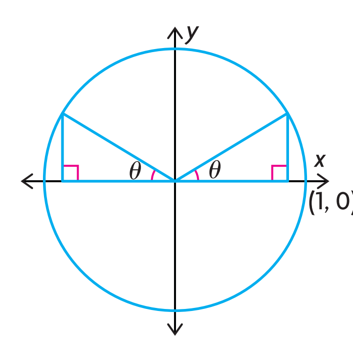
J. Any right triangle, where θ is the measure of one of the acute angles, has a complementary angle of $\left(\frac{\pi}{2} - \theta\right)$ for the other angle. Explain how you know that the cofunction **identity** $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ is true.

K. Write all the other cofunction identities between θ and $\left(\frac{\pi}{2} - \theta\right)$ based on the relationships in parts H and I. Verify each identity by graphing the corresponding functions on the graphing calculator.

Creating equivalent expressions using the principal and related angles

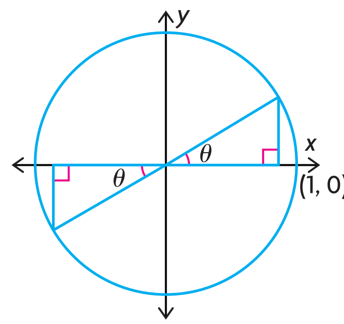
L. Explain how you can tell, from this diagram of a unit circle, that

i) $\sin(\pi - \theta) = \sin\theta$
ii) $\cos(\pi - \theta) = -\cos\theta$
iii) $\tan(\pi - \theta) = -\tan\theta$

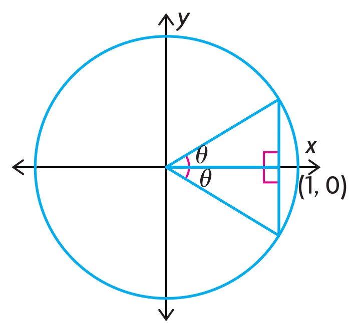


M. Write similar statements for the following diagrams.

i)



ii)



N. Summarize the strategies you used to identify and verify equivalent trigonometric expressions. Make a list of all the equivalent expressions you found.

Reflecting

O. How does a graphing calculator help you investigate the possible equivalence of two trigonometric expressions?

P. How can transformations be used to identify and confirm equivalent trigonometric expressions?

Q. How can the relationship between the acute angles in a right triangle be used to identify and confirm equivalent trigonometric expressions?

R. How can the relationship between a principal angle in standard position and the related acute angle be used to identify and confirm equivalent trigonometric expressions?

In Summary

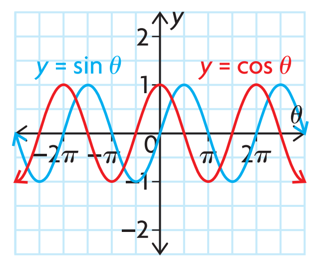
Key Ideas

- Because of their periodic nature, there are many equivalent trigonometric expressions.
- Two expressions may be equivalent if the graphs created by a graphing calculator of their corresponding functions coincide, producing only one visible graph over the entire domain of both functions. To demonstrate equivalency requires additional reasoning about the properties of both graphs.

Need to Know

- Horizontal translations that involve multiples of the period of a trigonometric function can be used to obtain two equivalent functions with the same graph. For example, the sine function has a period of 2π, so the graphs of $f(\theta) = \sin \theta$ and $f(\theta) = \sin(\theta + 2\pi)$ are the same. Therefore, $\sin \theta = \sin(\theta + 2\pi)$.
- Horizontal translations of $\frac{\pi}{2}$ that involve both a sine function and a cosine function can be used to obtain two equivalent functions with the same graph. Translating the cosine function $\frac{\pi}{2}$ to the right $\left(f(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)\right)$ results in the graph of the sine function, $f(\theta) = \sin \theta$.

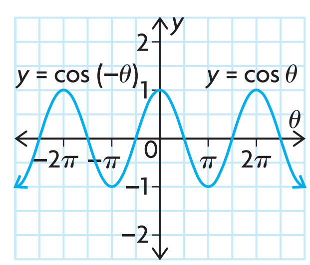
  Similarly, translating the sine function $\frac{\pi}{2}$ to the left $\left(f(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)\right)$ results in the graph of the cosine function, $f(\theta) = \cos \theta$.



$\sin \theta = \cos\left(\theta - \frac{\pi}{2}\right)$

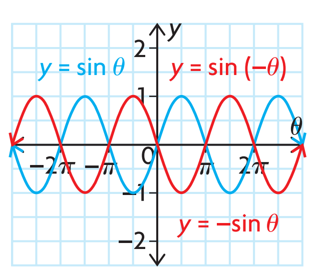
$\sin\left(\theta + \frac{\pi}{2}\right) = \cos \theta$

- Since $f(\theta) = \cos \theta$ is an even function, reflecting its graph across the y-axis results in two equivalent functions with the same graph.

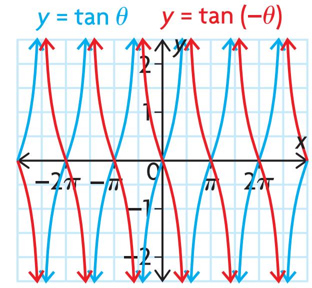


$\cos \theta = \cos(-\theta)$

- $f(\theta) = \sin \theta$ and $f(\theta) = \tan \theta$ are odd and have the property of rotational symmetry about the origin. Reflecting these functions across both the x-axis and the y-axis produces the same effect as rotating the function through 180° about the origin. Thus, the same graph is produced.



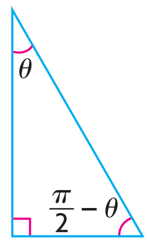
$\sin(-\theta) = -\sin \theta$



$\tan(-\theta) = -\tan \theta$

(continued)

- The cofunction identities describe trigonometric relationships between the complementary angles θ and $\left(\frac{\pi}{2} - \theta\right)$ in a right triangle.



$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

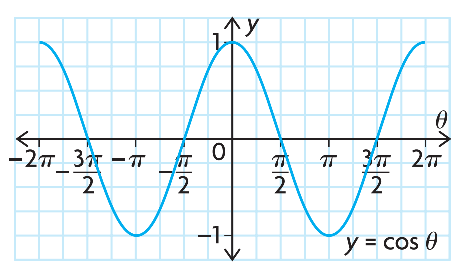
$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$$

- You can identify equivalent trigonometric expressions by comparing principal angles drawn in standard position in quadrants II, III, and IV with their related acute angle, θ, in quadrant I.

Principal Angle in Quadrant II	Principal Angle in Quadrant III	Principal Angle in Quadrant IV
$\sin(\pi - \theta) = \sin\theta$	$\sin(\pi + \theta) = -\sin\theta$	$\sin(2\pi - \theta) = -\sin\theta$
$\cos(\pi - \theta) = -\cos\theta$	$\cos(\pi + \theta) = -\cos\theta$	$\cos(2\pi - \theta) = \cos\theta$
$\tan(\pi - \theta) = -\tan\theta$	$\tan(\pi + \theta) = \tan\theta$	$\tan(2\pi - \theta) = -\tan\theta$

FURTHER Your Understanding

1. **a)** Use transformations and the cosine function to write three equivalent expressions for the following graph.



b) Use transformations and a different trigonometric function to write three equivalent expressions for the graph.

2. **a)** Classify the reciprocal trigonometric functions as odd or even, and then write the corresponding equation.

b) Use transformations to explain why each equation is true.

3. Use the cofunction identities to write an expression that is equivalent to each of the following expressions.

a) $\sin\frac{\pi}{6}$ **c)** $\tan\frac{3\pi}{8}$ **e)** $\sin\frac{\pi}{8}$

b) $\cos\frac{5\pi}{12}$ **d)** $\cos\frac{5\pi}{16}$ **f)** $\tan\frac{\pi}{6}$

4. a) Write the cofunction identities for the reciprocal trigonometric functions.

b) Use transformations to explain why each identity is true.

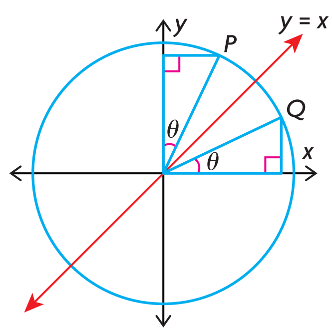
5. Write an expression that is equivalent to each of the following expressions, using the related acute angle.

a) $\sin \frac{7\pi}{8}$ **c)** $\tan \frac{5\pi}{4}$ **e)** $\sin \frac{13\pi}{8}$

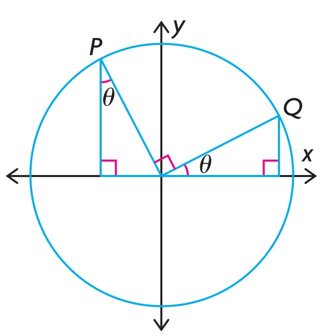
b) $\cos \frac{13\pi}{12}$ **d)** $\cos \frac{11\pi}{6}$ **f)** $\tan \frac{5\pi}{3}$

6. Show that each equation is true, using the given diagram.

a) $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$



b) $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta$



7. State whether each of the following are true or false. For those that are false, justify your decision.

a) $\cos (\theta + 2\pi) = \cos \theta$ **d)** $\tan (\pi - \theta) = \tan \theta$

b) $\sin (\pi - \theta) = -\sin \theta$ **e)** $\cot\left(\frac{\pi}{2} + \theta\right) = \tan \theta$

c) $\cos \theta = -\cos (\theta + 4\pi)$ **f)** $\sin (\theta + 2\pi) = \sin (-\theta)$

7.2 Compound Angle Formulas

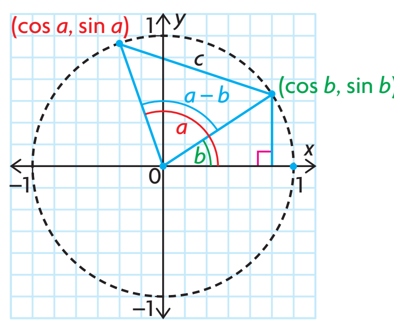
GOAL

Verify and use compound angle formulas.

INVESTIGATE the Math

compound angle
an angle that is created by adding or subtracting two or more angles

The cosine of the compound angle $(a - b)$ can be expressed in terms of the sines and cosines of a and b. Consider the following unit circle diagram:



By the cosine law, $c^2 = 1^2 + 1^2 - 2(1)(1)\cos(a - b)$

① $c^2 = 2 - 2\cos(a - b)$

However, c has endpoints of $(\cos a, \sin a)$ and $(\cos b, \sin b)$.

By the distance formula, $c = \sqrt{(\sin a - \sin b)^2 + (\cos a - \cos b)^2}$

Squaring both sides,

$c^2 = (\sin a - \sin b)^2 + (\cos a - \cos b)^2$

$c^2 = \sin^2 a - 2\sin a \sin b + \sin^2 b + \cos^2 a - 2\cos a \cos b + \cos^2 b$

$c^2 = \sin^2 a + \cos^2 a - 2\sin a \sin b - 2\cos a \cos b + \sin^2 b + \cos^2 b$

$c^2 = 1 - 2\sin a \sin b - 2\cos a \cos b + 1$

② $c^2 = 2 - 2\sin a \sin b - 2\cos a \cos b$

Equating ① and ②,

$2 - 2\cos(a - b) = 2 - 2\sin a \sin b - 2\cos a \cos b$

Solving for $\cos(a - b)$,

$\cos(a - b) = \sin a \sin b + \cos a \cos b$

? How can other formulas be developed to relate the primary trigonometric ratios of a compound angle to the trigonometric ratios of each angle in the compound angle?

A. Use a calculator and the special triangles to verify that the subtraction formula for cosine works if $a = 45°$ and $b = 30°$. Repeat for $a = \frac{\pi}{3}$ and $b = \frac{\pi}{6}$.

B. Use the subtraction formula for cosine to obtain an addition formula for cosine, $\cos(a + b)$, as follows:
 i) Rewrite the compound angle equation for $\cos(a - b)$.
 ii) Replace b with $(-b)$, and derive an equation for $\cos(a + b)$.
 iii) Simplify this equation, using your knowledge of even and odd functions, to write $\sin(-b)$ in terms of $\sin b$, and $\cos(-b)$ in terms of $\cos b$.

C. Use a calculator and the special triangles to verify your addition formula for cosine if $a = \frac{\pi}{3}$ and $b = \frac{\pi}{4}$.

D. To find an addition formula for sine, $\sin(a + b)$, use the cofunction identity $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$.
 i) Write $\sin(a + b) = \cos\left(\frac{\pi}{2} - (a + b)\right) = \cos\left(\left(\frac{\pi}{2} - a\right) - b\right)$.
 ii) Use the subtraction formula for cosine to expand and simplify this formula.

E. Use a calculator and the special triangles to verify your addition formula for sine by substituting $a = \frac{\pi}{3}$ and $b = \frac{\pi}{4}$.

F. Determine and verify a subtraction formula for sine, $\sin(a - b)$, using the addition formula you found in part D and the strategy you used in part B.

G. Recall that $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Use this identity to determine addition and subtraction formulas for $\tan(a + b)$ and $\tan(a - b)$. Use a calculator and the special triangles to verify your formulas if $a = \frac{\pi}{6}$ and $b = \frac{\pi}{4}$.

H. Make a list of all the compound angle formulas that you determined.

Reflecting

I. How did you use equivalent trigonometric expressions to simplify formulas in parts B, D, F, and G?

J. How did you use the special triangles to verify the addition and subtraction formulas you determined?

APPLY the Math

EXAMPLE 1 Selecting a strategy to determine the exact value of a trigonometric ratio

Determine the exact value of

a) $\cos(15°)$ **b)** $\tan\left(-\frac{5\pi}{12}\right)$

Solution

a) $\cos(15°)$

$= (\cos 45° - 30°)$

15° = 45° − 30°, so 15° can be expressed as the compound angle (45° − 30°).

$\cos(a - b)$

$= (\cos a)(\cos b) + (\sin a)(\sin b)$

$= (\cos 45°)(\cos 30°) + (\sin 45°)(\sin 30°)$

$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$

$= \frac{\sqrt{3} + 1}{2\sqrt{2}}$

Use the subtraction formula for cosine to expand this expression where $a = 45°$ and $b = 30°$. Then use the special triangles to evaluate it.

b) $\tan\left(-\frac{5\pi}{12}\right)$

$= \tan\left(-\frac{\pi}{4} - \frac{\pi}{6}\right)$

$-\frac{5\pi}{12} = \frac{-5(180°)}{12} = -75°$

$-75° = -45° - 30°$

So $-\frac{5\pi}{12}$ can be expressed as the compound angle $\left(-\frac{\pi}{4} - \frac{\pi}{6}\right)$.

$\tan(a - b)$

$= \frac{\tan a - \tan b}{1 + \tan a \tan b}$

$= \frac{\tan\left(-\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{6}\right)}{1 + \tan\left(-\frac{\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)}$

Use the subtraction formula for tangent to expand this expression where $a = -\frac{\pi}{4}$ and $b = \frac{\pi}{6}$. Then use the special triangles to evaluate it.

$= \frac{-1 - \frac{1}{\sqrt{3}}}{1 + (-1)\left(\frac{1}{\sqrt{3}}\right)}$

Simplify.

$= \frac{\frac{-\sqrt{3} - 1}{\sqrt{3}}}{\frac{\sqrt{3} - 1}{\sqrt{3}}}$

Divide by multiplying by the reciprocal.

$= \frac{-\sqrt{3} - 1}{\sqrt{3} - 1}$

Compound angle formulas can be used, both forward and backward, to evaluate and simplify trigonometric expressions.

EXAMPLE 2 Using compound angle formulas to simplify trigonometric expressions

Simplify each expression.

a) $\cos\frac{7\pi}{12}\cos\frac{5\pi}{12} + \sin\frac{7\pi}{12}\sin\frac{5\pi}{12}$

b) $\sin 2x \cos x - \cos 2x \sin x$

Solution

a) $\cos(a - b)$
$= (\cos a)(\cos b) + (\sin a)(\sin b)$

The expression given is the right side of the subtraction formula for cosine, where $a = \frac{7\pi}{12}$ and $b = \frac{5\pi}{12}$.

$\cos\frac{7\pi}{12}\cos\frac{5\pi}{12} + \sin\frac{7\pi}{12}\sin\frac{5\pi}{12}$
$= \cos\left(\frac{7\pi}{12} - \frac{5\pi}{12}\right)$
$= \cos\frac{\pi}{6}$
$= \frac{\sqrt{3}}{2}$

$\frac{7\pi}{12} - \frac{5\pi}{12} = \frac{2\pi}{12}$
$= \frac{\pi}{6}$

Use a special triangle to evaluate $\cos\frac{\pi}{6}$.

b) $\sin(a - b)$
$= (\sin a)(\cos b) - (\cos a)(\sin b)$

The expression given is the right side of the subtraction formula for sine, where $a = 2x$ and $b = x$.

$\sin 2x \cos x - \cos 2x \sin x$
$= \sin(2x - x)$
$= \sin x$

By expressing an angle as a sum or difference of angles in the special triangles, exact values of other angles can be determined.

EXAMPLE 3 Calculating trigonometric ratios of compound angles

Evaluate $\sin (a + b)$, where a and b are obtuse angles; $\sin a = \frac{3}{5}$ and $\sin b = \frac{5}{13}$.

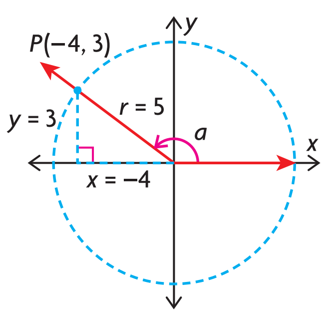
Solution

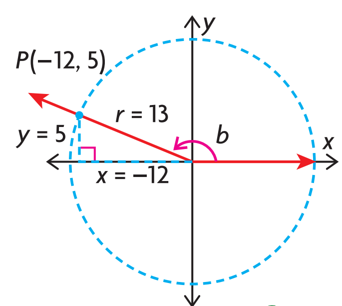
$$\sin a = \frac{3}{5} = \frac{y}{r} \quad \text{and} \quad \sin b = \frac{5}{13} = \frac{y}{r}$$

$$x^2 + y^2 = r^2 \qquad x^2 + y^2 = r^2$$
$$x^2 + 3^2 = 5^2 \qquad x^2 + 5^2 = 13^2$$
$$x^2 = 25 - 9 \qquad x^2 = 169 - 25$$
$$x = \pm\sqrt{16} \qquad x = \pm\sqrt{144}$$
$$x = -4 \qquad x = -12$$

Use the Pythagorean theorem to determine the x-coordinate of each point on the terminal arm. Since a and b are obtuse angles, their terminal arms lie in the second quadrant, where $\frac{\pi}{2} < a < \pi$ and $\frac{\pi}{2} < b < \pi$. In the second quadrant, x must be negative.

Sketch each angle in standard position.





$$\cos a = \frac{x}{r} = -\frac{4}{5} \qquad \cos b = \frac{x}{r} = -\frac{12}{13}$$

To determine $\sin (a + b)$, the sine and cosine of both a and b are required. Determine the cosine of a and b.

$$\begin{aligned}\sin (a + b) &= (\sin a)(\cos b) + (\cos a)(\sin b)\\ &= \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) + \left(-\frac{4}{5}\right)\left(\frac{5}{13}\right)\\ &= -\frac{36}{65} - \frac{20}{65}\\ &= -\frac{56}{65}\end{aligned}$$

Substitute the required trigonometric ratios into the compound angle formula for $\sin (a + b)$, and then evaluate.

Compound angle formulas can also be used to prove the equivalence of trigonometric expressions.

EXAMPLE 4 Identifying equivalent trigonometric expressions using compound angle formulas

Use compound angle formulas to show that $\sin(x - \pi)$, $\sin(x + \pi)$, and $\cos\left(x + \frac{\pi}{2}\right)$ are equivalent trigonometric expressions.

Solution

$$\begin{aligned}\sin(x - \pi) &= (\sin x)(\cos \pi) - (\cos x)(\sin \pi)\\ &= (\sin x)(-1) - (\cos x)(0)\\ &= -\sin x\end{aligned}$$

Use the subtraction formula for sine.

$$\begin{aligned}\sin(x + \pi) &= (\sin x)(\cos \pi) + (\cos x)(\sin \pi)\\ &= (\sin x)(-1) + (\cos x)(0)\\ &= -\sin x\end{aligned}$$

Use the addition formula for sine.

$$\begin{aligned}\cos\left(x + \frac{\pi}{2}\right) &= (\cos x)\left(\cos\frac{\pi}{2}\right) - (\sin x)\left(\sin\frac{\pi}{2}\right)\\ &= (\cos x)(0) - (\sin x)(1)\\ &= -\sin x\end{aligned}$$

Use the addition formula for cosine.

$$\sin(x - \pi) = \sin(x + \pi) = \cos\left(x + \frac{\pi}{2}\right)$$

They are all equivalent to the same expression, $-\sin x$.

In Summary

Key Idea

- The trigonometric ratios of compound angles are related to the trigonometric ratios of their component angles by the following compound angle formulas.

Addition Formulas	Subtraction Formulas
$\sin(a + b) = \sin a \cos b + \cos a \sin b$	$\sin(a - b) = \sin a \cos b - \cos a \sin b$
$\cos(a + b) = \cos a \cos b - \sin a \sin b$	$\cos(a - b) = \cos a \cos b + \sin a \sin b$
$\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$	$\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$

Need to Know

- You can use compound angle formulas to obtain exact values for trigonometric ratios.
- You can use compound angle formulas to show that some trigonometric expressions are equivalent.

CHECK Your Understanding

1. Rewrite each expression as a single trigonometric ratio.

a) $\sin a \cos 2a + \cos a \sin 2a$

b) $\cos 4x \cos 3x - \sin 4x \sin 3x$

2. Rewrite each expression as a single trigonometric ratio, and then evaluate the ratio.

a) $\dfrac{\tan 170° - \tan 110°}{1 + \tan 170° \tan 110°}$

b) $\cos \dfrac{5\pi}{12} \cos \dfrac{\pi}{12} + \sin \dfrac{5\pi}{12} \sin \dfrac{\pi}{12}$

3. Express each angle as a compound angle, using a pair of angles from the special triangles.

a) $75°$ **c)** $-\dfrac{\pi}{6}$ **e)** $105°$

b) $-15°$ **d)** $\dfrac{\pi}{12}$ **f)** $\dfrac{5\pi}{6}$

4. Determine the exact value of each trigonometric ratio.

a) $\sin 75°$ **c)** $\tan \dfrac{5\pi}{12}$ **e)** $\cos 105°$

b) $\cos 15°$ **d)** $\sin \left(-\dfrac{\pi}{12}\right)$ **f)** $\tan \dfrac{23\pi}{12}$

PRACTISING

5. Use the appropriate compound angle formula to determine the exact value of each expression.
K

a) $\sin \left(\pi + \dfrac{\pi}{6}\right)$ **c)** $\tan \left(\dfrac{\pi}{4} + \pi\right)$ **e)** $\tan \left(\dfrac{\pi}{3} - \dfrac{\pi}{6}\right)$

b) $\cos \left(\pi - \dfrac{\pi}{4}\right)$ **d)** $\sin \left(-\dfrac{\pi}{2} + \dfrac{\pi}{3}\right)$ **f)** $\cos \left(\dfrac{\pi}{2} + \dfrac{\pi}{3}\right)$

6. Use the appropriate compound angle formula to create an equivalent expression.

a) $\sin (\pi + x)$ **c)** $\cos \left(x + \dfrac{\pi}{2}\right)$ **e)** $\sin (x - \pi)$

b) $\cos \left(x + \dfrac{3\pi}{2}\right)$ **d)** $\tan (x + \pi)$ **f)** $\tan (2\pi - x)$

7. Use transformations to explain why each expression you created in question 6 is equivalent to the given expression.

8. Determine the exact value of each trigonometric ratio.

a) $\cos 75°$ **c)** $\cos \frac{11\pi}{12}$ **e)** $\tan \frac{7\pi}{12}$

b) $\tan(-15°)$ **d)** $\sin \frac{13\pi}{12}$ **f)** $\tan \frac{-5\pi}{12}$

9. If $\sin x = \frac{4}{5}$ and $\sin y = -\frac{12}{13}$, $0 < x < \frac{\pi}{2}$, $\frac{3\pi}{2} < y < 2\pi$, evaluate

a) $\cos(x + y)$ **c)** $\cos(x - y)$ **e)** $\tan(x + y)$

b) $\sin(x + y)$ **d)** $\sin(x - y)$ **f)** $\tan(x - y)$

10. A α and β are acute angles in quadrant I, with $\sin \alpha = \frac{7}{25}$ and $\cos \beta = \frac{5}{13}$. Without using a calculator, determine the values of $\sin(\alpha + \beta)$ and $\tan(\alpha + \beta)$.

11. Use compound angle formulas to verify each of the following cofunction identities.

a) $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ **b)** $\cos x = \sin\left(\frac{\pi}{2} - x\right)$

12. Simplify each expression.

a) $\sin(\pi + x) + \sin(\pi - x)$ **b)** $\cos\left(x + \frac{\pi}{3}\right) - \sin\left(x + \frac{\pi}{6}\right)$

13. T Simplify $\frac{\sin(f + g) + \sin(f - g)}{\cos(f + g) + \cos(f - g)}$.

14. C Create a flow chart to show how you would evaluate $\cos(a + b)$, given the values of $\sin a$ and $\sin b$, if both a and $b \in \left[0, \frac{\pi}{2}\right]$.

15. List the compound angle formulas you used in this lesson, and look for similarities and differences. Explain how you can use these similarities and differences to help you remember the formulas.

Extending

16. Prove $\sin C + \sin D = 2 \sin\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right)$.

17. Determine $\cot(x + y)$ in terms of $\cot x$ and $\cot y$.

18. Prove $\cos C + \cos D = 2 \cos\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right)$.

19. Prove $\cos C - \cos D = -2 \sin\left(\frac{C + D}{2}\right)\sin\left(\frac{C - D}{2}\right)$.

7.3 Double Angle Formulas

YOU WILL NEED

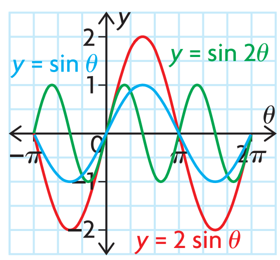
- graphing calculator

GOAL

Develop and use double angle formulas.

INVESTIGATE the Math

From your work with graphs of trigonometric functions, you already know that $f(\theta) = \sin 2\theta$ is not the same as $f(\theta) = 2 \sin \theta$.



$f(\theta) = \sin 2\theta$ is the graph of $y = \sin \theta$ compressed horizontally by a factor of $\frac{1}{2}$.

$f(\theta) = 2 \sin \theta$ is the graph of $y = \sin \theta$ stretched vertically by a factor of 2.

? How are the trigonometric ratios of an angle that has been doubled to 2θ related to the trigonometric ratios of the original angle θ?

A. Given $\sin 2\theta = \sin(\theta + \theta)$, use the appropriate compound angle formula to expand $\sin(\theta + \theta)$. Simplify both sides to develop a formula for $\sin 2\theta$.

B. Verify your double angle formula for sine by graphing each side as a function on a graphing calculator and examining the tables of values.

C. Verify that your double angle formula for sine works by evaluating both sides of the formula for $\theta = 45°$. Repeat for $\theta = \frac{\pi}{6}$.

D. Repeat parts A to C to develop a double angle formula for $\cos 2\theta$.

E. Use the identity $\sin^2 \theta + \cos^2 \theta = 1$ to eliminate $\sin \theta$ from the right side of your formula in part D. Verify that your new formula is correct by graphing and by substitution, as before.

F. Repeat part E, but this time eliminate $\cos\theta$ on the right side to develop an equivalent expression in terms of $\sin\theta$.

G. Repeat parts A to C to develop a double angle formula for $\tan 2\theta$.

H. Make a list of all the double angle formulas you developed.

Reflecting

I. How did you use compound angle formulas to develop double angle formulas?

J. Why were you able to develop three different formulas for $\cos 2\theta$?

K. How might you develop formulas for $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$?

APPLY the Math

EXAMPLE 1 Using double angle formulas to simplify and evaluate expressions

Simplify each of the following expressions and then evaluate.

a) $2\sin\frac{\pi}{8}\cos\frac{\pi}{8}$

b) $\dfrac{2\tan\frac{\pi}{6}}{1-\tan^2\frac{\pi}{6}}$

Solution

a)
$$2\sin x\cos x = \sin 2x$$
$$2\sin\frac{\pi}{8}\cos\frac{\pi}{8} = \sin 2\left(\frac{\pi}{8}\right)$$
$$= \sin\frac{\pi}{4}$$
$$= \frac{1}{\sqrt{2}}$$

This expression is the right side of the double angle formula for sine.

In this expression, $x = \frac{\pi}{8}$.

Use the special triangles to evaluate.

b)
$$\frac{2\tan x}{1-\tan^2 x} = \tan 2x,\text{ where }\tan x \neq \pm 1$$
$$\frac{2\tan\frac{\pi}{6}}{1-\tan^2\frac{\pi}{6}} = \tan 2\left(\frac{\pi}{6}\right)$$
$$= \tan\frac{\pi}{3}$$
$$= \sqrt{3}$$

This expression is similar to the right side of the double angle formula for tangent. In this expression, $x = \frac{\pi}{6}$.

Use the special triangles to evaluate $\tan\frac{\pi}{3}$.

If you know one of the primary trigonometric ratios for any angle, then you can determine the other two. You can then determine the primary trigonometric ratios for this angle doubled.

EXAMPLE 2 Selecting a strategy to determine the value of trigonometric ratios for a double angle

If $\cos\theta = -\frac{2}{3}$ and $0 \leq \theta \leq 2\pi$, determine the value of $\cos 2\theta$ and $\sin 2\theta$.

Solution

$$\cos 2\theta = 2\cos^2\theta - 1$$

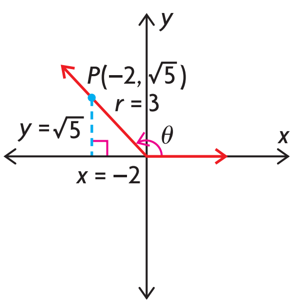
There are three double angle formulas for cosine. Since the value of $\cos\theta$ is given, choose the formula that is strictly in terms of cosine.

$$= 2\left(-\frac{2}{3}\right)^2 - 1$$

Let $\cos\theta = -\frac{2}{3}$ and evaluate.

$$= 2\left(\frac{4}{9}\right) - 1$$

$$= -\frac{1}{9}$$



If θ is in quadrant II and $\cos\theta = \frac{x}{r} = -\frac{2}{3}$, then

$$x^2 + y^2 = r^2$$

Since cosine is negative, the terminal arm of θ can lie in quadrant II or quadrant III. Since $r > 0$, x must be negative. Use the Pythagorean theorem to calculate y.

$$(-2)^2 + y^2 = 3^2$$

$$4 + y^2 = 9$$

$$y^2 = \pm 5$$

In quadrant II, y is positive.

$$y = \sqrt{5}$$

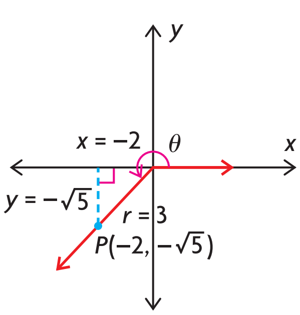
So, $\sin\theta = \frac{y}{r} = \frac{\sqrt{5}}{3}$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= 2\left(\frac{\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right)$$

Use the double angle formula for sine, and replace $\sin\theta$ and $\cos\theta$ with the values calculated.

$$= -\frac{4\sqrt{5}}{9}$$



If θ is in quadrant III, $y = -\sqrt{5}$.

Using the value of y that was calculated above, y is negative in quadrant III.

So $\sin\theta = \frac{y}{r} = \frac{-\sqrt{5}}{3}$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Use the double angle formula for sine, and replace $\sin\theta$ and $\cos\theta$ with the values calculated.

$$= 2\left(\frac{-\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right)$$

$$= \frac{4\sqrt{5}}{9}$$

EXAMPLE 3 Selecting a strategy to determine the primary trigonometric ratios for a double angle

Given $\tan\theta = -\frac{3}{4}$, where $\frac{3\pi}{2} \le \theta \le 2\pi$, calculate the value of $\cos 2\theta$.

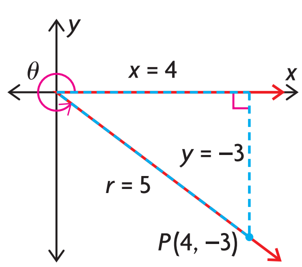
Solution

$$\begin{aligned} \tan\theta &= \frac{y}{x} = \frac{-3}{4} \\ x^2 + y^2 &= r^2 \\ 4^2 + (-3)^2 &= r^2 \\ 16 + 9 &= r^2 \\ \pm\sqrt{25} &= r \\ 5 &= r \end{aligned}$$

Since $\frac{3\pi}{2} \le \theta \le 2\pi$, the terminal arm of the angle lies in quadrant IV. Therefore, x is positive and y is negative. Use the Pythagorean theorem to determine r.

Since r is always positive, $r > 0$.

Draw θ in standard position.



$\sin\theta = \frac{y}{r} = \frac{-3}{5}$ and $\cos\theta = \frac{x}{r} = \frac{4}{5}$

Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$, determine the values of $\sin\theta$ and $\cos\theta$.

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \left(\frac{4}{5}\right)^2 - \left(\frac{-3}{5}\right)^2 \\ &= \frac{16}{25} - \frac{9}{25} \\ &= \frac{7}{25} \end{aligned}$$

Use one of the double angle formulas for $\cos 2\theta$, and substitute the values of $\sin\theta$ and $\cos\theta$.

The double angle formulas can be used to create other equivalent trigonometric relationships.

EXAMPLE 4 Using reasoning to derive other formulas from the double angle formulas

Develop a formula for $\sin \frac{x}{2}$.

Solution

$$\cos 2x = 1 - 2\sin^2 x$$
$$\cos 2\left(\frac{x}{2}\right) = 1 - 2\sin^2\left(\frac{x}{2}\right)$$

Since $\cos x = \cos 2\left(\frac{x}{2}\right)$, replace x with $\frac{x}{2}$ in the cosine double angle formula that only involves sine.

$$\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right)$$
$$2\sin^2\left(\frac{x}{2}\right) = 1 - \cos x$$
$$\sin^2\left(\frac{x}{2}\right) = \frac{1 - \cos x}{2}$$
$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos x}{2}}$$

Solve for $\sin\left(\frac{x}{2}\right)$ as follows:

- Add $2\sin^2\left(\frac{x}{2}\right)$ to both sides.
- Subtract $\cos x$ from both sides.
- Divide both sides by 2.
- Take the square root of both sides.

In Summary

Key Idea

- The double angle formulas show how the trigonometric ratios for a double angle, 2θ, are related to the trigonometric ratios for the original angle, θ.

Double Angle Formula for Sine

$\sin 2\theta = 2\sin\theta\cos\theta$

Double Angle Formulas for Cosine

$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$\cos 2\theta = 2\cos^2\theta - 1$
$\cos 2\theta = 1 - 2\sin^2\theta$

Double Angle Formula for Tangent

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Need to Know

- The double angle formulas can be derived from the appropriate compound angle formulas.
- You can use the double angle formulas to simplify expressions and to calculate exact values.
- The double angle formulas can be used to develop other equivalent formulas.

CHECK Your Understanding

1. Express each of the following as a single trigonometric ratio.

a) $2 \sin 5x \cos 5x$

b) $\cos^2 \theta - \sin^2 \theta$

c) $1 - 2 \sin^2 3x$

d) $\dfrac{2 \tan 4x}{1 - \tan^2 4x}$

e) $4 \sin \theta \cos \theta$

f) $2 \cos^2 \dfrac{\theta}{2} - 1$

2. Express each of the following as a single trigonometric ratio and then evaluate.

a) $2 \sin 45° \cos 45°$

b) $\cos^2 30° - \sin^2 30°$

c) $2 \sin \dfrac{\pi}{12} \cos \dfrac{\pi}{12}$

d) $\cos^2 \dfrac{\pi}{12} - \sin^2 \dfrac{\pi}{12}$

e) $1 - 2 \sin^2 \dfrac{3\pi}{8}$

f) $2 \tan 60° \cos^2 60°$

3. Use a double angle formula to rewrite each trigonometric ratio.

a) $\sin 4\theta$

b) $\cos 3x$

c) $\tan x$

d) $\cos 6\theta$

e) $\sin x$

f) $\tan 5\theta$

PRACTISING

4. **K** Determine the values of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, given $\cos \theta = \frac{3}{5}$ and $0 \leq \theta \leq \frac{\pi}{2}$.

5. Determine the values of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, given $\tan \theta = -\frac{7}{24}$ and $\frac{\pi}{2} \leq \theta \leq \pi$.

6. Determine the values of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, given $\sin \theta = -\frac{12}{13}$ and $\frac{3\pi}{2} \leq \theta \leq 2\pi$.

7. Determine the values of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, given $\cos \theta = -\frac{4}{5}$ and $\frac{\pi}{2} \leq \theta \leq \pi$.

8. **A** Determine the value of a in the following equation: $2 \tan x - \tan 2x + 2a = 1 - \tan 2x \tan^2 x$.

9. Jim needs to find the sine of $\frac{\pi}{8}$. If he knows that $\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$, how can he use this fact to find the sine of $\frac{\pi}{8}$? What is his answer?

10. Marion needs to find the cosine of $\frac{\pi}{12}$. If she knows that $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, how can she use this fact to find the cosine of $\frac{\pi}{12}$? What is her answer?

11. T **a)** Use a double angle formula to develop a formula for $\sin 4x$ in terms of x.

b) Use the formula you developed in part a) to verify that $\sin \frac{2\pi}{3} = \sin \frac{8\pi}{3}$.

12. Use the appropriate compound angle formula and double angle formula to develop a formula for

a) $\sin 3\theta$ in terms of $\cos \theta$ and $\sin \theta$

b) $\cos 3\theta$ in terms of $\cos \theta$ and $\sin \theta$

c) $\tan 3\theta$ in terms of $\tan \theta$

13. The angle x lies in the interval $\frac{\pi}{2} \le x \le \pi$, and $\sin^2 x = \frac{8}{9}$. Without using a calculator, determine the value of

a) $\sin 2x$

b) $\cos 2x$

c) $\cos \frac{x}{2}$

d) $\sin 3x$

14. C Create a flow chart to show how you would evaluate $\sin 2a$, given the value of $\sin a$, if $a \in \left[\frac{\pi}{2}, \pi\right]$.

15. Describe how you could use your knowledge of double angle formulas to sketch the graph of each function. Include a sketch with your description.

a) $f(x) = \sin x \cos x$

b) $f(x) = 2 \cos^2 x$

c) $f(x) = \dfrac{\tan x}{1 - \tan^2 x}$

Extending

16. Eliminate A from each pair of equations to find an equation that relates x to y.

a) $x = \tan 2A, y = \tan A$

b) $x = \cos 2A, y = \cos A$

c) $x = \cos 2A, y = \csc A$

d) $x = \sin 2A, y = \sec 4A$

17. Solve each equation for values of x in the interval $0 \le x \le 2\pi$.

a) $\cos 2x = \sin x$

b) $\sin 2x - 1 = \cos 2x$

18. Express each of the following in terms of $\tan \theta$.

a) $\sin 2\theta$

b) $\cos 2\theta$

c) $\dfrac{\sin 2\theta}{1 + \cos 2\theta}$

d) $\dfrac{1 - \cos 2\theta}{\sin 2\theta}$

7 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: How can you identify equivalent trigonometric expressions?

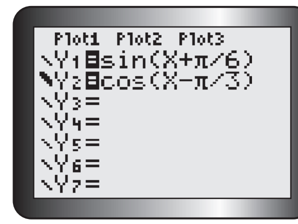
A1: Compare the graphs of the corresponding trigonometric functions on a graphing calculator. If the graphs appear to be identical, then the expressions may be equivalent.

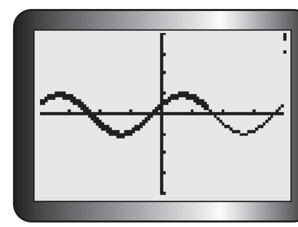
For example, to see if $\sin\left(x + \frac{\pi}{6}\right)$ is the same as $\cos\left(x - \frac{\pi}{3}\right)$, graph the functions $f(x) = \sin\left(x + \frac{\pi}{6}\right)$ and $g(x) = \cos\left(x - \frac{\pi}{3}\right)$ on the same screen. If you use a bold line for the second function, you will see it drawing in over the first graph.

Since the graphs appear to coincide, you can make the conjecture that $f(x) = g(x)$. It follows that $\sin\left(x + \frac{\pi}{6}\right) = \cos\left(x - \frac{\pi}{3}\right)$. This can be confirmed by analyzing both functions. Both functions have a period of 2π. As well, $f(x) = \sin\left(x + \frac{\pi}{6}\right)$ is the sine function translated $\frac{\pi}{6}$ to the left, while $g(x) = \cos\left(x - \frac{\pi}{3}\right)$ is the cosine function translated $\frac{\pi}{3}$ to the right. These transformations of the parent functions result in the same function over their entire domains.

Study Aid

- See Lesson 7.1.
- Try Mid-Chapter Review Questions 1 and 2.





A2: Use some of the following strategies:

- the reflective property of even and odd functions
- translations of a function by an amount that is equal to a multiple of its period
- combinations of other transformations
- the relationship between trigonometric ratios of complementary angles in a right triangle
- the relationship between a principal angle in standard position on the Cartesian plane and its related angles

A3: Use compound angle formulas.

For example, to identify a trigonometric expression that is equivalent to $\cos\left(x - \frac{\pi}{4}\right)$, use the subtraction formula for cosine.

$$\cos\left(x - \frac{\pi}{4}\right) = \cos x \cos \frac{\pi}{4} + \sin x \sin \frac{\pi}{4}$$
$$= (\cos x)\left(\frac{1}{\sqrt{2}}\right) + (\sin x)\left(\frac{1}{\sqrt{2}}\right)$$
$$= \frac{1}{\sqrt{2}}(\cos x + \sin x)$$

Study Aid

- See Lesson 7.2, Example 4.
- Try Mid-Chapter Review Questions 3 and 4.

Study Aid

- See Lesson 7.2, Example 1.
- Try Mid-Chapter Review Questions 5 and 6.

Q: How can you determine the exact values of trigonometric ratios for angles other than the special angles $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$, and their multiples?

A: You can combine special angles by adding or subtracting them, and then use compound angle formulas to determine trigonometric ratios for the new angle.

For example, consider $\frac{\pi}{4} + \frac{\pi}{3} = \frac{7\pi}{12}$.

Determine $\sin \frac{7\pi}{12}$ by finding

$$\begin{aligned}\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right) &= \sin\frac{\pi}{4}\cos\frac{\pi}{3} + \cos\frac{\pi}{4}\sin\frac{\pi}{3}\\ &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{1 + \sqrt{3}}{2\sqrt{2}}\end{aligned}$$

Study Aid

- See Lesson 7.3, Example 2.
- Try Mid-Chapter Review Questions 8 to 12.

Q: Given a trigonometric ratio for θ, how would you calculate trigonometric ratios for 2θ?

A: You can use double angle formulas.

For example, if you know that $\cos\theta = \frac{2}{5}$, you can calculate $\cos 2\theta$ using the formula

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{2}{5}\right)^2 - 1\\ &= \frac{8}{25} - 1\\ &= -\frac{17}{25}\end{aligned}$$

To calculate $\sin 2\theta$ and $\tan 2\theta$, you need to consider the quadrant in which θ lies. If $\cos\theta$ is positive, θ can be in quadrant I or quadrant IV. This means you need to calculate two answers for both $\sin 2\theta$ and $\tan 2\theta$.

PRACTICE Questions

Lesson 7.1

1. For each of the following trigonometric ratios, state an equivalent trigonometric ratio.

a) $\cos \frac{\pi}{16}$

b) $\sin \frac{7\pi}{9}$

c) $\tan \frac{9\pi}{10}$

d) $-\cos \frac{2\pi}{5}$

e) $-\sin \frac{9\pi}{7}$

f) $\tan \frac{3\pi}{4}$

2. Use the sine function to write an equation that is equivalent to $y = -6\cos\left(x + \frac{\pi}{2}\right) + 4$.

Lesson 7.2

3. Use a compound angle addition formula to determine a trigonometric expression that is equivalent to each of the following expressions.

a) $\cos\left(x + \frac{5\pi}{3}\right)$

b) $\sin\left(x + \frac{5\pi}{6}\right)$

c) $\tan\left(x + \frac{5\pi}{4}\right)$

d) $\cos\left(x + \frac{4\pi}{3}\right)$

4. Use a compound angle subtraction formula to determine a trigonometric expression that is equivalent to each of the following expressions.

a) $\sin\left(x - \frac{11\pi}{6}\right)$

b) $\tan\left(x - \frac{\pi}{3}\right)$

c) $\cos\left(x - \frac{7\pi}{4}\right)$

d) $\sin\left(x - \frac{2\pi}{3}\right)$

5. Evaluate each expression.

a) $\dfrac{\tan \frac{8\pi}{9} - \tan \frac{5\pi}{9}}{1 + \tan \frac{8\pi}{9} \tan \frac{5\pi}{9}}$

b) $\sin \frac{299\pi}{298} \cos \frac{\pi}{298} - \cos \frac{299\pi}{298} \sin \frac{\pi}{298}$

c) $\sin 50° \cos 20° - \cos 50° \sin 20°$

d) $\sin \frac{3\pi}{8} \cos \frac{\pi}{8} + \cos \frac{3\pi}{8} \sin \frac{\pi}{8}$

6. Simplify each expression.

a) $\dfrac{2\tan x}{1 - \tan^2 x}$

b) $\sin \frac{x}{5} \cos \frac{4x}{5} + \cos \frac{x}{5} \sin \frac{4x}{5}$

c) $\cos\left(\frac{\pi}{2} - x\right)$

d) $\sin\left(\frac{\pi}{2} + x\right)$

e) $\cos\left(\frac{\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} + x\right)$

f) $\tan\left(x - \frac{\pi}{4}\right)$

7. The expression $a\cos x + b\sin x$ can be expressed in the form $R\cos(x - \alpha)$, where $R = \sqrt{a^2 + b^2}$, $\cos\alpha = \frac{a}{R}$, and $\sin\alpha = \frac{b}{R}$. Use this information to write an expression that is equivalent to $\sqrt{3}\cos x - 3\sin x$.

Lesson 7.3

8. Evaluate each expression.

a) $2\cos^2 \frac{2\pi}{3} - 1$

b) $2\sin \frac{11\pi}{12} \cos \frac{11\pi}{12}$

c) $\cos^2 \frac{7\pi}{8} - \sin^2 \frac{7\pi}{8}$

d) $1 - 2\sin^2\left(\frac{\pi}{2}\right)$

9. The angle x lies in the interval $\pi \le x \le \frac{3\pi}{2}$, and $\cos^2 x = \frac{10}{11}$. Without using a calculator, determine the value of each trigonometric ratio.

a) $\sin x$

b) $\cos x$

c) $\sin 2x$

d) $\cos 2x$

10. Given $\sin x = \frac{3}{5}$ and $0 \le x \le \frac{\pi}{2}$, find $\sin 2x$ and $\cos 2x$.

11. Given $\sin x = \frac{5}{13}$ and $0 \le x \le \frac{\pi}{2}$, find $\sin 2x$.

12. Given $\cos x = -\frac{4}{5}$ and $\pi \le x \le \frac{3\pi}{2}$, find $\tan 2x$.

7.4 Proving Trigonometric Identities

YOU WILL NEED

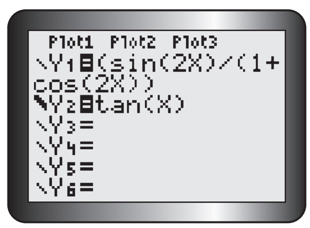
- graphing calculator

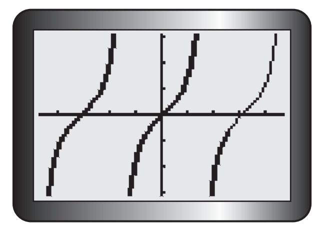
GOAL

Use equivalent trigonometric relationships to prove that an equation is an identity.

LEARN ABOUT the Math

When Alysia graphs the function $f(x) = \frac{\sin 2x}{1 + \cos 2x}$ using a graphing calculator, she sees that her graph looks the same as the graph for the tangent function $f(x) = \tan x$.





She makes a conjecture that $\frac{\sin 2x}{1 + \cos 2x} = \tan x$ is a trigonometric **identity**. In other words, she predicts that this equation is true for all values of x for which the expressions in the equation are defined.

? How can Alysia prove that her conjecture is true?

EXAMPLE 1 Using reasoning to prove an identity that involves double angles

Prove that $\frac{\sin 2x}{1 + \cos 2x} = \tan x$.

Solution

$$\text{LS} = \frac{\sin 2x}{1 + \cos 2x}$$

Begin with the left side (LS) because you can use double angle formulas to express the LS, using the same **argument** as the right side (RS).

$$= \frac{2 \sin x \cos x}{1 + 2\cos^2 x - 1}$$

$$= \frac{2 \sin x \cos x}{2\cos^2 x}$$

After applying the double angle formulas, simplify the denominator. Then divide the numerator and the denominator by $2 \cos x$.

$$= \frac{\sin x}{\cos x}$$

$$= \tan x = \text{RS}$$

Since both sides are equal,

$$\frac{\sin 2x}{1 + \cos 2x} = \tan x$$

The expressions are equivalent for all real numbers, except where $\cos 2x = -1$ and $\cos x = 0$.

Reflecting

A. Why was the left side of the identity simplified at the beginning of the solution?

B. Which formula for $\cos 2x$ was used, and why? Could another formula have been used instead?

C. If you replaced x with $\frac{\pi}{4}$ in Alysia's conjecture and you showed that both sides result in the same value, could you conclude that the equation is an identity? Explain.

APPLY the Math

EXAMPLE 2 Proving that an equation is not an identity

Prove that $\sin x + \sin 2x = \sin 3x$ is not an identity.

Solution

Let $x = \frac{\pi}{2}$.

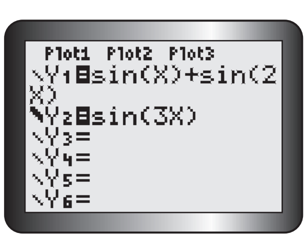
$$\begin{aligned} \text{LS} &= \sin\left(\frac{\pi}{2}\right) + \sin 2\left(\frac{\pi}{2}\right) \\ &= 1 + 0 \\ &= 1 \end{aligned}$$

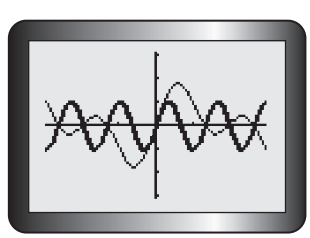
$$\begin{aligned} \text{RS} &= \sin 3\left(\frac{\pi}{2}\right) \\ &= -1 \end{aligned}$$

Choose any value for which both sides of the equation are defined, and evaluate both sides.

Since there is a value for which the left side does not equal the right side, the equation is not an identity.

$x = \frac{\pi}{2}$ is a **counterexample**—it disproves the equivalence of both sides of the equation.





Graphing both sides of the equation results in very different graphs.

EXAMPLE 3 Using reasoning to prove a cofunction identity

Prove that $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$.

Solution

$$\begin{aligned} \text{LS} &= \cos\left(\frac{\pi}{2} + x\right) \\ &= \cos\left(\frac{\pi}{2}\right)\cos x - \sin\left(\frac{\pi}{2}\right)\sin x \\ &= (0)\cos x - (1)\sin x \\ &= 0 - \sin x \\ &= -\sin x \\ &= \text{RS} \end{aligned}$$

Begin with the left side because a compound angle formula can be used to simplify the expression on the left side. Substitute the numerical values of $\cos\left(\frac{\pi}{2}\right)$ and $\sin\left(\frac{\pi}{2}\right)$.

Since both sides are equal,

$$\cos\left(\frac{\pi}{2} + x\right) = -\sin x$$

Because there is no denominator or square root on either side of the equation, the expressions are equivalent for all real numbers.

When you encounter a more complicated identity, you may be able to use several different strategies to prove the equivalence of the expressions.

EXAMPLE 4 Using reasoning to prove an identity that involves rational trigonometric expressions

Prove that $\frac{\cos(x - y)}{\cos(x + y)} = \frac{1 + \tan x \tan y}{1 - \tan x \tan y}$.

Solution

$$\begin{aligned} \text{RS} &= \frac{1 + \tan x \tan y}{1 - \tan x \tan y} \\ &= \frac{1 + \left(\frac{\sin x}{\cos x}\right)\left(\frac{\sin y}{\cos y}\right)}{1 - \left(\frac{\sin x}{\cos x}\right)\left(\frac{\sin y}{\cos y}\right)} \times \frac{(\cos x)(\cos y)}{(\cos x)(\cos y)} \\ &= \frac{(\cos x)(\cos y) + (\sin x)(\sin y)}{(\cos x)(\cos y) - (\sin x)(\sin y)} \\ &= \frac{\cos(x - y)}{\cos(x + y)} \\ &= \text{LS} \end{aligned}$$

Start with the right side. Replace $\tan x$ with $\frac{\sin x}{\cos x}$, and replace $\tan y$ with $\frac{\sin y}{\cos y}$. Then multiply the expression by $\frac{(\cos x)(\cos y)}{(\cos x)(\cos y)}$ (because this equals 1) to get one numerator and one denominator.

Rewrite the expressions in the numerator and the denominator using compound angle formulas.

Since both sides are equal,

$$\frac{\cos(x - y)}{\cos(x + y)} = \frac{1 + \tan x \tan y}{1 - \tan x \tan y}$$

The expressions are equivalent for all real numbers, except where $\cos(x + y) = 0$ and $\tan x \tan y = 1$.

Sometimes, you may need to factor if you want to prove that a given equation is an identity.

EXAMPLE 5 Using a factoring strategy to prove an identity

Prove that $\tan 2x - 2 \tan 2x \sin^2 x = \sin 2x$.

Solution

$$\begin{aligned} \text{LS} &= \tan 2x - 2 \tan 2x \sin^2 x \\ &= \tan 2x(1 - 2 \sin^2 x) \\ &= \tan 2x \cos 2x \\ &= \frac{\sin 2x}{\cos 2x}(\cos 2x) \\ &= \sin 2x, \cos 2x \neq 0 \\ &= \text{RS} \end{aligned}$$

Begin with the more complicated side.

Factor $\tan 2x$ out of the two terms.

The expression inside the brackets can be simplified using a double angle formula.

Write $\tan 2x$ as $\frac{\sin 2x}{\cos 2x}$, and simplify the resulting expression.

Since both sides are equal,

$\tan 2x - 2 \tan 2x \sin^2 x = \sin 2x$, $\cos 2x \neq 0$.

The expressions are equivalent for all real numbers, except where $\cos 2x = 0$. The left side involves the tangent function, which was expressed as a quotient, so the denominator cannot be 0.

In Summary

Key Ideas

- A trigonometric identity states the equivalence of two trigonometric expressions. It is written as an equation that involves trigonometric ratios, and the solution set is all real numbers for which the expressions on both sides of the equation are defined. As a result, the equation has an infinite number of solutions.
- Some trigonometric identities are the result of a definition, while others are derived from relationships that exist among trigonometric ratios.

Need to Know

- The following trigonometric identities are important for you to remember:

Identities Based on Definitions

Reciprocal Identities

$$\csc x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x}$$

Identities Derived from Relationships

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Double Angle Formulas

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Addition and Subtraction Formulas

$$\sin (x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin (x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos (x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos (x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

- You can verify the truth of a given trigonometric identity by graphing each side separately and showing that the two graphs are the same.
- To prove that a given equation is an identity, the two sides of the equation must be shown to be equivalent. This can be accomplished using a variety of strategies, such as
  - simplifying the more complicated side until it is identical to the other side, ormanipulating both sides to get the same expression
  - rewriting expressions using any of the identities stated above
  - using a common denominator or factoring, where possible

CHECK Your Understanding

1. Jared claims that $\sin x = \cos x$ is an identity, since $\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Use a counterexample to disprove his claim.

2. **a)** Use a graphing calculator to graph $f(x) = \sin x$ and $g(x) = \tan x \cos x$ for $-2\pi \le x \le 2\pi$.

b) Write a trigonometric identity based on your graphs.

c) Simplify one side of your identity to prove it is true.

d) This identity is true for all real numbers, except where $\cos x = 0$. Explain why.

3. Graph the appropriate functions to match each expression on the left with the equivalent expression on the right.

a)	$\sin x \cot x$	**A**	$\sin^2 x + \cos^2 x + \tan^2 x$
b)	$1 - 2\sin^2 x$	**B**	$1 + 2\sin x \cos x$
c)	$(\sin x + \cos x)^2$	**C**	$\cos x$
d)	$\sec^2 x$	**D**	$2\cos^2 x - 1$

4. Prove algebraically that the expressions you matched in question 3 are equivalent.

PRACTISING

5. **K** Give a counterexample to show that each equation is not an identity.

a) $\cos x = \frac{1}{\cos x}$

b) $1 - \tan^2 x = \sec^2 x$

c) $\sin(x + y) = \cos x \cos y + \sin x \sin y$

d) $\cos 2x = 1 + 2\sin^2 x$

6. **A** Graph the expression $\frac{1 - \tan^2 x}{1 + \tan^2 x}$, and make a conjecture about another expression that is equivalent to this expression.

7. Prove your conjecture in question 6.

8. Prove that $\frac{1 + \tan x}{1 + \cot x} = \frac{1 - \tan x}{\cot x - 1}$.

9. Prove each identity.

a) $\frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin \theta \cos \theta} = 1 - \tan \theta$

b) $\tan^2 x - \sin^2 x = \sin^2 x \tan^2 x$

c) $\tan^2 x - \cos^2 x = \frac{1}{\cos^2 x} - 1 - \cos^2 x$

d) $\frac{1}{1 + \cos \theta} + \frac{1}{1 - \cos \theta} = \frac{2}{\sin^2 \theta}$

10. Prove each identity.

a) $\cos x \tan^3 x = \sin x \tan^2 x$

b) $\sin^2 \theta + \cos^4 \theta = \cos^2 \theta + \sin^4 \theta$

c) $(\sin x + \cos x)\left(\dfrac{\tan^2 x + 1}{\tan x}\right) = \dfrac{1}{\cos x} + \dfrac{1}{\sin x}$

d) $\tan^2 \beta + \cos^2 \beta + \sin^2 \beta = \dfrac{1}{\cos^2 \beta}$

e) $\sin\left(\dfrac{\pi}{4} + x\right) + \sin\left(\dfrac{\pi}{4} - x\right) = \sqrt{2} \cos x$

f) $\sin\left(\dfrac{\pi}{2} - x\right) \cot\left(\dfrac{\pi}{2} + x\right) = -\sin x$

11. Prove each identity.

T

a) $\dfrac{\cos 2x + 1}{\sin 2x} = \cot x$

b) $\dfrac{\sin 2x}{1 - \cos 2x} = \cot x$

c) $(\sin x + \cos x)^2 = 1 + \sin 2x$

d) $\cos^4 \theta - \sin^4 \theta = \cos 2\theta$

e) $\cot \theta - \tan \theta = 2 \cot 2\theta$

f) $\cot \theta + \tan \theta = 2 \csc 2\theta$

g) $\dfrac{1 + \tan x}{1 - \tan x} = \tan\left(x + \dfrac{\pi}{4}\right)$

h) $\csc 2x + \cot 2x = \cot x$

i) $\dfrac{2 \tan x}{1 + \tan^2 x} = \sin 2x$

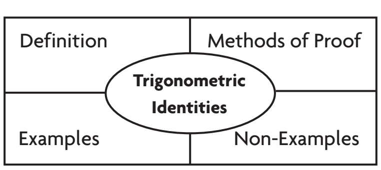
j) $\sec 2t = \dfrac{\csc t}{\csc t - 2 \sin t}$

k) $\csc 2\theta = \dfrac{1}{2}(\sec \theta)(\csc \theta)$

l) $\sec t = \dfrac{\sin 2t}{\sin t} - \dfrac{\cos 2t}{\cos t}$

12. Graph the expression $\dfrac{\sin x + \sin 2x}{1 + \cos x + \cos 2x}$, and make a conjecture about another expression that is equivalent to this expression.

13. Prove your conjecture in question 12.



14. Copy the chart shown, and complete it to summarize what you know about trigonometric identities.

C

15. Your friend wants to know whether the equation $2 \sin x \cos x = \cos 2x$ is an identity. Explain how she can determine whether it is an identity. If it is an identity, explain how she can prove this. If it is not an identity, explain how she can change one side of the equation to make it an identity.

Extending

16. Each of the following expressions can be written in the form $a \sin 2x + b \cos 2x + c$. Determine the values of a, b, and c.

a) $2 \cos^2 x + 4 \sin x \cos x$

b) $-2 \sin x \cos x - 4 \sin^2 x$

17. Express $8 \cos^4 x$ in the form $a \cos 4x + b \cos 2x + c$. State the values of the constants a, b, and c.

7.5 Solving Linear Trigonometric Equations

GOAL

Solve linear trigonometric equations algebraically and graphically.

YOU WILL NEED

- graphing calculator

LEARN ABOUT the Math

In Lesson 7.4, you learned how to prove that a given trigonometric equation is an identity. Not all trigonometric equations are identities, however. To see the difference between an equation that is an identity and an equation that is not, consider the following two equations on the domain $0 \leq x \leq 2\pi$: $\sin^2 x + \cos^2 x = 1$ and $2 \sin x - 1 = 0$.

The first equation is true for all values of x in the given domain, so it is an identity.

The second equation is true for only some values of x, so it is not an identity.

? How can you solve a trigonometric equation that is not an identity?

EXAMPLE 1 Selecting a strategy to determine the solutions for a linear trigonometric equation

You are given the equation $2 \sin x + 1 = 0$, $0 \leq x \leq 2\pi$.

a) Determine all the solutions in the specified interval.
b) Verify the solutions using graphing technology.

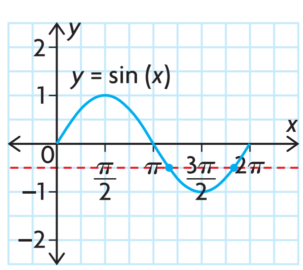
Solution

a) $2 \sin x + 1 = 0$

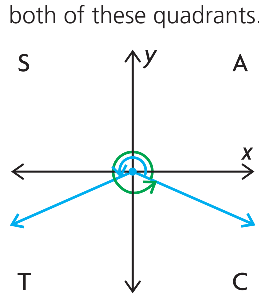
$$2 \sin x = -1$$

$$\sin x = -\frac{1}{2}$$

Rearrange the equation to isolate $\sin x$.



Sketch a graph of the sine function to estimate where its value is $-\frac{1}{2}$. From the graph, one solution is possible when $\pi \leq x \leq \frac{3\pi}{2}$ and another solution is possible when $\frac{3\pi}{2} \leq x \leq 2\pi$. Therefore, the terminal arms of the two angles lie in quadrants III and IV. This makes sense since r is positive and y is negative, so the sine ratio is negative for angles in both of these quadrants. This is confirmed by the CAST rule.



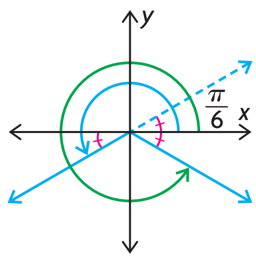
Two solutions are possible in the specified interval, $0 \leq x \leq 2\pi$, since the sine graph will complete one cycle in this interval.

Determine the related acute angle.

$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\frac{\pi}{6}$ is a special angle.
Using the special triangle that contains $\frac{\pi}{6}$ and $\frac{\pi}{3}$, $\sin \frac{\pi}{6} = \frac{1}{2}$.

Use the related angle to determine the required solutions in the given interval.

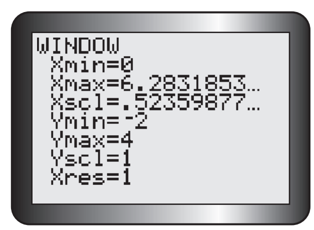


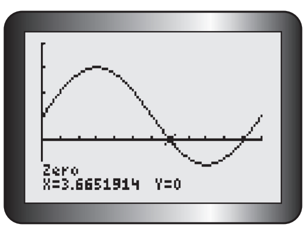
The solution in quadrant III is $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$.

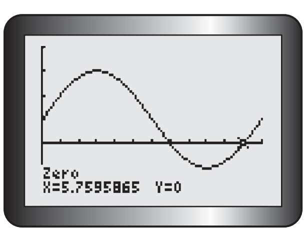
The solution in quadrant IV is $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

b) Graph $f(x) = 2 \sin x + 1$ in radian mode, for $0 \le x \le 2\pi$, and determine the zeros.

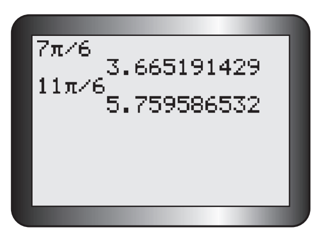
Use the window settings that match the domain for Xmin and Xmax. Use a scale of $\frac{\pi}{6}$.







To verify the solutions found in part a), express the solutions as decimals.



The zeros are located at approximately 3.665 191 4 and 5.759 586 5. These values are very close to $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

Reflecting

A. How was solving the equation $2 \sin x + 1 = 0$ like solving the equation $2x + 1 = 0$? How was it different?

B. Once $\sin x$ was isolated in Example 1, how was the sign of the trigonometric ratio used to determine the quadrants in which the solutions were located?

C. The interval in Example 1 was $0 \le x \le 2\pi$. If the interval had been $x \in \mathbf{R}$, how many solutions would the equation have had? Explain.

APPLY the Math

EXAMPLE 2 Using an algebraic strategy to determine the approximate solutions for a linear trigonometric equation

Solve $3(\tan\theta + 1) = 2$, where $0° \le \theta \le 360°$, correct to one decimal place.

Solution

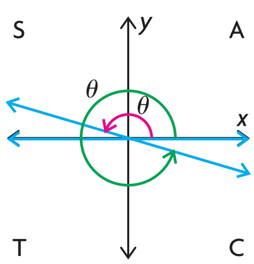
$$3(\tan\theta + 1) = 2$$

$$\tan\theta + 1 = \frac{2}{3}$$

$$\tan\theta = \frac{2}{3} - 1$$

Rearrange the equation to isolate $\tan\theta$.

$$\tan\theta = -\frac{1}{3}$$

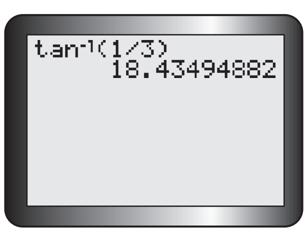


Since the tangent ratio is negative, x can be negative when y is positive, and vice versa.

The tangent ratio is negative in quadrants II and IV. The terminal arm of the angles lies in these two quadrants

There are two solutions for θ in the interval $0° \le \theta \le 360°$.

Determine the related acute angle using the inverse tangent function.



Evaluate $\tan^{-1}\left(\frac{1}{3}\right)$ using a calculator in degree mode, and round your answer to one decimal place.

$\tan^{-1}\left(\frac{1}{3}\right) \doteq 18.4°$, so the related acute angle is about 18.4°.

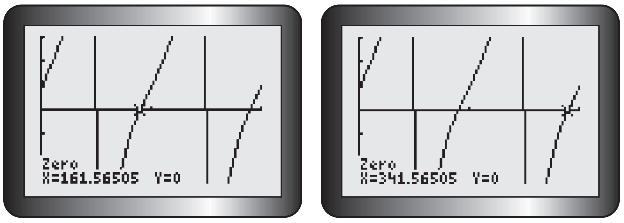
Subtract 18.4° from 180° to obtain the solution in quadrant II.
$\theta \doteq 180° - 18.4° = 161.6°$

If β is the related angle, the principal angle in quadrant II is $180° - \beta$. The principal angle in quadrant IV is $360° - \beta$.

Subtract 18.4° from 360° to obtain the solution in quadrant IV.
$\theta \doteq 360° - 18.4° = 341.6°$

θ is about 161.6° or 341.6°.

Verify the solutions by graphing $f(\theta) = 3(\tan\theta + 1) - 2$ in degree mode and determining the zeros in the given domain.



Choose window settings to match the domain $0 \le \theta \le 360°$.

The results confirm the solutions.

EXAMPLE 3 Solving a problem that involves a linear trigonometric equation

Today, the high tide in Matthews Cove, New Brunswick, is at midnight. The water level at high tide is 7.5 m. The depth, d metres, of the water in the cove at time t hours is modelled by the equation $d(t) = 4 + 3.5\cos\frac{\pi}{6}t$. Jenny is planning a day trip to the cove tomorrow, but the water needs to be at least 2 m deep for her to manoeuvre her sailboat safely. How can Jenny determine the times when it will be safe for her to sail into Matthews Cove?

Solution

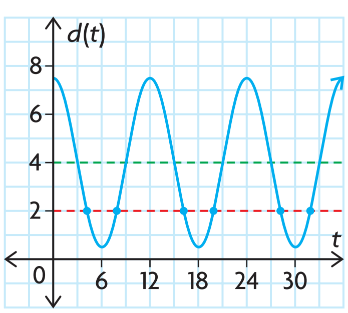
Draw a rough sketch of the depth function for at least the next 24 h, assuming that $t = 0$ is the high tide at midnight.

For the function $f(x) = a\cos kx + c$, the amplitude is a, the period is $\frac{2\pi}{k}$, and the horizontal axis is the line $y = c$. For the function $d(t) = 4 + 3.5\cos\frac{\pi}{6}t$,

$a = 3.5$

$c = 4$

$$\text{period} = \frac{2\pi}{\frac{\pi}{6}} = 2\pi \times \frac{6}{\pi} = 12$$



From the graph, the water level will be near 2 m around 4 a.m., 8 a.m., 4 p.m., and 8 p.m.

It looks like the best time for her to enter the cove is around 8 a.m., and she needs to leave the cove around 4 p.m.

Determine the times when the water level is above 2 m and the times when the level equals 2 m.

$$4 + 3.5\cos\frac{\pi}{6}t = 2$$

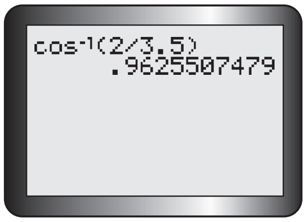
$$3.5\cos\frac{\pi}{6}t = 2 - 4$$

$$\cos\frac{\pi}{6}t = \frac{-2}{3.5}$$

To get a better approximation of the times, solve the equation for $d(t) = 2$ to determine the related acute angle.

Since $4 + 3.5\cos\frac{\pi}{6}t = 2$ is a linear trigonometric equation, isolate $\cos\frac{\pi}{6}t$.

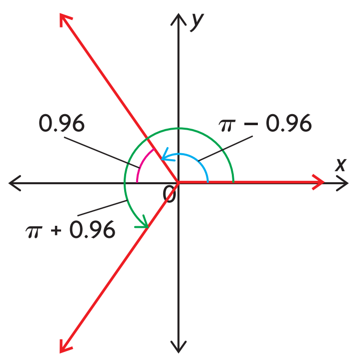
Determine the related acute angle.



Using a calculator in radian mode, determine the inverse cosine of $\frac{2}{3.5}$ to find the related acute angle.

$\frac{\pi}{6}t \doteq 0.96$

The related acute angle is about 0.96.



The cosine ratio is negative, so x is negative and r is positive. The terminal arms of $\frac{\pi}{6}t$ must lie in quadrants II and III.

To find the value of $\frac{\pi}{6}t$ in quadrant II, subtract the related acute angle from π.
$\pi - 0.96 = 2.18$

To find the value of $\frac{\pi}{6}t$ in quadrant III, add the related acute angle to π.
$\pi + 0.96 = 4.1$

The value of $\frac{\pi}{6}t$ is about 2.18 in quadrant II and about 4.1 in quadrant III.

To find the approximate times when the depth is 2 m, solve the following equations.

$$\frac{\pi}{6}t = 2.18 \quad \text{or} \quad \frac{\pi}{6}t = 4.1$$

Since Jenny is sailing tomorrow, the domain is $0 \le t \le 24$.

$$t = \frac{6}{\pi}(2.18) \qquad t = \frac{6}{\pi}(4.1)$$

$$t \doteq 4.16 \qquad t \doteq 7.83$$

$$t = 4.16 + 12 \qquad t = 7.83 + 12$$

$$t = 16.16 \qquad t = 19.83$$

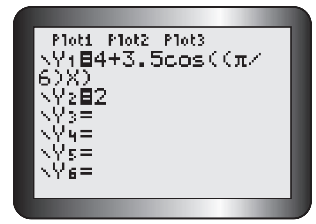
You can generate more solutions by adding 12, the period of the cosine function.

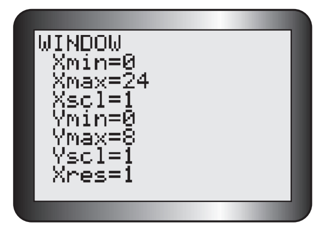
Jenny can safely sail into the cove when the water level is higher than 2 m. This occurs tomorrow, during the day, between 7:50 a.m. and 4:10 p.m.

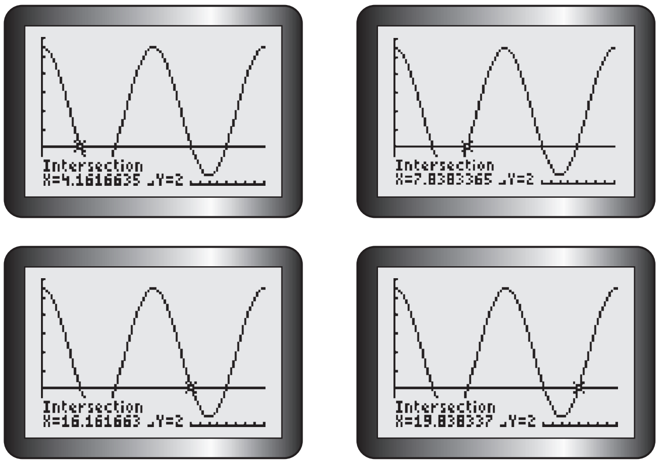
Multiply the digits to the right of the decimal by 60 to convert from a fraction of an hour to minutes. Tomorrow, the water level will be 2 m at about 4:10 a.m., 7:50 a.m., 4:10 p.m., and 7:50 p.m.

The water level is higher than 2 m when the tide function graph is above the line $d = 2$.

To verify the solution, graph $d(t)$ and the horizontal line $d = 2$ for the 24 h following midnight. Then determine the points of intersection.







There is no need to convert the values of t into hours and minutes, since the values on the graph can be compared with the calculated solutions.

The values of t are very close to the calculated values. Therefore, the solution is reasonable.

EXAMPLE 4 Selecting a strategy to solve a linear trigonometric equation that involves double angles

Solve $2 \sin \theta \cos \theta = \cos 2\theta$ for θ in the interval $0 \le \theta \le 2\pi$.

Solution

$$2 \sin \theta \cos \theta = \cos 2\theta$$
$$\sin 2\theta = \cos 2\theta$$

Use the sin 2θ double angle formula to express the equation using the same argument.

$$\frac{\sin 2\theta}{\cos 2\theta} = \frac{\cos 2\theta}{\cos 2\theta}$$
$$\tan 2\theta = 1$$

Divide both sides by cos 2θ to express the equation using a single trigonometric function.

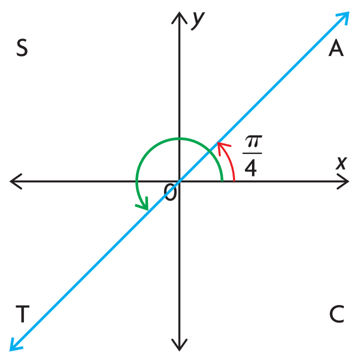
Solve $\tan 2\theta = 1$.

The related acute angle of 2θ is $\tan^{-1}(1) = \frac{\pi}{4}$.

Determine the related angle for 2θ by evaluating $\tan^{-1}(1)$.

Use the 1, 1, $\sqrt{2}$ special triangle to determine the inverse tangent of 1.

The tangent ratio is positive in quadrants I and III.



Since the tangent ratio is positive, x and y must have the same sign. This means that the terminal arm of 2θ lies in quadrant I or quadrant III.

The value of 2θ in quadrant I is $\frac{\pi}{4}$.

The value of 2θ in quadrant III is $\frac{5\pi}{4}$.

To find the value of 2θ in quadrant III, add the related angle to π.

$$\pi + \frac{\pi}{4} = \frac{5\pi}{4}.$$

To determine θ, solve the following equations.

$$2\theta = \frac{\pi}{4} \quad \text{or} \quad 2\theta = \frac{5\pi}{4}$$

$$\theta = \frac{\pi}{8} \qquad \theta = \frac{5\pi}{8}$$

$$\theta = \frac{\pi}{8} + \frac{\pi}{2} = \frac{5\pi}{8} \text{ (already determined)}$$

$$\theta = \frac{5\pi}{8} + \frac{\pi}{2} = \frac{9\pi}{8}$$

$$\theta = \frac{9\pi}{8} + \frac{\pi}{2} = \frac{13\pi}{8}$$

The period of $\tan 2\theta$ is $\frac{\pi}{2}$, so adding this to the two solutions will generate the other solutions in the given domain, $0 \le \theta \le 2\pi$.

Solutions for θ are $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}$, or $\frac{13\pi}{8}$.

In Summary

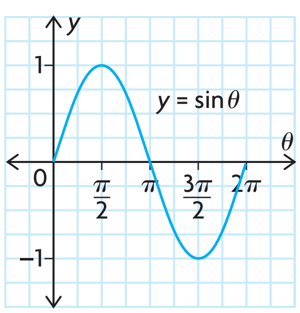
Key Idea

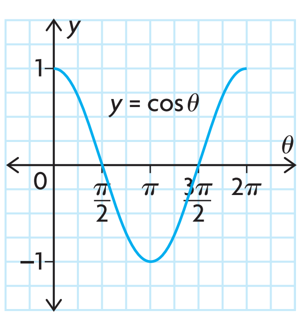
- The same strategies can be used to solve linear trigonometric equations when the variable is measured in degrees or radians.

Need to Know

- Because of their periodic nature, trigonometric equations have an infinite number of solutions. When we use a trigonometric model, we usually want solutions within a specified interval.
- To solve a linear trigonometric equation, use special triangles, a calculator, a sketch of the graph, and/or the CAST rule.
- A scientific or graphing calculator provides very accurate estimates of the value for an inverse trigonometric function. The inverse trigonometric function of a positive ratio yields the related angle. Use the related acute angle and the period of the corresponding function to determine all the solutions in the given interval.
- You can use a graphing calculator to verify the solutions for a linear trigonometric equation by
  - graphing the appropriate functions on the graphing calculator and determining the points of intersection
  - graphing an equivalent single function and determining its zeros

CHECK Your Understanding





1. Use the graph of $y = \sin\theta$ to estimate the value(s) of θ in the interval $0 \le \theta \le 2\pi$.

a) $\sin\theta = 1$ **b)** $\sin\theta = -1$ **c)** $\sin\theta = 0.5$ **d)** $\sin\theta = -0.5$ **e)** $\sin\theta = 0$ **f)** $\sin\theta = \frac{\sqrt{3}}{2}$

2. Use the graph of $y = \cos\theta$ to estimate the value(s) of θ in the interval $0 \le \theta \le 2\pi$.

a) $\cos\theta = 1$ **b)** $\cos\theta = -1$ **c)** $\cos\theta = 0.5$ **d)** $\cos\theta = -0.5$ **e)** $\cos\theta = 0$ **f)** $\cos\theta = \frac{\sqrt{3}}{2}$

3. Solve $\sin x = \frac{\sqrt{3}}{2}$, where $0 \le x \le 2\pi$.

a) How many solutions are possible?
b) In which quadrants would you find the solutions?
c) Determine the related acute angle for the equation.
d) Determine all the solutions for the equation.

4. Solve $\cos x = -0.8667$, where $0° \leq x \leq 360°$.

a) How many solutions are possible?
b) In which quadrants would you find the solutions?
c) Determine the related angle for the equation, to the nearest degree.
d) Determine all the solutions for the equation, to the nearest degree.

5. Solve $\tan \theta = 2.7553$, where $0 \leq \theta \leq 2\pi$.

a) How many solutions are possible?
b) In which quadrants would you find the solutions?
c) Determine the related angle for the equation, to the nearest hundredth.
d) Determine all the solutions for the equation, to the nearest hundredth.

PRACTISING

6. Determine the solutions for each equation, where $0 \leq \theta \leq 2\pi$.
K

a) $\tan \theta = 1$
b) $\sin \theta = \frac{1}{\sqrt{2}}$
c) $\cos \theta = \frac{\sqrt{3}}{2}$
d) $\sin \theta = -\frac{\sqrt{3}}{2}$
e) $\cos \theta = -\frac{1}{\sqrt{2}}$
f) $\tan \theta = \sqrt{3}$

7. Using a calculator, determine the solutions for each equation on the interval $0° \leq \theta \leq 360°$. Express your answers to one decimal place.

a) $2 \sin \theta = -1$
b) $3 \cos \theta = -2$
c) $2 \tan \theta = 3$
d) $-3 \sin \theta - 1 = 1$
e) $-5 \cos \theta + 3 = 2$
f) $8 - \tan \theta = 10$

8. Using a calculator, determine the solutions for each equation, to two decimal places, on the interval $0 \leq x \leq 2\pi$.

a) $3 \sin x = \sin x + 1$
b) $5 \cos x - \sqrt{3} = 3 \cos x$
c) $\cos x - 1 = -\cos x$
d) $5 \sin x + 1 = 3 \sin x$

9. Using a calculator, determine the solutions for each equation, to two decimal places, on the interval $0 \leq x \leq 2\pi$.

a) $2 - 2 \cot x = 0$
b) $\csc x - 2 = 0$
c) $7 \sec x = 7$
d) $2 \csc x + 17 = 15 + \csc x$
e) $2 \sec x + 1 = 6$
f) $8 + 4 \cot x = 10$

10. Using a calculator, determine the solutions for each equation, to two decimal places, on the interval $0 \leq x \leq 2\pi$.

a) $\sin 2x = \frac{1}{\sqrt{2}}$
b) $\sin 4x = \frac{1}{2}$
c) $\sin 3x = -\frac{\sqrt{3}}{2}$
d) $\cos 4x = -\frac{1}{\sqrt{2}}$
e) $\cos 2x = -\frac{1}{2}$
f) $\cos \frac{x}{2} = \frac{\sqrt{3}}{2}$

11. A city's daily high temperature, in degrees Celsius, can be modelled by the function $t(d) = -28 \cos \frac{2\pi}{365}d + 10$, where d is the day of the year and $1 =$ January 1. On days when the temperature is approximately 32 °C or above, the air conditioners at city hall are turned on. During what days of the year are the air conditioners running at city hall?

A

12. The height, in metres, of a nail in a water wheel above the surface of the water, as a function of time, can be modelled by the function $h(t) = -4 \sin \frac{\pi}{4}(t - 1) + 2.5$, where t is the time in seconds. During what periods of time is the nail below the water in the first 24 s that the wheel is rotating?

13. Solve $\sin\left(x + \frac{\pi}{4}\right) = \sqrt{2} \cos x$ for $0 \leq x \leq 2\pi$.

T

14. Sketch the graph of $y = \sin 2\theta$ for $0 \leq \theta \leq 2\pi$. On the graph, clearly indicate all the solutions for the trigonometric equation $\sin 2\theta = -\frac{1}{\sqrt{2}}$.

C

15. Explain why the value of the function $f(x) = 25 \sin \frac{\pi}{50}(x + 20) - 55$ at $x = 3$ is the same as the value of the function at $x = 7$.

16. Create a table like the one below to compare the algebraic and graphical strategies for solving a trigonometric equation. In what ways are the strategies similar, and in what ways are they different? Use examples in your comparison.

	Method for Solving	
	Algebraic Strategy	**Graphical Strategy**
Similarities		
Differences		

Extending

17. Solve the trigonometric equation $2 \sin x \cos x + \sin x = 0$. (*Hint:* You may find it helpful to factor the left side of the equation.)

18. Solve each equation for $0 \leq x \leq 2\pi$.

a) $\sin 2x - 2 \cos^2 x = 0$ **b)** $3 \sin x + \cos 2x = 2$

7.6 Solving Quadratic Trigonometric Equations

GOAL

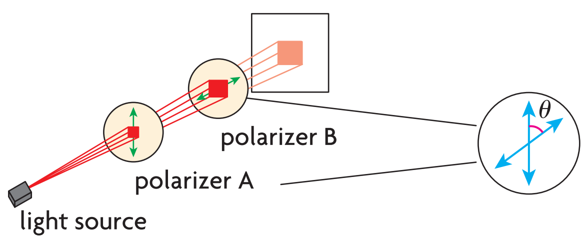
Solve quadratic trigonometric equations using graphs and algebra.

YOU WILL NEED

- graphing calculator

LEARN ABOUT the Math

A polarizing material is used in camera lens filters, LCD televisions, and sunglasses to reduce glare. In these examples, two polarizers are used to reduce the intensity of the light that enters your eyes.



The amount of the reduction in light intensity, I, depends on θ, the acute angle formed between the axis of polarizer A and the axis of polarizer B. Malus's law states that $I = I_0 \cos^2 \theta$, where I_0 is the intensity of the initial beam of light and I is the intensity of the light emerging from the polarizing material.

? At what angle to the axis of polarizer A should polarizer B be placed to reduce the light intensity by 97%?

EXAMPLE 1 Solving a quadratic trigonometric equation using an algebraic strategy

Use Malus's law to determine the angle between polarizer A and polarizer B that will reduce the light intensity by 97%.

Solution

Malus's law is $I = I_0 \cos^2 \theta$.
Solve the equation $0.03\, I_0 = I_0 \cos^2 \theta$.

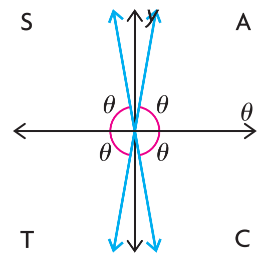
If the light intensity is reduced by 97%, then it is $1 - 0.97$ or 0.03 of the initial intensity. Therefore, $I = 0.03 I_0$.

$$\frac{0.03I_0}{I_0} = \frac{I_0 \cos^2 \theta}{I_0}$$

Divide both sides by I_0 to isolate $\cos \theta$.

$$0.03 = \cos^2 \theta$$
$$\pm\sqrt{0.03} = \sqrt{\cos^2 \theta}$$
$$\pm 0.1732 = \cos \theta$$
$$\cos \theta = 0.1732 \text{ or } \cos \theta = -0.1732$$

Take the square root of both sides.



Since the cosine ratio has both positive and negative values, solving both equations will result in values for θ that lie in all four quadrants.

This means that there are four possible solutions. These solutions, however, are all related by the acute related angle.

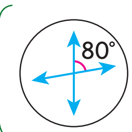
Only the acute angle is necessary. This is the angle in quadrant I.

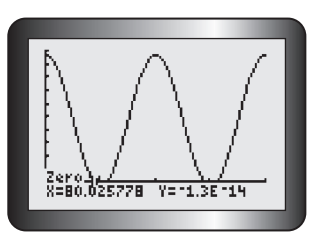
$$\cos \theta = 0.1732$$
$$\theta = \cos^{-1}(0.1732)$$

To determine the related acute angle, use a calculator in degree mode and determine the inverse cosine of 0.1732.

$$\theta \doteq 80°$$

To reduce the light intensity by 97%, the axis of polarizing material B must be placed at an angle of about 80° to the axis of polarizing material A.





To verify the solution, graph $f(x) = \cos^2 \theta - 0.03$ in degree mode and determine its first zero.

The graph confirms the calculated solution.

Reflecting

A. Compare the number of solutions between 0° and 360° for the equation $\cos^2 x = 0.03$ with the number of solutions for a linear trigonometric equation, such as $\cos x = 0.03$. Explain the difference, using both graphical and algebraic analyses.

B. Why were some of the solutions for the trigonometric equation $\cos^2 x = 0.03$ omitted in the context of Example 1?

C. How would the equation change if the intensity of light in an LCD television was reduced by 25%? What angle would be needed between the axis of polarizer A and the axis of polarizer B for this situation?

APPLY the Math

EXAMPLE 2 Selecting a factoring strategy to solve quadratic trigonometric equations

Solve each equation for x in the interval $0 \le x \le 2\pi$. Verify your solutions by graphing.

a) $\sin^2 x - \sin x = 2$ **b)** $2\sin^2 x - 3\sin x + 1 = 0$

Solution

a)
$$\sin^2 x - \sin x = 2$$
$$\sin^2 x - \sin x - 2 = 0$$
$$(\sin x - 2)(\sin x + 1) = 0$$
$\sin x = 2$ or $\sin x = -1$

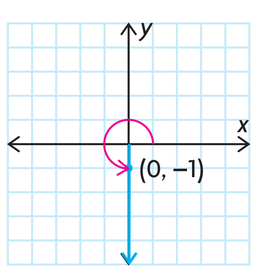
Subtract 2, so you have 0 on the right side. This is a quadratic equation in $\sin x$. Factor.

Solve both of these equations.

The equation $\sin x = 2$ has no solutions.

Since the graph of $y = \sin x$ has the range $\{y \in \mathbf{R} \mid -1 \le y \le 1\}$, the values of $\sin x$ cannot exceed 1.

The equation $\sin x = -1$ has only one solution in the interval $0 \le x \le 2\pi$.

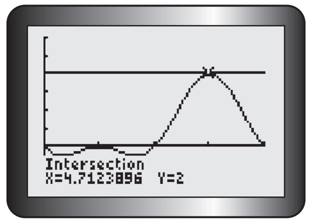


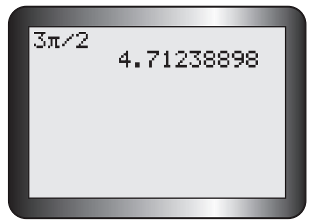
Since $\sin x = \frac{y}{r} = \frac{-1}{1}$, the point $(0, -1)$ lies on the terminal arm of angle x.

The solution is $x = \frac{3\pi}{2}$.

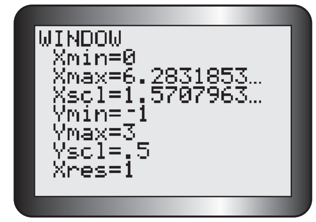
Tech **Support**

For help using the graphing calculator to determine points of intersection, see Technical Appendix, T-12.





To verify the solution, graph $f(x) = \sin^2 x - \sin x$ and $g(x) = 2$ in the required interval. Then determine the points of intersection.



You can see that there is only one solution in the interval $0 \le x \le 2\pi$.

Since $\frac{3\pi}{2} \doteq 4.712\ 388\ 98$, this verifies the previous solution.

b)

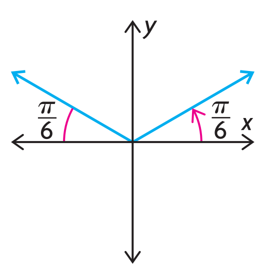
$$2\sin^2 x - 3\sin x + 1 = 0$$
$$(2\sin x - 1)(\sin x - 1) = 0$$
$$\sin x = \frac{1}{2} \text{ or } \sin x = 1$$

Factor the left side.

$\sin x = \frac{1}{2}$ has two solutions in $0 \le x \le 2\pi$.

$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ is the solution in quadrant I and is also the related acute angle.

Use the 1, 2, $\sqrt{3}$ special triangle to determine that $\sin\frac{\pi}{6} = \frac{1}{2}$.



Since $\sin x$ is positive, both y and r are positive. The solutions lie in quadrants I and II.

To determine the solution in quadrant II, subtract the related angle from π.

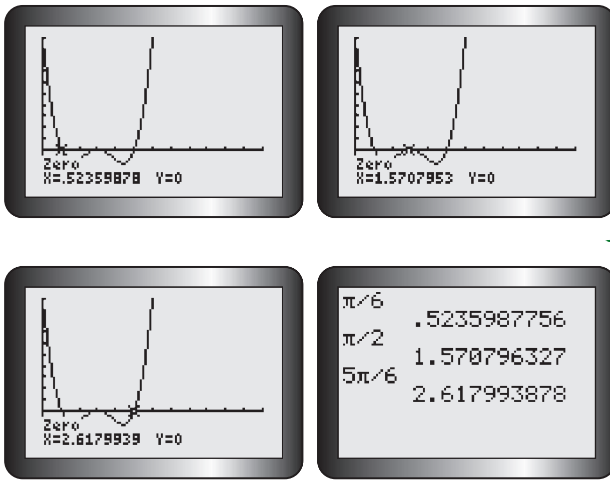
$$\pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

The solution in quadrant II is $\frac{5\pi}{6}$.

$\sin x = 1$ has one solution.
This occurs when $x = \frac{\pi}{2}$.

$\sin^{-1}(1) = \frac{\pi}{2}$

Graph $f(x) = 2\sin^2 x - 3\sin x + 1$, and determine the zeros to verify the solutions.



Limit the window to the interval $[0, 2\pi]$ so you only consider the required solutions.

If you set Xscl to $\frac{\pi}{6}$, you can see that the zeros match the solutions already obtained.

The solutions match those obtained algebraically.

EXAMPLE 3 Selecting a strategy using identities to solve quadratic trigonometric equations

For each equation, use a trigonometric identity to create a quadratic equation. Then solve the equation for x in the interval $[0, 2\pi]$.

a) $2\sec^2 x - 3 + \tan x = 0$ **b)** $3\sin x + 3\cos 2x = 2$

Solution

a)

$$2\sec^2 x - 3 + \tan x = 0$$

$$2(1 + \tan^2 x) - 3 + \tan x = 0$$

Use the Pythagorean identity $1 + \tan^2 x = \sec^2 x$ to create an equation with only $\tan x$ and $\tan^2 x$ in it.

$$2 + 2\tan^2 x - 3 + \tan x = 0$$

$$2\tan^2 x + \tan x - 1 = 0$$

Expand and combine terms. Factor.

$$(2\tan x - 1)(\tan x + 1) = 0$$

$$2\tan x - 1 = 0 \text{ or } \tan x + 1 = 0$$

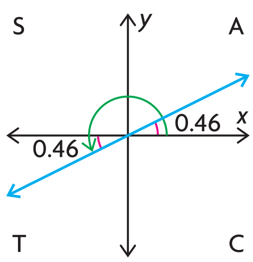
Set each factor equal to 0 to solve the equations.

$$\tan x = \frac{1}{2} \quad \text{or} \quad \tan x = -1$$

$\tan x = \frac{1}{2}$ has solutions in quadrants I and III.

$\tan^{-1}\left(\frac{1}{2}\right) \doteq 0.46$

This is the solution in quadrant I and is also the related angle.

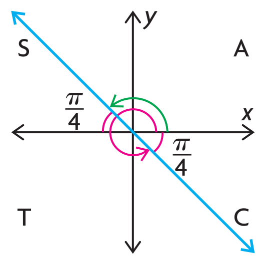


The solution in quadrant III is $\pi + 0.46 \doteq 3.60$

$\tan x = -1$ has solutions in quadrants II and IV.

$\tan^{-1}(1) = \frac{\pi}{4}$

The related angle is $\frac{\pi}{4}$.



The solution in quadrant II is $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

The solution in quadrant IV is $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$.

Use the CAST rule to help determine the solutions in the required interval, $0 \le x \le 2\pi$.

Solutions to the equation are $x \doteq 0.46, \frac{3\pi}{4}, 3.60$, or $\frac{7\pi}{4}$ radians, rounded to two decimal places where not exact.

Round answers that are not exact.

b)

$$3 \sin x + 3 \cos 2x = 2$$
$$3 \sin x + 3(1 - 2 \sin^2 x) = 2$$
$$3 \sin x + 3 - 6 \sin^2 x = 2$$
$$0 = 2 - 3 \sin x - 3 + 6 \sin^2 x$$
$$0 = 6 \sin^2 x - 3 \sin x - 1$$

To create a single trigonometric function (such as sin x) with the same argument, use the double angle formula $\cos 2x = 1 - 2 \sin^2 x$. Rearrange the equation so that one side equals 0.

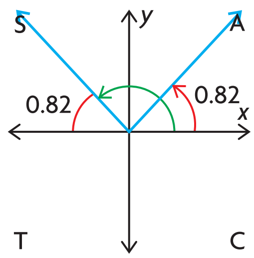
$$0 = 6a^2 - 3a - 1$$
$$a = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(6)(-1)}}{2(6)}$$
$$a = \frac{3 \pm \sqrt{33}}{12}$$
$$a \doteq 0.73 \text{ or } a \doteq -0.23$$
$$\sin x = 0.73 \text{ or } \sin x = -0.23$$

This is not factorable, so substitute $a = \sin x$ and use the quadratic formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a = 6$, $b = -3$, and $c = -1$.

$\sin x = 0.73$ has solutions in quadrants I and II.

$\sin^{-1}(0.73) \doteq 0.82$

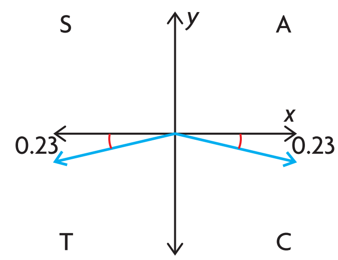
This is the solution in quadrant I and is also the related angle.



The other solution is $\pi - 0.82 = 2.32$.

$\sin x = -0.23$ has solutions in quadrants III and IV.

$\sin^{-1}(0.23) \doteq 0.23$. The related angle is 0.23.



The solution in quadrant III is $\pi + 0.23 = 3.37$.

The solution in quadrant IV is $2\pi - 0.23 = 6.05$.

Use the CAST rule to help determine the solutions in the required interval, $0 \leq x \leq 2\pi$.

The solutions are approximately 0.82, 2.32, 3.37, or 6.05.

In Summary

Key Ideas

- In some applications, the formula contains a square of a trigonometric ratio. This leads to a quadratic trigonometric equation that can be solved algebraically or graphically.
- A quadratic trigonometric equation may have multiple solutions in the interval $0 \leq x \leq 2\pi$. Some of the solutions may be inadmissible, however, in the context of the problem.

Need to Know

- You can often factor a quadratic trigonometric equation and then solve the resulting two linear trigonometric equations. In cases where the equation cannot be factored, use the quadratic formula and then solve the resulting linear trigonometric equations.

Note: The solutions to $ax^2 + bx + c = 0$ are determined by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

- You may need to use a Pythagorean identity, compound angle formula, or double angle formula to create a quadratic equation that contains only a single trigonometric function whose arguments all match.

CHECK Your Understanding

1. Factor each expression.

a) $\sin^2 \theta - \sin \theta$

b) $\cos^2 \theta - 2 \cos \theta + 1$

c) $3 \sin^2 \theta - \sin \theta - 2$

d) $4 \cos^2 \theta - 1$

e) $24 \sin^2 x - 2 \sin x - 2$

f) $49 \tan^2 x - 64$

2. Solve the first equation in each pair of equations for y and/or z. Then use the same strategy to solve the second equation for x in the interval $0 \leq x \leq 2\pi$.

a) $y^2 = \frac{1}{3}$, $\tan^2 x = \frac{1}{3}$

b) $y^2 + y = 0$, $\sin^2 x + \sin x = 0$

c) $y - 2yz = 0$, $\cos x - 2 \cos x \sin x = 0$

d) $yz = y$, $\tan x \sec x = \tan x$

3. a) Solve the equation $6y^2 - y - 1 = 0$.

b) Solve $6 \cos^2 x - \cos x - 1 = 0$ for $0 \leq x \leq 2\pi$.

PRACTISING

4. Solve for θ, to the nearest degree, in the interval $0° \leq \theta \leq 360°$.

K

a) $\sin^2 \theta = 1$

b) $\cos^2 \theta = 1$

c) $\tan^2 \theta = 1$

d) $4 \cos^2 \theta = 1$

e) $3 \tan^2 \theta = 1$

f) $2 \sin^2 \theta = 1$

5. Solve each equation for x, where $0° \leq x \leq 360°$.

a) $\sin x \cos x = 0$

b) $\sin x (\cos x - 1) = 0$

c) $(\sin x + 1) \cos x = 0$

d) $\cos x (2 \sin x - \sqrt{3}) = 0$

e) $(\sqrt{2} \sin x - 1)(\sqrt{2} \sin x + 1) = 0$

f) $(\sin x - 1)(\cos x + 1) = 0$

6. Solve each equation for x, where $0 \leq x \leq 2\pi$.

a) $(2 \sin x - 1) \cos x = 0$

b) $(\sin x + 1)^2 = 0$

c) $(2 \cos x + \sqrt{3}) \sin x = 0$

d) $(2 \cos x - 1)(2 \sin x + \sqrt{3}) = 0$

e) $(\sqrt{2} \cos x - 1)(\sqrt{2} \cos x + 1) = 0$

f) $(\sin x + 1)(\cos x - 1) = 0$

7. Solve for θ to the nearest hundredth, where $0 \leq \theta \leq 2\pi$.

a) $2 \cos^2 \theta + \cos \theta - 1 = 0$

b) $2 \sin^2 \theta = 1 - \sin \theta$

c) $\cos^2 \theta = 2 + \cos \theta$

d) $2 \sin^2 \theta + 5 \sin \theta - 3 = 0$

e) $3 \tan^2 \theta - 2 \tan \theta = 1$

f) $12 \sin^2 \theta + \sin \theta - 6 = 0$

8. Solve each equation for x, where $0 \leq x \leq 2\pi$.

a) $\sec x \csc x - 2 \csc x = 0$

b) $3 \sec^2 x - 4 = 0$

c) $2 \sin x \sec x - 2\sqrt{3} \sin x = 0$

d) $2 \cot x + \sec^2 x = 0$

e) $\cot x \csc^2 x = 2 \cot x$

f) $3 \tan^3 x - \tan x = 0$

9. Solve each equation in the interval $0 \le x \le 2\pi$. Round to two decimal places, if necessary.

a) $5 \cos 2x - \cos x + 3 = 0$

b) $10 \cos 2x - 8 \cos x + 1 = 0$

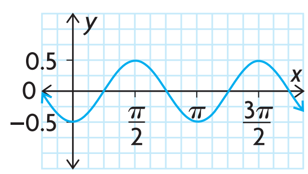
c) $4 \cos 2x + 10 \sin x - 7 = 0$

d) $-2 \cos 2x = 2 \sin x$

10. Solve the equation $8 \sin^2 x - 8 \sin x + 1 = 0$ in the interval $0 \le x \le 2\pi$.

11. The quadratic trigonometric equation $\cot^2 x - b \cot x + c = 0$ has the solutions $\frac{\pi}{6}, \frac{\pi}{4}, \frac{7\pi}{6}$, and $\frac{5\pi}{4}$ in the interval $0 \le x \le 2\pi$. What are the values of b and c?

12. The graph of the quadratic trigonometric equation $\sin^2 x - c = 0$ is shown. What is the value of c?



13. **A** Natasha is a marathon runner, and she likes to train on a 2π km stretch of rolling hills. The height, in kilometres, of the hills above sea level, relative to her home, can be modelled by the function $h(d) = 4 \cos^2 d - 1$, where d is the distance travelled in kilometres. At what intervals in the stretch of rolling hills is the height above sea level, relative to Natasha's home, less than zero?

14. **T** Solve the equation $6 \sin^2 x = 17 \cos x + 11$ for x in the interval $0 \le x \le 2\pi$.

15. **a)** Solve the equation $\sin^2 x - \sqrt{2} \cos x = \cos^2 x + \sqrt{2} \cos x + 2$ for x in the interval $0 \le x \le 2\pi$.

b) Write a general solution for the equation in part a).

16. **C** Explain why it is possible to have different numbers of solutions for quadratic trigonometric equations. Give examples to illustrate your explanation.

Extending

17. Given that $f(x) = \frac{\tan x}{1 - \tan x} - \frac{\cot x}{1 - \cot x}$, determine all the values of a in the interval $0 \le a \le 2\pi$, such that $f(x) = \tan (x + a)$.

18. Solve the equation $2 \cos 3x + \cos 2x + 1 = 0$.

19. Solve $3 \tan^2 2x = 1$, $0° \le x \le 360°$.

20. Solve $\sqrt{2} \sin \theta = \sqrt{3} - \cos \theta$, $0 \le \theta \le 2\pi$.

7 Chapter Review

FREQUENTLY ASKED Questions

Study Aid

- See Lesson 7.4, Examples 1 to 5.
- Try Chapter Review Questions 7, 8, and 9.

Q: What is the difference between a trigonometric equation and a trigonometric identity, and how can you prove that a given equation is an identity?

A: A trigonometric equation is true for one, several, or many values of the variable it contains. A trigonometric identity is an equation that involves trigonometric ratios and is true for *all* values of the variables for which the expressions on both sides are defined.

To prove that an equation is an identity, you can use algebraic manipulation on one or both sides of the equation until one side is identical to the other side. This often involves a variety of strategies, such as

- rewriting the expressions using known identities
- rewriting the expressions using compound angle formulas and double angle formulas
- using a common denominator or factoring where possible

To prove that an equation is *not* an identity, you can use a counterexample. If any value, when substituted, results in LS $\neq$ RS, then the equation is *not* an identity.

Study Aid

- See Lesson 7.5, Examples 1 to 4.
- Try Chapter Review Question 10.

Q: How can you solve a linear trigonometric equation?

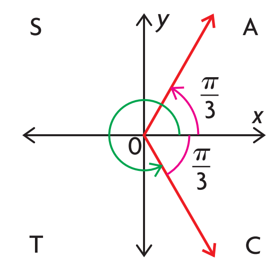
A1: You can solve a linear trigonometric equation algebraically, using special triangles, a calculator, a sketch of the graph of the corresponding function, and/or the CAST rule.

For example, to solve $2(\cos 2x + 1) = 3$ for $0 \leq x \leq 2\pi$, first rearrange the equation to isolate $\cos 2x$.

$$2 \cos 2x + 2 = 3$$
$$2 \cos 2x = 1$$
$$\cos 2x = \frac{1}{2}$$

Evaluate $\cos^{-1}\left(\frac{1}{2}\right)$ to determine the related acute angle of $2x$.

Using the 1, 2, $\sqrt{3}$ special triangle, the related angle is $\frac{\pi}{3}$.



Cosine is positive in quadrants I and IV.

$2x = \frac{\pi}{3}$ in quadrant I, so $x = \frac{\pi}{6}$.

$2x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ in quadrant IV, so $x = \frac{5\pi}{6}$.

$$\frac{\pi}{6} + \pi = \frac{7\pi}{6}$$

$$\frac{5\pi}{6} + \pi = \frac{11\pi}{6}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

Cos $2x$ has a period of π, so add π to these solutions to determine the other solutions in the given domain.

A2: You can solve a linear trigonometric equation, or verify the solutions, using a graphing calculator.

One way to solve the equation $2(\cos 2x + 1) = 3$ is to enter $Y1 = 2(\cos 2x + 1)$ and $Y2 = 3$ and determine the intersection points.

Another way to solve the equation is to enter $Y1 = 2(\cos 2x + 1) - 3$ and determine the zeros.

Q: What strategies can you use to solve a quadratic trigonometric equation?

A1: You can often factor a quadratic trigonometric equation, and then solve the resulting two linear trigonometric equations.

For example, to solve $2 \tan^2 x - \tan x - 6 = 0$, factor the left side so that $(2 \tan x + 3)(\tan x - 2) = 0$. Solve the two linear equations, $2 \tan x + 3 = 0$ and $\tan x - 2 = 0$.

If it is not factorable, you can use the quadratic formula, then solve the resulting two linear equations.

A2: You may need to use a Pythagorean identity, compound angle formula, or double angle formula to create a quadratic equation that contains only a single trigonometric function whose arguments all match.

A3: You can use a graphing calculator to solve or verify the solutions. Graph the functions defined by the two sides of the equation and determine the intersection points. You can also create a single function of the form $f(x) = 0$, graph it, and determine its zeros.

Study *Aid*

- See Lesson 7.6, Examples 1, 2, and 3.
- Try Chapter Review Questions 11, 12, and 13.

PRACTICE Questions

Lesson 7.1

1. State a trigonometric ratio that is equivalent to each of the following trigonometric ratios.

a) $\sin \frac{3\pi}{10}$

b) $\cos \frac{6\pi}{7}$

c) $-\sin \frac{13\pi}{7}$

d) $-\cos \frac{8\pi}{7}$

2. Write an equation that is equivalent to $y = -5 \sin \left(x - \frac{\pi}{2}\right) - 8$, using the cosine function.

Lesson 7.2

3. Use a compound angle formula to determine a trigonometric expression that is equivalent to each of the following expressions.

a) $\sin \left(x - \frac{4\pi}{3}\right)$

b) $\cos \left(x + \frac{3\pi}{4}\right)$

c) $\tan \left(x + \frac{\pi}{3}\right)$

d) $\cos \left(x - \frac{5\pi}{4}\right)$

4. Evaluate each expression.

a) $\dfrac{\tan \frac{\pi}{12} + \tan \frac{7\pi}{4}}{1 - \tan \frac{\pi}{12} \tan \frac{7\pi}{4}}$

b) $\cos \frac{\pi}{9} \cos \frac{19\pi}{18} - \sin \frac{\pi}{9} \sin \frac{19\pi}{18}$

Lesson 7.3

5. Simplify each expression.

a) $2 \sin \frac{\pi}{12} \cos \frac{\pi}{12}$

b) $\cos^2 \frac{\pi}{12} - \sin^2 \frac{\pi}{12}$

c) $1 - 2 \sin^2 \frac{3\pi}{8}$

d) $\dfrac{2 \tan \frac{\pi}{6}}{1 - \tan^2 \frac{\pi}{6}}$

6. Determine the values of $\sin 2x$, $\cos 2x$, and $\tan 2x$, given

a) $\sin x = \frac{3}{5}$, and x is acute

b) $\cot x = -\frac{7}{24}$, and x is obtuse

c) $\cos x = \frac{12}{13}$, and $\frac{3\pi}{2} \le x \le 2\pi$

Lesson 7.4

7. Determine whether each of the following is a trigonometric equation or a trigonometric identity.

a) $\tan 2x = \dfrac{2 \sin x \cos x}{1 - 2 \sin^2 x}$

b) $\sec^2 x - \tan^2 x = \cos x$

c) $\csc^2 x - \cot^2 x = \sin^2 x + \cos^2 x$

d) $\tan^2 x = 1$

8. Prove that $\frac{1 - \sin^2 x}{\cot^2 x} = 1 - \cos^2 x$ is a trigonometric identity.

9. Prove that $\frac{2 \sec^2 x - 2 \tan^2 x}{\csc x} = \sin 2x \sec x$ is a trigonometric identity.

Lesson 7.5

10. Solve each trigonometric equation in the interval $0 \le x \le 2\pi$.

a) $\dfrac{2}{\sin x} + 10 = 6$

b) $-\dfrac{5 \cot x}{2} + \dfrac{7}{3} = -\dfrac{1}{6}$

c) $3 + 10 \sec x - 1 = -18$

Lesson 7.6

11. a) Solve the equation $y^2 - 4 = 0$.

b) Solve $\csc^2 x - 4 = 0$ in the interval $0 \le x \le 2\pi$.

12. Solve each equation for x in the interval $0 \le x \le 2\pi$.

a) $2 \sin^2 x - \sin x - 1 = 0$

b) $\tan^2 x \sin x - \dfrac{\sin x}{3} = 0$

c) $\cos^2 x + \left(\dfrac{1 - \sqrt{2}}{2}\right) \cos x - \dfrac{\sqrt{2}}{4} = 0$

d) $25 \tan^2 x - 70 \tan x = -49$

13. Solve the equation $\frac{1}{1 + \tan^2 x} = -\cos x$ for x in the interval $0 \le x \le 2\pi$.

7 Chapter Self-Test

1. Prove that $\frac{1 - 2\sin^2 x}{\cos x + \sin x} + 2\sin\frac{x}{2}\cos\frac{x}{2} = \cos x$.
2. Solve the following equation: $\cos 2x + 2\sin^2 x - 3 = -2$, where $0 \le x \le 2\pi$.
3. Determine the solution(s) for each of the following equations, where $0 \le x \le 2\pi$.

   a) $\cos x = \frac{\sqrt{3}}{2}$ **b)** $\tan x = -\sqrt{3}$ **c)** $\sin x = -\frac{\sqrt{2}}{2}$
4. The quadratic trigonometric equation $a\cos^2 x + b\cos x - 1 = 0$ has the solutions $\frac{\pi}{3}$, π, and $\frac{5\pi}{3}$ in the interval $0 \le x \le 2\pi$. What are the values of a and b?
5. The depth of the ocean at a swim buoy can be modelled by the function $d(t) = 4 + 2\sin\left(\frac{\pi}{6}t\right)$, where d is the depth of water in metres and t is the time in hours, if $0 \le t \le 24$. Consider a day when $t = 0$ represents midnight. Determine when the depth of water is 3 m.
6. Nina needs to find the cosine of $\frac{11\pi}{4}$. If she knows the sine and cosine of π, as well as the sine and cosine of $\frac{7\pi}{4}$, how can she find the cosine of $\frac{11\pi}{4}$? What is her answer?
7. Solve $3\sin x + 2 = 1.5$, where $0 \le x \le 2\pi$.
8. The tangent of the acute angle α is 0.75, and the tangent of the acute angle β is 2.4. Without using a calculator, determine the value of $\sin(\alpha - \beta)$ and $\cos(\alpha + \beta)$.
9. The angle x lies in the interval $\frac{\pi}{2} \le x \le \pi$, and $\sin^2 x = \frac{4}{9}$. Determine the value of each of the following. Round your answers to four decimal places.

   a) $\sin 2x$ **c)** $\cos\frac{x}{2}$

   b) $\cos 2x$ **d)** $\sin 3x$
10. Use the graph of $f(x) = \cos x$ to estimate the solution of each of the following trigonometric equations in the interval $-2\pi \le x \le 2\pi$.

    a) $2 - 14\cos x = -5$

    b) $9 - 22\cos x - 1 = 19$

    c) $2 + 7.5\cos x = -5.5$