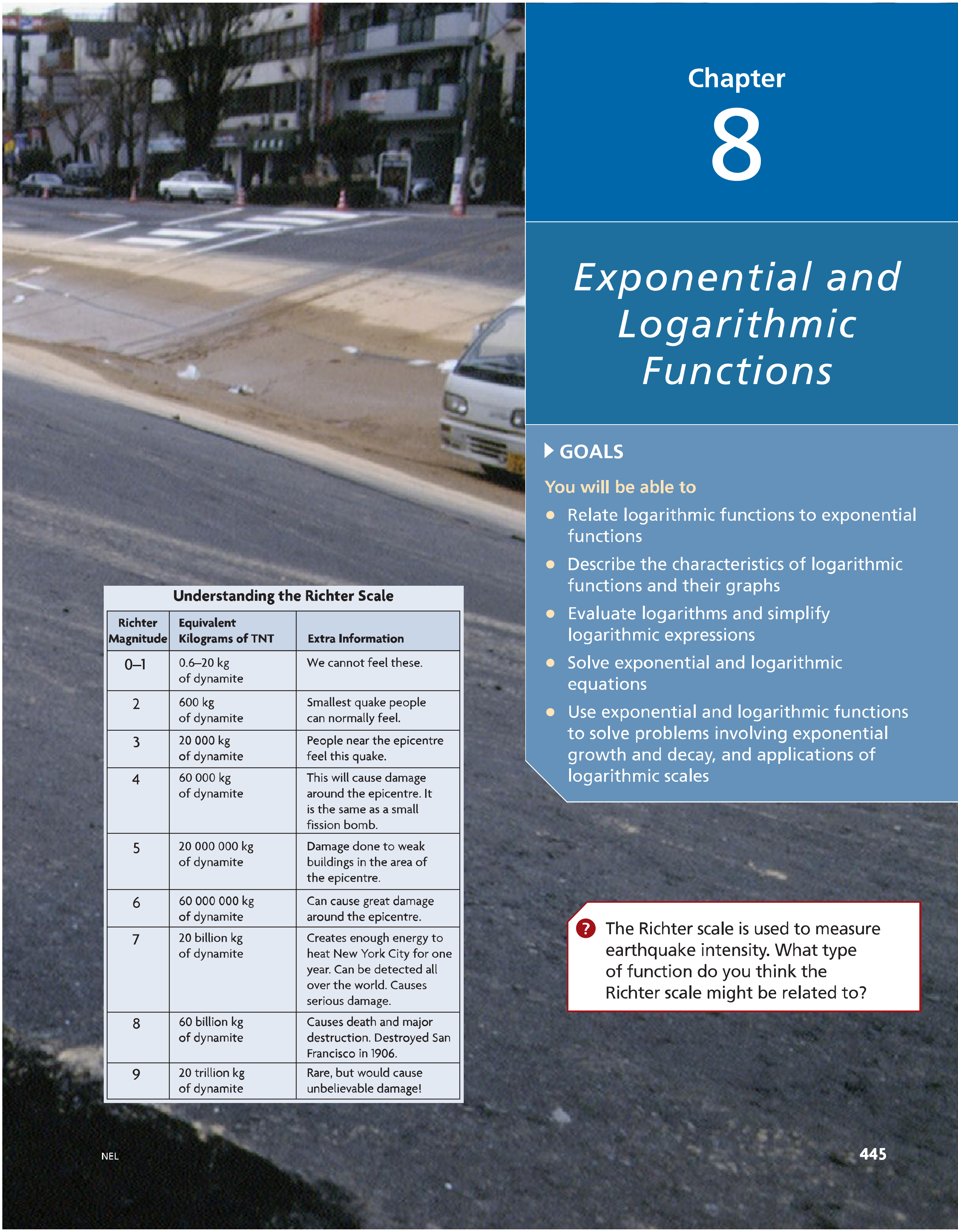


Chapter

8

Exponential and Logarithmic Functions

GOALS

You will be able to

- Relate logarithmic functions to exponential functions
- Describe the characteristics of logarithmic functions and their graphs
- Evaluate logarithms and simplify logarithmic expressions
- Solve exponential and logarithmic equations
- Use exponential and logarithmic functions to solve problems involving exponential growth and decay, and applications of logarithmic scales

Understanding the Richter Scale

Richter Magnitude	Equivalent Kilograms of TNT	Extra Information
0–1	0.6–20 kg of dynamite	We cannot feel these.
2	600 kg of dynamite	Smallest quake people can normally feel.
3	20 000 kg of dynamite	People near the epicentre feel this quake.
4	60 000 kg of dynamite	This will cause damage around the epicentre. It is the same as a small fission bomb.
5	20 000 000 kg of dynamite	Damage done to weak buildings in the area of the epicentre.
6	60 000 000 kg of dynamite	Can cause great damage around the epicentre.
7	20 billion kg of dynamite	Creates enough energy to heat New York City for one year. Can be detected all over the world. Causes serious damage.
8	60 billion kg of dynamite	Causes death and major destruction. Destroyed San Francisco in 1906.
9	20 trillion kg of dynamite	Rare, but would cause unbelievable damage!

? The Richter scale is used to measure earthquake intensity. What type of function do you think the Richter scale might be related to?

8 Getting Started

Study Aid

- For help, see the Review of Essential Skills found at the Nelson Advanced Functions website.

Question	Appendix
1, 2, 3	R-1
4, 8	R-7, R-8
6, 7	R-7

SKILLS AND CONCEPTS You Need

1. Rewrite each expression in an equivalent form, and then evaluate.

a) 5^{-2} **c)** $36^{\frac{1}{2}}$ **e)** $-121^{\frac{1}{2}}$

b) 11^0 **d)** $125^{\frac{1}{3}}$ **f)** $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$

2. Simplify each expression, and then evaluate.

a) $(3^5)(3^2)$ **c)** $\frac{10^9}{10^6}$ **e)** $(8^{\frac{1}{3}})^2$

b) $(-2)^{12}(-2)^{-10}$ **d)** $\frac{(7^6)(7^{-3})}{7^{-1}}$ **f)** $\frac{(4^{\frac{3}{4}})(4^{\frac{1}{4}})}{4^{\frac{1}{2}}}$

3. Simplify.

a) $(2m)^3$ **c)** $(16x^6)^{\frac{1}{2}}$ **e)** $(-d^4)\left(\frac{c}{d}\right)^2$

b) $(a^4b^5)^{-2}$ **d)** $\frac{x^5y^2}{x^2y}$ **f)** $\left((x^3)^{-\frac{1}{3}}\right)^{-1}$

4. Sketch a graph of each of the following exponential functions. State the domain, range, y-intercept, and the equation of the horizontal asymptote of each function.

a) $y = 2^x$ **b)** $y = \left(\frac{1}{2}\right)^x$ **c)** $y = 3^{2x} - 2$

5. a) Determine the equation of the inverse of each of the following functions.

i) $f(x) = 3x - 6$ **iii)** $f(x) = 6x^3$

ii) $f(x) = x^2 - 5$ **iv)** $f(x) = (x - 4)^2 + 3$

b) Which of the inverses you found in part a) are also functions?

6. A bacteria culture doubles every 4 h. If there are 100 bacteria in the culture initially, determine how many bacteria there will be after

a) 12 h **c)** 3.5 days

b) 1 day **d)** 1 week

7. The population of a town is declining at a rate of 1.2% per year. If the population was 15 000 in 2005, what will the population be in 2020?

8. Use a table like this to compare the graphs of $y = 3(2^x)$ and $y = 3\left(\frac{1}{2}\right)^x$.

Similarities	Differences

APPLYING What You Know

Underwater Light Intensity

For every metre below the surface of the ocean, the light intensity at the surface is reduced by 2.4%. A particular underwater camera requires at least 40% of the light at the surface of the ocean to operate.

? What is the maximum depth at which the camera can successfully take photographs underwater?

A. Explain why the function $P = 100(0.976)^m$ gives the percent of light remaining at a depth of m metres below the surface of the ocean.

B. Graph P as a function of m.

C. Determine a reasonable domain and range for this function. What restrictions might have to be placed on the domain and range?

D. Determine the light intensity at a depth of 12 m.

E. At what depth is the light intensity reduced to 40% of the intensity at the surface of the ocean? Explain how you determined your answer.

F. The water in the western end of Lake Ontario is murky, and the light intensity is reduced by 3.6%/m. Write the function that represents the percent, P, of light remaining at a depth of m metres below the surface.

G. Graph the function you created in part F.

H. Compare this graph with your graph in part B. How are the graphs alike? How are they different?

I. What is the maximum depth at which the camera could take photographs in the murky water of Lake Ontario?

8.1 Exploring the Logarithmic Function

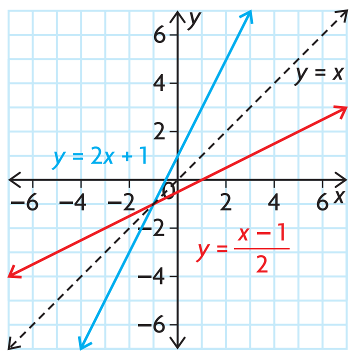
YOU WILL NEED

- graph paper

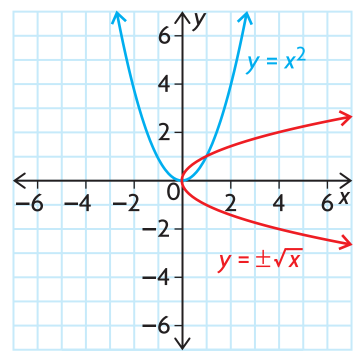
GOAL

Investigate the inverse of the exponential function.

EXPLORE the Math



The inverse of a linear function, such as $f(x) = 2x + 1$, is linear.



The inverse of a quadratic function, such as $g(x) = x^2$, has a shape that is congruent to the shape of the original function.

? What does the graph of the inverse of an exponential function like $y = 2^x$ look like, and what are its characteristics?

A. Consider the function $h(x) = 2^x$. Create a table of values, using integer values for the domain $-3 \leq x \leq 4$.

B. On graph paper, graph the exponential function in part A. State the domain and range of this function.

C. Interchange x and y in the equation for h to obtain the equation of the inverse relation. Create a table of values for this inverse relation. How does each y-value of this relation relate to the base, 2, and its corresponding x-value?

D. On the same axes that you used to graph the exponential function in part B, graph the inverse. Is the inverse a function? Explain.

E. Graph the line $y = x$ on the same axes. How do the graphs of the exponential function $h(x) = 2^x$ and the graph of the logarithmic function $h^{-1}(x) = \log_2 x$ relate to this line?

F. Repeat parts A to E, first using $j(x) = 10^x$ and then using $k(x) = \left(\frac{1}{2}\right)^x$.

G. State the domain and range of the inverses of $h(x)$, $j(x)$, and $k(x)$.

H. How is the range of each logarithmic function related to the domain of its corresponding exponential function? How is the domain of the logarithmic function related to the range of the corresponding exponential function?

I. How would you describe these logarithmic functions? Create a summary table that includes information about intercepts, asymptotes, and shapes of the graphs.

logarithmic function

The inverse of the exponential function $y = a^x$ is the function with exponential equation $x = a^y$. We write y as a function of x using the logarithmic form of this equation, $y = \log_a x$. As with the exponential function, $a > 0$ and $a \neq 1$.

Reflecting

J. What point is common to the graphs of all three logarithmic functions?

K. How are the graphs of an exponential function and the logarithmic function with the same base related?

L. How are the graphs of $h(x) = 2^x$ and $k(x) = \left(\frac{1}{2}\right)^x$ related? How are the graphs of $h^{-1}(x) = \log_2 x$ and $k^{-1}(x) = \log_{\frac{1}{2}} x$ related?

M. How does the value of a in $y = a^x$ influence the graph of $y = \log_a x$? How might you have predicted this?

N. The graph of $h^{-1}(x) = \log_2 x$ includes the point $(8, 3)$. Therefore, $3 = \log_2 8$. What is the value of $\log_2 16$ What meaning does $\log_2 x$ have? More generally, what meaning does the expression $\log_a x$ have?

In Summary

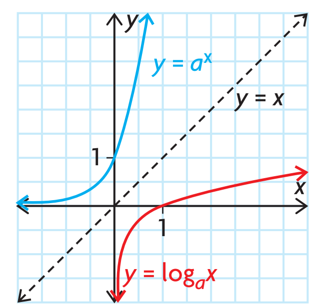
Key Ideas

- The inverse of the exponential function $y = a^x$ is also a function. It can be written as $x = a^y$. (This is the exponential form of the inverse.) An equivalent form of $x = a^y$ is $y = \log_a x$. (This is the logarithmic form of the inverse and is read as "the **logarithm** of x to the base a.") The function $y = \log_a x$ is called the logarithmic function.
- Since $x = a^y$ and $y = \log_a x$ are equivalent, a logarithm is an exponent. The expression $\log_a x$ means "the exponent that must be applied to base $\boldsymbol{a}$ to get the value of x." For example, $\log_2 8 = 3$ since $2^3 = 8$.

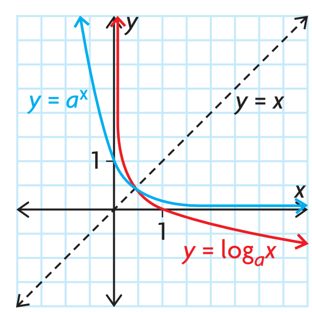
Need to Know

- The general shape of the graph of the logarithmic function depends on the value of the base.

  When $a > 1$, the exponential function is an increasing function, and the logarithmic function is also an increasing function.



  When $0 < a < 1$, the exponential function is a decreasing function and the logarithmic function is also a decreasing function.



- The y-axis is the vertical asymptote for the logarithmic function. The x-axis is the horizontal asymptote for the exponential function.
- The x-intercept of the logarithmic function is 1, while the y-intercept of the exponential function is 1.
- The domain of the logarithmic function is $\{x \in \mathbf{R} \mid x > 0\}$, since the range of the exponential function is $\{y \in \mathbf{R} \mid y > 0\}$.
- The range of the logarithmic function is $\{y \in \mathbf{R}\}$, since the domain of the exponential function is $\{x \in \mathbf{R}\}$.

FURTHER *Your Understanding*

1. Sketch a graph of the inverse of each exponential function.

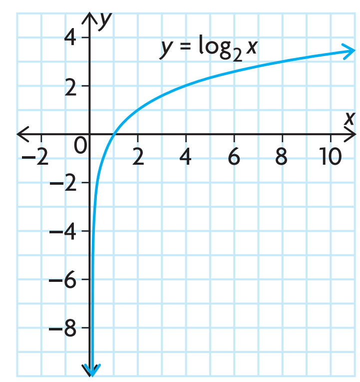
   a) $f(x) = 4^x$ c) $f(x) = \left(\frac{1}{3}\right)^x$

   b) $f(x) = 8^x$ d) $f(x) = \left(\frac{1}{5}\right)^x$

2. Write the equation of each inverse function in question 1 in
   i) exponential form
   ii) logarithmic form

3. Compare the key features of the graphs in question 1.

4. Explain how you can use the graph of $y = \log_2 x$ (at right) to help you determine the solution to $2^y = 8$.



5. Write the equation of the inverse of each exponential function in exponential form.

   a) $y = 3^x$ c) $y = \left(\frac{1}{4}\right)^x$

   b) $y = 10^x$ d) $y = m^x$

6. Write the equation of the inverse of each exponential function in question 5 in logarithmic form.

7. Write the equation of each of the following logarithmic functions in exponential form.

   a) $y = \log_5 x$ c) $y = \log_3 x$

   b) $y = \log_{10} x$ d) $y = \log_{\frac{1}{4}} x$

8. Write the equation of the inverse of each logarithmic function in question 7 in exponential form.

9. Evaluate each of the following:

   a) $\log_2 4$ c) $\log_4 64$ e) $\log_2\left(\frac{1}{2}\right)$

   b) $\log_3 27$ d) $\log_5 1$ f) $\log_3 \sqrt{3}$

10. Why can $\log_3(-9)$ not be evaluated?

11. For each of the following logarithmic functions, write the coordinates of the five points that have y-values of -2, -1, 0, 1, 2.

    a) $y = \log_2 x$ b) $y = \log_{10} x$

8.2 Transformations of Logarithmic Functions

YOU WILL NEED

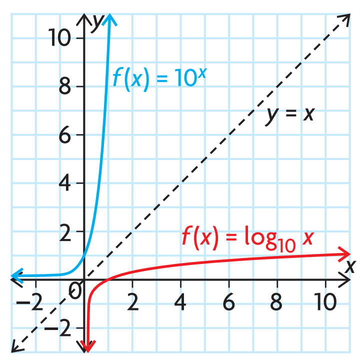
- graphing calculator

GOAL

Determine the effects of varying the parameters of the graph of $y = a \log_{10}(k(x - d)) + c$.

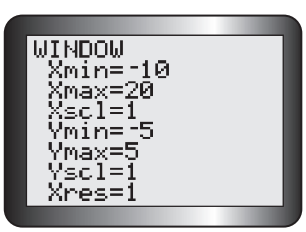
INVESTIGATE the Math

The function $f(x) = \log_{10}x$ is an example of a logarithmic function. It is the inverse of the exponential function $f(x) = 10^x$.



? How does varying the parameters of a function in the form $g(x) = a \log_{10}(k(x - d)) + c$ affect the graph of the parent function, $f(x) = \log_{10}x$?

A. The log button on a graphing calculator represents $\log_{10}x$. Graph $y = \log_{10}x$ on a graphing calculator. Use the window setting shown.



B. Consider the following functions:

- $y = \log_{10}(x - 2)$
- $y = \log_{10}(x - 4)$
- $y = \log_{10}(x + 4)$

Make a conjecture about the type of transformation that must be applied to the graph of $y = \log_{10}x$ to graph each of these functions.

C. Graph the functions in part B along with the graph of $y = \log_{10}x$. Compare each of these graphs with the graph of $y = \log_{10}x$. Was your conjecture correct? Summarize the transformations that are applied to $y = \log_{10}x$ to obtain $y = \log_{10}(x - d)$.

Communication | *Tip*

If there is no value of a in a logarithmic function ($\log_a x$), the base is understood to be 10; that is, $\log x = \log_{10}x$. Logarithms with base 10 are called common logarithms.

D. Examine the following functions:

- $y = \log_{10}x + 3$
- $y = \log_{10}x - 4$

Make a conjecture about the type of transformation that must be applied to the graph of $y = \log_{10}x$ to graph each of these functions.

E. Delete all but the first function in the equation editor, and enter the functions in part D. Graph the functions. Compare each of these graphs with the graph of $y = \log_{10}x$. Was your conjecture correct? Summarize the transformations that are applied to $y = \log_{10}x$ to obtain $y = \log_{10}x + c$.

F. State the transformations that you would need to apply to $y = \log_{10}x$ to graph the function $y = \log_{10}(x - d) + c$.

G. Make a conjecture about the transformations that you would need to apply to $y = \log_{10}x$ to graph each of the following functions:

- $y = 2\log_{10}x$
- $y = \frac{1}{3}\log_{10}x$
- $y = -2\log_{10}x$

H. Delete all but the first function in the equation editor, and enter the functions in part G. Graph the functions. Compare each of these graphs with the graph of $y = \log_{10}x$. Was your conjecture correct? Summarize the transformations that are applied to $y = \log_{10}x$ to obtain $y = a\log_{10}x$.

I. Make a conjecture about the transformations that you would need to apply to $y = \log_{10}x$ to graph each of the following functions:

- $y = \log_{10}(2x)$
- $y = \log_{10}\left(\frac{1}{5}x\right)$
- $y = \log_{10}(-2x)$

J. Delete all but the first function in the equation editor, and enter the functions in part I. Graph the functions. Compare each of these graphs with the graph of $y = \log_{10}x$. Was your conjecture correct? Summarize the transformations that are applied to $y = \log_{10}x$ to obtain $y = \log_{10}(kx)$.

K. What transformations must be applied to $y = \log_{10}x$ to graph $y = a\log_{10}(kx)$?

Reflecting

L. Describe the domain and range of $y = \log_{10}(x - d)$, $y = \log_{10}x + c$, $y = \log_{10}(kx)$, and $y = a\log_{10}x$.

M. How do the algebraic representations of the functions resulting from transformations of logarithmic functions compare with the algebraic representations of the functions resulting from transformations of polynomial, trigonometric, and exponential functions?

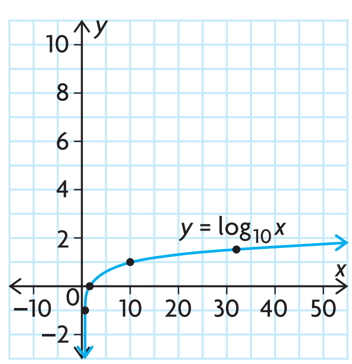
N. Identify the transformations that are related to the parameters *a, k, d,* and *c* in the general logarithmic function $y = a\,(\log_{10}k(x - d)) + c$.

APPLY the Math

EXAMPLE 1 Connecting transformations of a logarithmic function to key points of $y = \log_{10}x$

Use transformations to sketch the function $y = -2\log_{10}(x - 4)$. State the domain and range.

Solution



Sketch $y = \log_{10}x$. Choose some points on the graph, such as $\left(\frac{1}{10}, -1\right)$, (1, 0), (10, 1), and the estimated point (32, 1.5). Use these points as key points to help graph the transformed function. The vertical asymptote is the *y*-axis, $x = 0$. Apply transformations in the same order used for all functions: stretches/compressions/reflections first, followed by translations.

$(x, y) \rightarrow (x, -2y)$

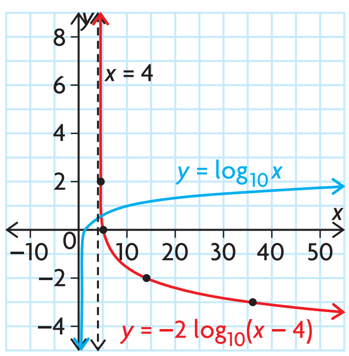
Parent Function $y = \log_{10}x$	Stretched/Reflected Function $y = -2\log_{10}x$
$\left(\frac{1}{10}, -1\right)$	$\left(\frac{1}{10}, -2(-1)\right) = \left(\frac{1}{10}, 2\right)$
(1, 0)	$(1, -2(0)) = (1, 0)$
(10, 1)	$(10, -2(1)) = (10, -2)$
(32, 1.5)	$(32, -2(1.5)) = (32, -3)$

The parent function is changed by multiplying all the *y*-coordinates by −2, resulting in a vertical stretch of factor 2 and a reflection in the *x*-axis.

$(x, -2y) \rightarrow (x + 4, -2y)$

Stretched/Reflected Function $y = -2\log_{10}x$	**Final Transformed Function** $y = -2\log_{10}(x - 4)$
$\left(\frac{1}{10}, 2\right)$	$\left(\frac{1}{10} + 4, 2\right) = \left(4\frac{1}{10}, 2\right)$
$(1, 0)$	$(1 + 4, 0) = (5, 0)$
$(10, -2)$	$(10 + 4, -2) = (14, -2)$
$(32, -3)$	$(32 + 4, -3) = (36, -3)$

Adding 4 to the *x*-coordinate of each of the transformed points results in a horizontal translation 4 units to the right.



Plot the new points and draw the graph.
The vertical asymptote is now $x = 4$ because of the translation to the right.

Domain $= \{x \in \mathbf{R} \mid x > 4\}$

The values of *x* must all be greater than 4 since the curve is to the right of the vertical asymptote.

Range $= \{y \in \mathbf{R}\}$

The range of the original function was not changed by the transformations.

EXAMPLE 2 Connecting a geometric description of a function to an algebraic representation

The logarithmic function $y = \log_{10}x$ has been vertically compressed by a factor of $\frac{2}{3}$, horizontally stretched by a factor of 4, and then reflected in the y-axis. It has also been horizontally translated so that the vertical asymptote is $x = -2$ and then vertically translated 3 units down. Write an equation of the transformed function, and state its domain and range.

Solution

$y = a\log_{10}(k(x - d)) + c$ ← Write the general form of the logarithmic equation.

$y = \frac{2}{3}\log_{10}\left(-\frac{1}{4}(x + 2)\right) - 3$ ←

Since the function has been vertically compressed by a factor of $\frac{2}{3}$, $a = \frac{2}{3}$.

Since the function has been horizontally stretched by a factor of 4, $\frac{1}{k} = 4$, so $k = \frac{1}{4}$.

The function has been reflected in the y-axis, so k is negative.

The vertical asymptote of the parent function is $x = 0$.

Since the asymptote of the transformed function is $x = -2$, the parent function has been horizontally translated 2 units left, so $d = -2$.

The function has been vertically translated 3 units down, so $c = -3$.

Domain $= \{x \in \mathbf{R} \mid x < -2\}$ ← The curve is to the left of the vertical asymptote, so the domain is $x < -2$.

Range $= \{y \in \mathbf{R}\}$ ← The range is the same as the range of the parent function.

In Summary

Key Ideas

- A logarithmic function of the form $f(x) = a \log_{10}(k(x - d)) + c$ can be graphed by applying the appropriate transformations to the parent function, $f(x) = \log_{10}x$.
- To graph a transformed logarithmic function, apply the stretches/compressions/reflections given by parameters a and k first. Then apply the vertical and horizontal translation given by the parameters c and d.

Need to Know

- Consider a logarithmic function of the form $f(x) = a \log_{10}(k(x - d)) + c$.

Transformations of the Parent Function
$\lvert a \rvert$ gives the vertical stretch/compression factor. If $a < 0$, there is also a reflection in the x-axis.
$\left\lvert \frac{1}{k} \right\rvert$ gives the horizontal stretch/compression factor. If $k < 0$, there is also a reflection in the y-axis.
d gives the horizontal translation.
c gives the vertical translation.

- The vertical asymptote changes when a horizontal translation is applied. The domain of a transformed logarithmic function depends on where the vertical asymptote is located and whether the function is to the left or the right of the vertical asymptote. If the function is to the left of the asymptote $x = d$, the domain is $x < d$. If it is to the right of the asymptote, the domain is $x > d$.
- The range of a transformed logarithmic function is always $\{y \in \mathbf{R}\}$.

CHECK Your Understanding

1. Each of the following functions is a transformation of $f(x) = \log_{10}x$. Describe the transformation that must be applied to $f(x)$ to graph $g(x)$.

a) $g(x) = 3 \log_{10}x$

b) $g(x) = \log_{10}(2x)$

c) $g(x) = \log_{10}x - 5$

d) $g(x) = \log_{10}(x + 4)$

2. a) State the coordinates of the images of the points $\left(\frac{1}{10}, -1\right)$, (1, 0), and (10, 1) for each of the functions in question 1.

b) State the domain and range of each transformed function, $g(x)$, in question 1.

3. Given the parent function $f(x) = \log_{10}x$, state the equation of the function that results from each of the following pairs of transformations:

a) vertical stretch by a factor of 5, vertical translation 3 units up

b) reflection in the x-axis, horizontal compression by a factor of $\frac{1}{3}$

c) horizontal translation 4 units left, vertical translation 3 units down

d) reflection in the x-axis, horizontal translation 4 units right

PRACTISING

4. Let $f(x) = \log_{10}x$. For each function $g(x)$

K
a) state the transformations that must be applied to f to produce the graph of g.

b) State the coordinates of the points on g that are images of the points (1, 0) and (10, 1) on the graph of f.

c) State the equation of the asymptote.

d) State the domain and range.

i) $g(x) = -4\log_{10}x + 5$

ii) $g(x) = \frac{1}{2}\log_{10}(x - 6) + 3$

iii) $g(x) = \log_{10}(3x) - 4$

iv) $g(x) = 2\log_{10}[-2(x + 2)]$

v) $g(x) = \log_{10}(2x + 4)$

vi) $g(x) = \log_{10}(-x - 2)$

5. Sketch the graph of each function using transformations. State the domain and range.

a) $f(x) = 3\log_{10}x + 3$

b) $g(x) = -\log_{10}(x - 6)$

c) $h(x) = \log_{10}2x$

d) $j(x) = \log_{10}0.5x - 1$

e) $k(x) = 4\log_{10}\left(\frac{1}{6}x\right) - 2$

f) $r(x) = \log_{10}(-2x - 4)$

6. Compare the functions $f(x) = 10^{\left(\frac{x}{3}\right)} + 1$ and $g(x) = 3\log_{10}(x - 1)$.

7. **a)** Describe how the graphs of $f(x) = \log_3x$, $g(x) = \log_3(x + 4)$, and $h(x) = \log_3x + 4$ are similar yet different, without drawing the graphs.

b) Describe how the graphs of $f(x) = \log_3x$, $m(x) = 4\log_3x$, and $n(x) = \log_34x$ are similar yet different, without drawing the graphs.

8. The function $f(x) = \log_{10}x$ has the point $(10, 1)$ on its graph.

A
If $f(x)$ is vertically stretched by a factor of 3, reflected in the x-axis, horizontally stretched by a factor of 2, horizontally translated 5 units to the right, and vertically translated 2 units up, determine

a) the equation of the transformed function

b) the coordinates of the image point transformed from $(10, 1)$

c) the domain and range of the transformed function

9. State the transformations that are needed to turn $y = 4\log_{10}(x - 4)$

T
into $y = -2\log_{10}(x + 1)$.

10. Describe three characteristics of the function $y = \log_{10}x$ that remain

C
unchanged under the following transformations: a vertical stretch by a factor of 4 and a horizontal compression by a factor of 2.

Extending

11. Sketch the graph of $f(x) = \frac{-2}{\log_2(x + 2)}$.

8.3 Evaluating Logarithms

GOAL

Evaluate logarithmic expressions, and approximate the logarithm of a number to any base.

YOU WILL NEED

- graphing calculator

LEARN ABOUT the Math

Jackson knows that a rumour spreads very quickly. He tells three people a rumour. By the end of the next hour, each of these people has told three more people. Each person who hears the rumour tells three more people in the next hour. Jackson has written an algebraic model, $N(t) = 3^{t+1}$, to represent the number of people who hear the rumour within a particular hour, where $N(t)$ is the number of people told during hour t and $t = 1$ corresponds to the hour during which the first three people heard the rumour and started telling others.

? In which hour will an additional 2187 people hear the rumour?

EXAMPLE 1 Selecting a strategy to solve a problem

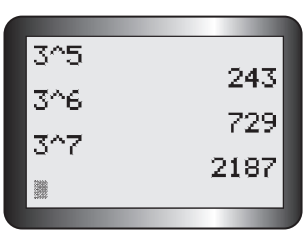
Determine the hour in which an additional 2187 people will hear the rumour.

Solution A: Using a guess-and-check strategy to solve an exponential equation

$N(t) = 2187$

$2187 = 3^{t+1}$

Substitute 2187 for $N(t)$ in the equation.
It is easier to solve the equation if both sides are written as powers with the same base. Using guess and check, write 2187 as a power of 3.



$3^7 = 3^{t+1}$

$7 = t + 1$

$6 = t$

Both sides will be equal when both powers of 3 have the same exponent. Equate the exponents, and solve for t.

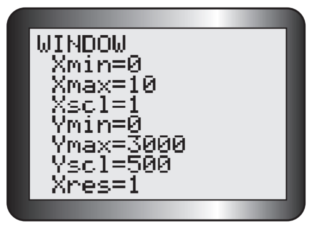
Another 2187 students will hear the rumour during the 6th hour.

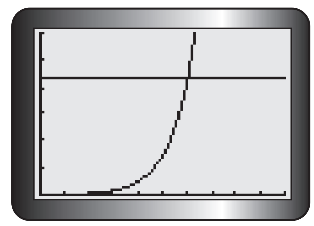
Solution B: Using a graphing calculator to solve an exponential equation

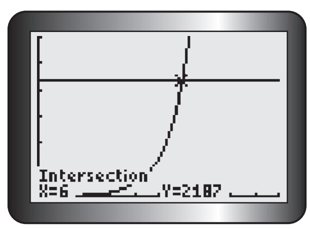
$N(t) = 2187$

$2187 = 3^{t+1}$

A graph can be used to solve the equation.
Enter $y = 3^{x+1}$ in Y1 of the equation editor and $y = 2817$ in Y2. Graph using a window that corresponds to the domain and range in this situation.







The point of intersection for the two functions is the solution to the equation. Use the intersect operation to determine this point.

Another 2187 students will hear the rumour during the 6th hour.

Solution C: Rewriting an exponential equation in logarithmic form

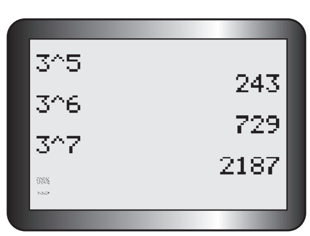
$N(t) = 2187$

$2187 = 3^{t+1}$

Determine the value of the exponent t, when $N(t) = 2187$. To solve for t, rewrite the equation in logarithmic form.

$t + 1 = \log_3 2187$

Since a logarithm is an exponent, evaluate $\log_3 2187$ by determining the exponent to which the base 3 must be raised to get 2187. Use guess and check.



$t + 1 = 7$

$t = 6$

Another 2187 students will hear the rumour during the 6th hour.

Reflecting

A. Solutions A and B used the exponential form of the model, but different strategies. Which one of these strategies will only work for some equations? Explain why.

B. Solution C used the logarithmic form of the model. Is there any advantage of rewriting the model in this form? Explain.

C. If you had to solve the equation $3^{t+1} = 1000$, which strategy would you use? Explain your reasons.

APPLY the Math

EXAMPLE 2 Using reasoning to evaluate logarithmic expressions

Use the definition of a logarithm to determine the value of each expression.

a) $\log_4 64$

b) $\log_3\left(\frac{1}{27}\right)$

c) $\log_2(-4)$

d) $\log_5 \sqrt[3]{25}$

Solution

a) $\log_4 64 = x$ — Determine the exponent to which 4 must be raised to get 64.

$4^x = 64$ — Rewrite the equation in exponential form.

$4^x = 4^3$ — Rewrite 64 as a power of 4.

$x = 3$ — The exponent is 3.

b) $\log_3\left(\frac{1}{27}\right) = x$ — Determine the exponent to which 3 must be raised to get $\frac{1}{27}$.

$3^x = \frac{1}{27}$ — Rewrite the equation in exponential form.

$3^x = 3^{-3}$ — Since $\frac{1}{27} = \frac{1}{3^3}$, $\frac{1}{27}$ can be replaced with 3^{-3}.

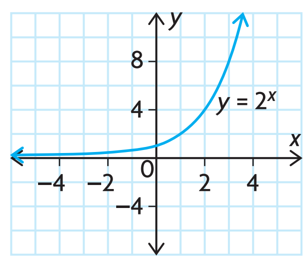
$x = -3$ — The exponent is -3.

c) $\log_2(-4) = x$ — Determine the exponent to which 2 must be raised to get -4.

$2^x = -4$ — Rewrite the equation in exponential form.

There is no solution. — Since 2 is a positive number, there will never be a negative result when 2 is raised to an exponent. The domain of any logarithmic function is $x > 0$.

Recall that the range of $y = 2^x$ is $\{y \in \mathbf{R} \mid y > 0\}$.



d) $\log_5 \sqrt[3]{25} = x$ — Determine the exponent to which 5 must be raised to get $\sqrt[3]{25}$.

$$5^x = \sqrt[3]{25}$$
$$5^x = \sqrt[3]{5^2}$$
$$5^x = 5^{\frac{2}{3}}$$

Rewrite the radical using the equivalent fractional exponent.

$$x = \frac{2}{3}$$

The exponent is $\frac{2}{3}$.

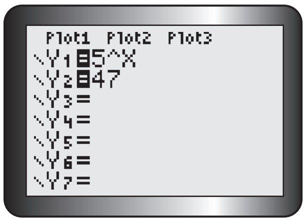
EXAMPLE 3 Selecting a strategy to estimate the logarithm of a number

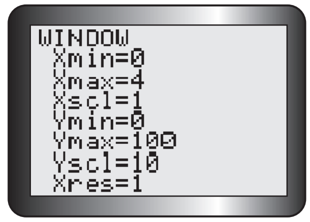
Determine the approximate value of $\log_5 47$.

Solution A: Using graphing technology

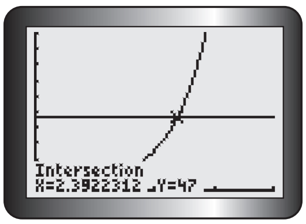
$\log_5 47 = x$ — Determine the exponent to which 5 must be raised to get 47.

$5^x = 47$ — Rewrite the equation in exponential form.





Graph the functions $y = 5^x$ and $y = 47$ using a suitable window.



Determine the point of intersection to estimate the value of *x*.

$x \doteq 2.39$

Solution B: Using guess and check

$\log_5 47 = x$

$5^x = 47$ — Rewrite the equation in exponential form.

$5^2 = 25$ and $5^3 = 125$ — The exponent must be between 2 and 3.

$5^{2.5} \doteq 55.9$ — Try 2.5. The result is too high.

$5^{2.25} \doteq 37.38$ — Try halfway between 2 and 2.5. The result is too low.

$5^{2.375} \doteq 45.71$ — Try halfway between 2.25 and 2.5. The result is getting close.

$5^{2.4} \doteq 47.59$ — Next try 2.4. The result is a little bit too high.

$5^{2.3875} \doteq 46.64$ — Average 2.4 and 2.375. The result is very close.

$5^{2.39375} \doteq 47.12$ — Refine the guess by averaging 2.4 and 2.3875.

The value is approximately 2.39.

Tech *Support*

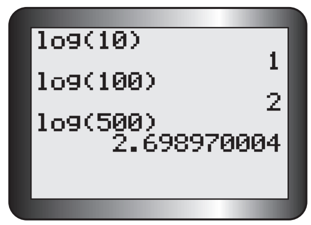
For help using the graphing calculator to find points of intersection, see Technical Appendix, T-12.

EXAMPLE 4 Selecting a strategy to evaluate common logarithms

Use the log key on a calculator to evaluate the following logarithms. Explain how the calculator determined the values.

a) log 10 **b)** log 100 **c)** log 500

Solution



Notice that no base is given with the logarithms. Recall that $\log x = \log_{10} x$.

a) $\log_{10} 10 = x$

$10^x = 10$, so $x = 1$

b) $\log_{10} 100 = x$

$10^x = 100$, so $x = 2$

c) $\log_{10} 500 = x$

$10^x = 500$, so $x \doteq 2.7$

Let x represent the value of each expression. Rewrite each equation in exponential form.

The calculator determined the exponents that must be applied to base 10 to get 10, 100, and 500.

EXAMPLE 5 Examining some general properties of logarithms

Evaluate each of the following logarithms.

a) $\log_6 1$ **b)** $\log_5 5^x$ **c)** $6^{\log_6 x}$

Solution

a) $\log_6 1 = 0$

The value of the expression is the exponent to which 6 must be raised to get 1. A power equals 1 only when its exponent is 0.

$\log_6 1 = x$

$6^x = 1$

$6^x = 6^0$

$x = 0$

To verify, let the expression equal x and rewrite the expression in exponential form.

b) $\log_5 5^x = x$

The value of the expression is the exponent to which 5 must be raised to get 5^x. The exponent must be x.

$$\begin{aligned} \log_5 5^x &= y \\ 5^y &= 5^x \\ y &= x \\ \log_5 5^x &= x \end{aligned}$$

To verify, let the expression equal y and rewrite the expression in exponential form.

c) $6^{\log_6 x}$

This expression is written in exponential form. Let the expression equal y, and rewrite it in logarithmic form.

$$\begin{aligned} 6^{\log_6 x} &= y \\ \log_6 y &= \log_6 x \\ y &= x \\ 6^{\log_6 x} &= x \end{aligned}$$

The left side equals the right side only if x and y are equal.

In Summary

Key Ideas

- Simple exponential equations can be solved using a variety of strategies:
  - expressing both sides as powers with a common base and then equating the exponents
  - graphing both sides of the equation using graphing technology and then determining the point of intersection
  - rewriting the equation in logarithmic form and simplifying
- A logarithm is an exponent. The logarithm of a number to a given base is the exponent to which the base must be raised to get the number.

Need to Know

- Logarithms of negative numbers do not exist, because a negative number cannot be written as a power of a positive base.
- A logarithm written with any base can be estimated with a calculator, using graphing technology, or guess and check.
- The expression $\log x$ is called a common logarithm. It means $\log_{10} x$, and it can be evaluated using the log key on a calculator.
- The following are some properties of logarithms, where $a > 0$ and $a \neq 1$:
  - $\log_a 1 = 0$
  - $\log_a a^x = x$
  - $a^{\log_a x} = x$

CHECK Your Understanding

1. Express in logarithmic form.

a) $4^2 = 16$
b) $3^4 = 81$
c) $8^0 = 1$
d) $6^{-2} = \frac{1}{36}$
e) $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$
f) $8^{\frac{1}{3}} = 2$

2. Express in exponential form.

a) $\log_2 8 = 3$
b) $\log_5 \frac{1}{25} = -2$
c) $\log_3 81 = 4$
d) $\log_{\frac{1}{6}} 216 = -3$
e) $\log_6 \sqrt{6} = \frac{1}{2}$
f) $\log_{10} 1 = 0$

3. Evaluate.

a) $\log_5 5$
b) $\log_7 1$
c) $\log_2 \left(\frac{1}{4}\right)$
d) $\log_7 \sqrt{7}$
e) $\log_{\frac{2}{3}} \left(\frac{8}{27}\right)$
f) $\log_2 \sqrt[3]{2}$

PRACTISING

4. Solve for x. Round your answers to two decimal places, if necessary.

a) $\log \left(\frac{1}{10}\right) = x$
b) $\log 1 = x$
c) $\log (1\,000\,000) = x$
d) $\log 25 = x$
e) $\log x = 0.25$
f) $\log x = -2$

5. Evaluate.

a) $\log_6 \sqrt{6}$
b) $\log_5 125 - \log_5 25$
c) $\log_3 81 + \log_4 64$
d) $\log_2 \frac{1}{4} - \log_3 1$
e) $\log_5 \sqrt[3]{5}$
f) $\log_3 \sqrt{27}$

6. **K** Use your knowledge of logarithms to solve each of the following equations for x.

a) $\log_5 x = 3$
b) $\log_x 27 = 3$
c) $\log_4 \frac{1}{64} = x$
d) $\log_{\frac{1}{4}} x = -2$
e) $\log_5 x = \frac{1}{2}$
f) $\log_4 x = 1.5$

7. Graph $f(x) = 3^x$. Use your graph to estimate each of the following logarithms.

a) $\log_3 17$ **b)** $\log_3 36$ **c)** $\log_3 112$ **d)** $\log_3 143$

8. Estimate the value of each of the following logarithms to two decimal places.

a) $\log_4 32$ **b)** $\log_6 115$ **c)** $\log_3 212$ **d)** $\log_{11} 896$

9. Evaluate.

a) $\log_3 3^5$
b) $5^{\log_5 25}$
c) $4^{\log_4 \frac{1}{16}}$
d) $\log_m m^n$
e) $a^{\log_a b}$
f) $\log_{\frac{1}{10}} 1$

10. Evaluate $\log_2 16^{\frac{1}{3}}$.

11. **A** The number of mold spores in a petri dish increases by a factor of 10 every week. If there are initially 40 spores in the dish, how long will it take for there to be 2000 spores?

12. **Half-life** is the time it takes for half of a sample of a radioactive element to decay. The function $M(t) = P\left(\frac{1}{2}\right)^{\frac{t}{h}}$ can be used to calculate the mass remaining if the half-life is h and the initial mass is P. The half-life of radium is 1620 years.

a) If a laboratory has 5 g of radium, how much will there be in 150 years?
b) How many years will it take until the laboratory has only 4 g of radium?

13. The function $s(d) = 0.159 + 0.118 \log d$ relates the slope, s, of a beach to the average diameter, d, in millimetres, of the sand particles on the beach. Which beach has a steeper slope: beach A, which has very fine sand with $d = 0.0625$, or beach B, which has very coarse sand with $d = 1$? Justify your decision.

14. The function $S(d) = 93 \log d + 65$ relates the speed of the wind, S, in miles per hour, near the centre of a tornado to the distance that the tornado travels, d, in miles.

a) If a tornado travels a distance of about 50 miles, estimate its wind speed near its centre.
b) If a tornado has sustained winds of approximately 250 mph, estimate the distance it can travel.

15. The astronomer Johannes Kepler (1571–1630) determined that the time, D, in days, for a planet to revolve around the Sun is related to the planet's average distance from the Sun, k, in millions of kilometres. This relation is defined by the equation $\log D = \frac{3}{2} \log k - 0.7$. Verify that Kepler's equation gives a good approximation of the time it takes for Earth to revolve around the Sun, if Earth is about 150 000 000 km from the Sun.

16. Use Kepler's equation from question 15 to estimate the period of revolution of each of the following planets about the Sun, given its distance from the Sun.

a) Uranus, 2854 million kilometres
b) Neptune, 4473 million kilometres

17. The doubling function $y = y_0 2^{\frac{t}{D}}$ can be used to model exponential growth when the doubling time is D. The bacterium *Escherichia coli* has a doubling period of 0.32 h. A culture of *E. coli* starts with 100 bacteria.
T

a) Determine the equation for the number of bacteria, y, in x hours.
b) Graph your equation.
c) Graph the inverse.
d) Determine the equation of the inverse. What does this equation represent?
e) How many hours will it take for there to be 450 bacteria in the culture? Explain your strategy.

18. To evaluate a logarithm whose base is not 10 you can use the following relationship (which will be developed in section 8.5):

$$\log_a b = \frac{\log b}{\log a}$$

Use this to evaluate each of the following to four decimal places.

a) $\log_5 5$
b) $\log_2 10$
c) $\log_5 45$
d) $\log_8 92$
e) $\log_4 0.5$
f) $\log_7 325$

19. Consider the expression $\log_5 a$.
C

a) For what values of a will this expression yield positive numbers?
b) For what values of a will this expression yield negative numbers?
c) For what values of a will this expression be undefined?

Extending

20. Simplify.

a) $3^{\log_3 27} + 10^{\log_{10} 1000}$
b) $5^{\log_5 8} - 3^{\log_3 5 + \log_3 7}$

21. Determine the inverse of each relation.

a) $y = \sqrt[3]{x}$
b) $y = 3(2)^x$
c) $y = (0.5)^{x+2}$
d) $y = 3\log_2(x - 3) + 2$

22. Graph each function and its inverse. State the domain, range, and asymptote of each. Determine the equation of the inverse.

a) $y = 3\log(x + 6)$
b) $y = -2\log_5 3x$
c) $y = 2 + 3\log x$
d) $y = 20(8)^x$
e) $y = 2(3)^{x+2}$
f) $y = -5^x - 3$

23. For the function $y = \log_{10} x$, where $0 < x < 1000$, how many integer values of y are possible if $y > -20$?

8.4 Laws of Logarithms

GOAL

Recognize the connection between the laws of exponents and the laws of logarithms, and use the laws of logarithms to simplify expressions.

LEARN ABOUT the Math

Since the logarithm function with base a is the inverse of the exponential function with base a, it makes sense that each exponent law should have a corresponding logarithmic law. You have seen that the exponential property $a^0 = 1$ has the corresponding logarithmic property $\log_a 1 = 0$.

Recall the following exponent laws:

- product law: $a^x \times a^y = a^{x+y}$
- quotient law: $a^x \div a^y = a^{x-y}$
- power law: $(a^x)^y = a^{xy}$

? What are the corresponding laws of logarithms for these exponent laws?

EXAMPLE 1 Connecting the product laws

Determine an equivalent expression for $\log_a(mn)$, where a, m, and n are positive numbers and $a \neq 1$.

Solution

Let $m = a^x$ and $n = a^y$.

Since a, m, and n are all positive, m and n can be expressed as powers of a.

$mn = (a^x)(a^y) = a^{x+y}$

Substitute the expressions for m and n into the product mn. Simplify using the product law for exponents.

$\log_a(mn) = \log_a(a^{x+y})$

These expressions must be equal since $mn = a^{x+y}$, as shown above. On the right side of this equation, the exponent that must be applied to a to get a^{x+y} is $x + y$.

$\log_a(mn) = x + y$

$m = a^x$ so $\log_a m = x$

$n = a^y$ so $\log_a n = y$

Write the powers involving m and n in logarithmic form. Substitute the logarithmic expressions into the equation $\log_a(mn) = x + y$.

$\log_a(mn) = \log_a m + \log_a n$

The logarithm of a product is equal to the sum of the logarithms of the factors.

EXAMPLE 2 Connecting the quotient laws

Determine an equivalent expression for $\log_a\left(\frac{m}{n}\right)$, where a, m, and n are positive numbers and $a \neq 1$.

Solution

Let $m = a^x$ and $n = a^y$.

Since a, m, and n are all positive, m and n can be expressed as powers of a.

$\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}$

Substitute the expression for m and n into the quotient $\frac{m}{n}$. Simplify using the quotient law for exponents.

$\log_a\left(\frac{m}{n}\right) = \log_a(a^{x-y})$

$\log_a\left(\frac{m}{n}\right) = x - y$

These expressions must be equal since $\frac{m}{n} = a^{x-y}$, as shown above. On the right side of this equation, the exponent that must be applied to a to get a^{x-y} is $x - y$.

$m = a^x$ so $\log_a m = x$

$n = a^y$ so $\log_a n = y$

Write the powers involving m and n in logarithmic form. Substitute the logarithmic expressions into the equation $\log_a\left(\frac{m}{n}\right) = x - y$.

$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.

EXAMPLE 3 Connecting the power laws

Determine an equivalent expression for $\log_a(m^n)$, where a, m, and n are positive numbers and $a \neq 1$.

Solution

Let $m = a^x$. ← Since a and m are positive, m can be expressed as a power of a.

$m^n = (a^x)^n = a^{nx}$ ← Substitute the expression for m into the power m^n. Simplify using the power law for exponents.

$\log_a(m^n) = \log_a(a^{nx})$
$\log_a(m^n) = nx$ ← These expressions must be equal since $m^n = a^{nx}$, as shown above. On the right side of this equation, the exponent that must be applied to a to get a^{nx} is nx.

$m = a^x$, so $\log_a m = x$ ← Write the power involving m in logarithmic form. Substitute the logarithmic expressions into the equation $\log_a(m^n) = nx$.

$\log_a(m^n) = n \log_a m$ ← The logarithm of a power of a number is equal to the exponent multiplied by the logarithm of the number.

Reflecting

A. Which exponent law is related to each logarithm law? How can this be seen in the operations used in each pair of related laws?

B. Why does it make sense that each exponent law has a related logarithm law?

C. Can $\log_2 5 + \log_3 7$ be expressed as a single logarithm using any of the logarithm laws? Explain.

D. Can $\log_6 12 - \log_4 8$ be expressed as a single logarithm using any of the logarithm laws? Explain.

APPLY the Math

EXAMPLE 4 Selecting strategies to simplify logarithmic expressions

Simplify each logarithmic expression.

a) $\log_3 6 + \log_3 4.5$ **b)** $\log_2 48 - \log_2 3$ **c)** $\log_5 \sqrt[3]{25}$

Solution

a) $\log_3 6 + \log_3 4.5$ ← Since the logarithms have the same base, the sum can be simplified.

$= \log_3(6 \times 4.5)$ ← The sum of the logarithms of two numbers is the logarithm of their product.

$= \log_3 27$ ← The exponent that must be applied to 3 to get 27 is 3.

$= 3$

b) $\log_2 48 - \log_2 3$ ← These logarithms have the same base, so the difference of the logarithms of the two numbers can be written as the logarithm of their quotient.

$= \log_2\left(\frac{48}{3}\right)$

$= \log_2 16$ ← The exponent that must be applied to 2 to get 16 is 4.

$= 4$

c) $\log_5 \sqrt[3]{25}$

$= \log_5 25^{\frac{1}{3}}$ ← Change the cube root into a rational exponent.

$= \frac{1}{3}\log_5 25$ ← The logarithm of a power is the same as the exponent multiplied by the logarithm of the base of the power.

$= \frac{1}{3} \times 2$ ← Evaluate $\log_5 25$, and then multiply the result by the fraction.

$= \frac{2}{3}$

Communication *Tip*

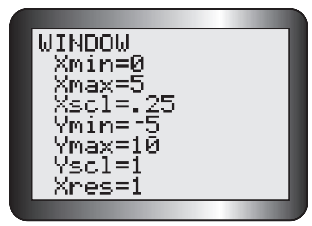
The laws of logarithms are generalizations that simplify the calculation of logarithms with the same base, much like the laws of exponents simplify the calculation of powers with the same base. The laws of logarithms and the laws of exponents can be used both forward and backward to simplify and evaluate expressions.

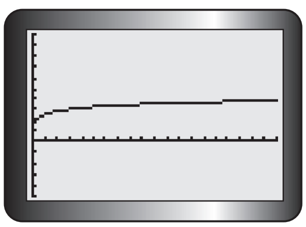
EXAMPLE 5 Connecting laws of logarithms to graphs of logarithmic functions

Graph the functions $f(x) = \log(1000x)$ and $g(x) = 3 + \log x$. How do the graphs compare? Explain your findings algebraically.

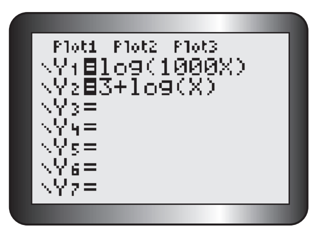
Solution

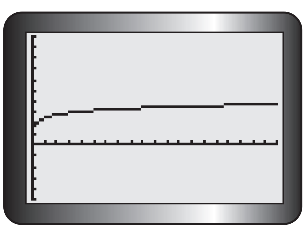
Graph the function $f(x)$ in Y1 with a graphing calculator, using the following window settings.





Add the function $g(x)$ in Y2 using the same window. The two graphs are identical on the screen.





The graphs are equivalent.

$$f(x) = \log(1000x)$$

Notice that $\log(1000x)$ is the logarithm of a product.

$$= \log 1000 + \log x$$

Rewrite the logarithm of the product as the sum of the logarithms of the factors.

$$= 3 + \log x$$

Evaluate log 1000.

$$= g(x)$$

The result is equivalent to the function $g(x)$.

EXAMPLE 6 Selecting strategies to simplify logarithmic expressions

Use the properties of logarithms to express $\log_a\sqrt{\dfrac{x^3y^2}{w}}$ in terms of $\log_a x$, $\log_a y$, and $\log_a w$.

Solution

$\log_a\sqrt{\dfrac{x^3y^2}{w}} = \log_a\left(\dfrac{x^3y^2}{w}\right)^{\frac{1}{2}}$ ← Express the square root using the rational exponent of $\frac{1}{2}$.

$= \dfrac{1}{2}\log_a\left(\dfrac{x^3y^2}{w}\right)$ ← Use the power law of logarithms to write an equivalent expression.

$= \dfrac{1}{2}(\log_a x^3y^2 - \log_a w)$ ← Express the logarithm of the quotient of x^3y^2 and w as a difference.

$= \dfrac{1}{2}(\log_a x^3 + \log_a y^2 - \log_a w)$ ← Express the logarithm of the product of x^3y^2 as a sum.

$= \dfrac{1}{2}\log_a x^3 + \dfrac{1}{2}\log_a y^2 - \dfrac{1}{2}\log_a w$ ← Expand using the distributive property.

$= \dfrac{1}{2} \times 3\log_a x + \dfrac{1}{2} \times 2\log_a y - \dfrac{1}{2}\log_a w$ ← Use the power law of logarithms again to write an equivalent expression where appropriate.

$= \dfrac{3}{2}\log_a x + \log_a y - \dfrac{1}{2}\log_a w$ ← Simplify.

In Summary

Key Ideas

- The laws of logarithms are directly related to the laws of exponents, since logarithms are exponents.
- The laws of logarithms can be used to simplify logarithmic expressions if all the logarithms have the same base.

Need to Know

- The laws of logarithms are as follows, where $a > 0$, $x > 0$, $y > 0$, and $a \neq 1$:
  - **product law of logarithms:** $\log_a xy = \log_a x + \log_a y$
  - **quotient law of logarithms:** $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
  - **power law of logarithms:** $\log_a x^r = r\log_a x$

CHECK Your Understanding

1. Write each expression as a sum or difference of logarithms.

a) $\log(45 \times 68)$

b) $\log_m pq$

c) $\log\left(\frac{123}{31}\right)$

d) $\log_m\left(\frac{p}{q}\right)$

e) $\log_2(14 \times 9)$

f) $\log_4\left(\frac{81}{30}\right)$

2. Express each of the following as a logarithm of a product or quotient.

a) $\log 5 + \log 7$

b) $\log_3 4 - \log_3 2$

c) $\log_m a + \log_m b$

d) $\log x - \log y$

e) $\log_6 7 + \log_6 8 + \log_6 9$

f) $\log_4 10 + \log_4 12 - \log_4 20$

3. Express each of the following in the form $r \log_a x$.

a) $\log 5^2$

b) $\log_3 7^{-1}$

c) $\log_m p^q$

d) $\log \sqrt[3]{45}$

e) $\log_7(36)^{0.5}$

f) $\log_5 \sqrt[5]{125}$

PRACTISING

4. **K** Use the laws of logarithms to simplify and then evaluate each expression.

a) $\log_3 135 - \log_3 5$

b) $\log_5 10 + \log_5 2.5$

c) $\log 50 + \log 2$

d) $\log_4 4^7$

e) $\log_2 224 - \log_2 7$

f) $\log \sqrt{10}$

5. Describe how the graphs of $y = \log_2(4x)$, $y = \log_2(8x)$, and $y = \log_2\left(\frac{x}{2}\right)$ are related to the graph of $y = \log_2 x$.

6. Evaluate the following logarithms.

a) $\log_{25} 5^3$

b) $\log_6 54 + \log_6 2 - \log_6 3$

c) $\log_6 6\sqrt{6}$

d) $\log_2\sqrt{36} - \log_2\sqrt{72}$

e) $\log_3 54 + \log_3\left(\frac{3}{2}\right)$

f) $\log_8 2 + 3\log_8 2 + \frac{1}{2}\log_8 16$

7. Use the laws of logarithms to express each of the following in terms of $\log_b x$, $\log_b y$, and $\log_b z$.

a) $\log_b xyz$

b) $\log_b\left(\frac{z}{xy}\right)$

c) $\log_b x^2y^3$

d) $\log_b\sqrt{x^5yz^3}$

8. Explain why $\log_5 3 + \log_5 \frac{1}{3} = 0$.

9. Write each expression as a single logarithm.

a) $3 \log_5 2 + \log_5 7$

b) $2 \log_3 8 - 5 \log_3 2$

c) $2 \log_2 3 + \log_2 5$

d) $\log_3 12 + \log_3 2 - \log_3 6$

e) $\log_4 3 + \frac{1}{2} \log_4 8 - \log_4 2$

f) $2 \log 8 + \log 9 - \log 36$

10. **A** Use the laws of logarithms to express each side of the equation as a single logarithm. Then compare both sides of the equation to solve.

a) $\log_2 x = 2 \log_2 7 + \log_2 5$

b) $\log x = 2 \log 4 + 3 \log 3$

c) $\log_4 x + \log_4 12 = \log_4 48$

d) $\log_7 x = 2 \log_7 25 - 3 \log_7 5$

e) $\log_3 x = 2 \log_3 10 - \log_3 25$

f) $\log_5 x - \log_5 8 = \log_5 6 + 3 \log_5 2$

11. Write each expression as a single logarithm. Assume that all the variables represent positive numbers.

a) $\log_2 x + \log_2 y + \log_2 z$

b) $\log_5 u - \log_5 v + \log_5 w$

c) $\log_6 a - (\log_6 b + \log_6 c)$

d) $\log_2 x^2 - \log_2 xy + \log_2 y^2$

e) $1 + \log_3 x^2$

f) $3 \log_4 x + 2 \log_4 x - \log_4 y$

12. Write $\frac{1}{2} \log_a x + \frac{1}{2} \log_a y - \frac{3}{4} \log_a z$ as a single logarithm. Assume that all the variables represent positive numbers.

13. Describe the transformations that take the graph of $f(x) = \log_2 x$ to the graph of $g(x) = \log_2 (8x^3)$.

14. **T** Use different expressions to create two logarithmic functions that have the same graph. Demonstrate algebraically why these functions have the same graph.

15. **C** Explain how the laws of logarithms can help you evaluate $\log_3 \left(\frac{\sqrt[5]{27}}{2187} \right)$.

Extending

16. Explain why $\log_x x^{m-1} + 1 = m$.

17. If $\log_b x = 0.3$, find the value of $\log_b x\sqrt{x}$.

18. Use graphing technology to draw the graphs of $y = \log x + \log 2x$ and $y = \log 2x^2$. Although the graphs are different, simplifying the first expression using the laws of logarithms produces the second expression. Explain why the graphs are different.

19. Create a pair of equivalent expressions that demonstrate each of the laws of logarithms. Prove that these expressions are equivalent.

8 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: In what ways can the equation of the inverse of an exponential function be written?

A: One way that the inverse of an exponential function can be written is in exponential form. For example, the inverse of the exponential function $y = a^x$ is $x = a^y$. Another way that the inverse can be written is in logarithmic form. For example, $x = a^y$ can be written as $y = \log_a x$. This means that a logarithm is an exponent. Specifically, $\log_a x$ means "the exponent that must be applied to a to get x." Since $x = a^y$ is equivalent to $y = \log_a x$, this exponent is y.

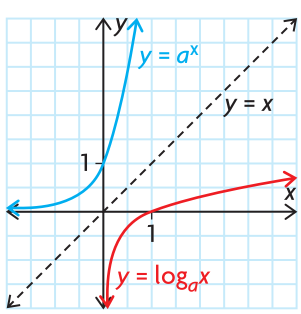
Study *Aid*

- See Lesson 8.1.
- Try Mid-Chapter Review Questions 1 and 2.

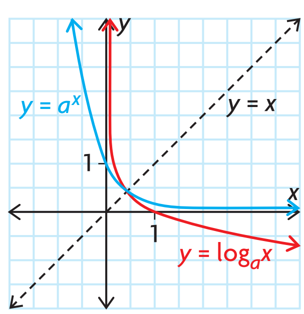
Q: What does the graph of a logarithmic function of the form $y = \log_a x$ look like and what are its characteristics?

A: The general shape of the graph of a logarithmic function depends on the value of its base.

When $a > 1$, the exponential function is an increasing function, and the logarithmic function is also an increasing function.



When $0 < a < 1$, the exponential function is a decreasing function, and the logarithmic function is also a decreasing function.



- The y-axis is the vertical asymptote for the logarithmic function. The x-axis is the horizontal asymptote for the exponential function.
- The x-intercept of the logarithmic function is 1, while the y-intercept of the exponential function is 1.
- The domain of the logarithmic function is $\{x \in \mathbf{R} \mid x > 0\}$, since the range of the exponential function is $\{y \in \mathbf{R} \mid y > 0\}$.
- The range of the logarithmic function is $\{y \in \mathbf{R}\}$, since the domain of the exponential function is $\{x \in \mathbf{R}\}$.

Study *Aid*

- See Lesson 8.2, Examples 1 and 2.
- Try Mid-Chapter Review Questions 3 and 4.

Q: How does varying the parameters of the equation $y = a \log(k(x - d)) + c$ affect the graph of the parent function, $y = \log x$?

A: The value of the parameter a determines whether there is a vertical stretch or compression. The value of k determines whether there is a horizontal stretch or compression. The value of d indicates a horizontal translation, and the value of c indicates a vertical translation. If a is negative, there is a reflection of the parent function $y = \log x$ in the x-axis. If k is negative, there is a reflection of the parent function $y = \log x$ in the y-axis.

Study *Aid*

- See Lesson 8.3, Examples 2 and 3.
- Try Mid-Chapter Review Questions 7, 8, and 9.

Q: How do you evaluate a logarithm?

A1: A logarithm of a number indicates the exponent to which the base must be raised to get the number.

For example, $\log_4 64$ means "the exponent to which you must raise 4 to get 64." The answer is 3.

A2: If the logarithm involves base 10, a calculator can be used to determine its value; $\log_{10} 25 = \log 25 \doteq 1.3979$.

A3: If the logarithm has a base other than 10, use the relationship $\log_a b = \frac{\log b}{\log a}$ and a calculator to determine its value;

$\log_2 15 = \frac{\log 15}{\log 2} \doteq 3.9069$.

Study *Aid*

- See Lesson 8.4, Examples 4, 5, and 6.
- Try Mid-Chapter Review Questions 10 to 13.

Q: How do you simplify expressions that contain logarithms?

A: If the logarithms are written with the same base, you can simplify them using the laws of logarithms that correspond to the relevant exponent laws.

The log of a product can be expressed as a sum of the logs; for example, $\log_5(6 \times 7) = \log_5 6 + \log_5 7$.

The log of a quotient can be expressed as the difference of the logs; for example, $\log_7\left(\frac{25}{6}\right) = \log_7 25 - \log_7 6$.

The logarithm of a power can be expressed as the product of the exponent of the power and the logarithm of the base of the power; for example, $\log_3 4^6 = 6 \log_3 4$.

PRACTICE Questions

Lesson 8.1

1. Express in logarithmic form.

a) $y = 5^x$ c) $x = 10^y$

b) $y = \left(\frac{1}{3}\right)^x$ d) $m = p^q$

2. Express in exponential form.

a) $y = \log_3 x$ c) $k = \log m$

b) $y = \log x$ d) $t = \log_s r$

Lesson 8.2

3. Describe the transformations of the parent function $y = \log x$ that result in $f(x)$.

a) $f(x) = 2 \log x - 4$

b) $f(x) = -\log 3x$

c) $f(x) = \frac{1}{4} \log \frac{1}{4}x$

d) $f(x) = \log [2(x - 2)]$

e) $f(x) = \log (x + 5) + 1$

f) $f(x) = 5 \log (-x) - 3$

4. Given the parent function $y = \log_3 x$, write the equation of the function that results from each set of transformations.

a) vertical stretch by a factor of 4, followed by a reflection in the x-axis

b) horizontal translation 3 units to the left, followed by a vertical translation 1 unit up

c) vertical compression by a factor of $\frac{2}{3}$, followed by a horizontal stretch by a factor of 2

d) vertical stretch by a factor of 3, followed by a reflection in the y-axis and a horizontal translation 1 unit to the right

5. State the coordinates of the image point of (9, 2) for each of the transformed functions in question 4.

6. How does the graph of $f(x) = 2 \log_2 x + 2$ compare with the graph of $g(x) = \log_2 x$?

Lesson 8.3

7. Evaluate.

a) $\log_3 81$ c) $\log_5 1$

b) $\log_4 \frac{1}{16}$ d) $\log_{\frac{2}{3}} \frac{27}{8}$

8. Evaluate to three decimal places.

a) $\log 4$ c) $\log 135$

b) $\log 45$ d) $\log 300$

9. Evaluate the value of each expression to three decimal places.

a) $\log_2 21$ c) $\log_7 141$

b) $\log_5 117$ d) $\log_{11} 356$

Lesson 8.4

10. Express as a single logarithm.

a) $\log 7 + \log 4$ c) $\log_3 11 + \log_3 4 - \log_3 6$

b) $\log 5 - \log 2$ d) $\log_p q + \log_p q$

11. Evaluate.

a) $\log_{11} 33 - \log_{11} 3$

b) $\log_7 14 + \log_7 3.5$

c) $\log_5 100 + \log_5 \frac{1}{4}$

d) $\log_{\frac{1}{2}} 72 - \log_{\frac{1}{2}} 9$

e) $\log_4 \sqrt[3]{16}$

f) $\log_3 9\sqrt{27}$

12. Describe how the graph of $f(x) = \log x^3$ is related to the graph of $g(x) = \log x$.

13. Use a calculator to evaluate each expression to two decimal places.

a) $\log 4^8$ d) $\log 200 \div \log 50$

b) $\log \sqrt{40}$ e) $(\log 20)^2$

c) $\log 9^4$ f) $5 \log 5$

8.5 Solving Exponential Equations

GOAL

Solve exponential equations in one variable using a variety of strategies.

LEARN ABOUT the Math

All radioactive substances decrease in mass over time.

Jamie works in a laboratory that uses radioactive substances. The laboratory received a shipment of 200 g of radioactive radon, and 16 days later, 12.5 g of the radon remained.

? What is the half-life of radon?

EXAMPLE 1 Selecting a strategy to solve an exponential equation

Calculate the half-life of radon.

Solution A: Solving the equation algebraically by writing both sides with the same base

$$M(t) = P\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

Write the formula for half-life, where h is the half-life period.

$$12.5 = 200\left(\frac{1}{2}\right)^{\frac{16}{h}}$$

Substitute the known values $M(t)$ (for the remaining mass), P (the original mass), and t (the time in days).

$$\frac{12.5}{200} = \left(\frac{1}{2}\right)^{\frac{16}{h}}$$

Isolate h by dividing both sides of the equation by 200.

$$\frac{1}{16} = \left(\frac{1}{2}\right)^{\frac{16}{h}}$$

Express the left side of the equation as a fraction.

$$\left(\frac{1}{2}\right)^{4} = \left(\frac{1}{2}\right)^{\frac{16}{h}}$$

Both sides can be written with the same base.

$$4 = \frac{16}{h}$$

The powers are equal when the exponents are equal. Solve the resulting equation.

$$4h = 16$$
$$h = 4$$

Multiply both sides by h, and then divide by 4 to solve for h.

The half-life of radon is 4 days.

Solution B: Solving the equation algebraically by taking the logarithm of both sides

$$M(t) = P\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

Write the formula for half-life.

$$12.5 = 200\left(\frac{1}{2}\right)^{\frac{16}{h}}$$

Substitute the given values.

$$\frac{12.5}{200} = \left(\frac{1}{2}\right)^{\frac{16}{h}}$$

Divide both sides of the equation by 200.

$$0.0625 = (0.5)^{\frac{16}{h}}$$

Express the fractions as decimals.

$$\log(0.0625) = \log(0.5)^{\frac{16}{h}}$$

If two quantities are equal, then the logs of the quantities will also be equal.

$$\log(0.0625) = \frac{16}{h}\log(0.5)$$

Use the power rule for logarithms to rewrite the right side of the equation without an exponent.

$$h\log(0.0625) = 16\log(0.5)$$

Multiply both sides by h.

$$h = \frac{16\log(0.5)}{\log(0.0625)}$$
$$h = 4$$

Divide both sides of the equation by $\log(0.0625)$ and evaluate the result with a calculator.

The half-life of radon is 4 days.

Solution C: Solving the equation graphically using graphing technology

$$M(t) = P\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

Write the formula for half-life.

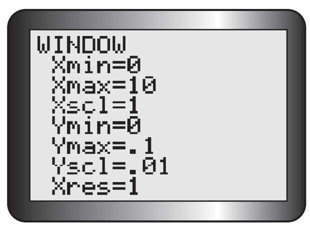
$$12.5 = 200\left(\frac{1}{2}\right)^{\frac{16}{h}}$$

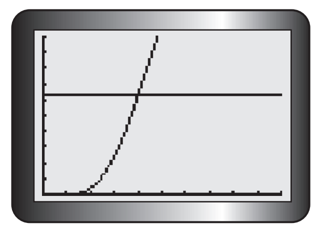
Substitute the given values.

$$\frac{12.5}{200} = \left(\frac{1}{2}\right)^{\frac{16}{h}}$$

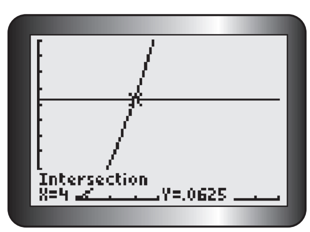
To solve for h, divide both sides of the equation by 200.

$$0.0625 = (0.5)^{\frac{16}{h}}$$





Express the fractions as decimals. Enter the right side of the equation into Y1 of the equation editor. Enter the left side into Y2. Graph using a window that corresponds to the domain and range in this context.



Determine the point of intersection of the graphs using the intersect operation. The *x*-coordinate of the point of intersection is the solution to the equation.

The half-life of radon is 4 days.

Tech *Support*

For help using the graphing calculator to determine points of intersection, see Technical Appendix, T-12.

Reflecting

A. Why did the strategy that was used in Solution A result in an exact answer? Will this strategy always result in an exact answer? Explain.

B. Which of the strategies used in the three different solutions will always result in an exact answer? Explain.

C. Which of the three strategies do you prefer? Justify your preference.

APPLY the Math

EXAMPLE 2 Selecting a strategy to solve an exponential equation with more than one power

Solve $2^{x+2} - 2^x = 24$.

Solution

$$2^{x+2} - 2^x = 24$$

The terms on the left side of the equation cannot be combined.

$$2^x(2^2 - 1) = 24$$

Divide out the common factor of 2^x on the left side of the equation.

$$2^x(4 - 1) = 24$$
$$2^x(3) = 24$$
$$2^x = 8$$
$$2^x = 2^3$$
$$x = 3$$

Simplify the expression in brackets. Divide both sides by 3. Express the right side of the equation as a power of 2.

EXAMPLE 3 Using logarithms to solve a problem

An investment of \$2500 grows at a rate of 4.8% per year, compounded annually. How long will it take for the investment to be worth \$4000? Recall that the formula for compound interest is $A = P(1 + i)^n$.

Solution

$$A = P(1 + i)^n$$
$$4000 = 2500(1.048)^n$$

Substitute the known values ($P = 2500$, $i = 0.048$, and $A = 4000$) into the formula. The variable n represents the number of years.

$$\frac{4000}{2500} = (1.048)^n$$

Divide both sides of the equation by 2500.

$$1.6 = (1.048)^n$$

Express the result as a decimal.

$$\log(1.6) = \log(1.048)^n$$

Take the log of both sides to solve for n.

$$\log(1.6) = n\log(1.048)$$

Use the power rule for logarithms to rewrite the equation.

$$n = \frac{\log 1.6}{\log(1.048)} \doteq 10.025$$

Divide both sides of the equation by $\log(1.048)$ to solve for n.

It will take approximately 10.025 years for the investment to be worth \$4000.

EXAMPLE 4 Selecting a strategy to solve an exponential equation with different bases

Solve $2^{x+1} = 3^{x-1}$ to three decimal places.

Solution

$$2^{x+1} = 3^{x-1}$$

Both sides of the equation cannot be written with the same base.

$$\log (2^{x+1}) = \log (3^{x-1})$$

Take the log of both sides of the equation.

$$(x + 1) \log 2 = (x - 1) \log 3$$

Use the power rule for logarithms to rewrite both sides of the equation with no exponents.

$$x \log 2 + \log 2 = x \log 3 - \log 3$$

Expand using the distributive property.

$$\log 2 + \log 3 = x \log 3 - x \log 2$$

Collect like terms to solve the equation.

$$\frac{\log 2 + \log 3}{\log 3 - \log 2} = \frac{x (\log 3 - \log 2)}{\log 3 - \log 2}$$

Divide out the common factor of x on the right side. Then divide both sides by $\log 3 - \log 2$.

$$\frac{\log 2 + \log 3}{\log 3 - \log 2} = x$$

Evaluate using a calculator.

$$4.419 \doteq x$$

Round the answer to the required number of decimal places.

In Summary

Key Ideas

- Two exponential expressions with the same base are equal when their exponents are equal. For example, if $a^m = a^n$, then $m = n$, where $a > 0$, $a \neq 1$, and $m, n \in \mathbf{R}$.
- If two expressions are equal, taking the log of both expressions maintains their equality. For example, if $M = N$, then $\log_a M = \log_a N$, where $M, N > 0$, $a > 0$, $a \neq 1$.

Need to Know

- To solve an exponential equation algebraically, take the logarithm of both sides of the equation using a base of 10, and then use the power rule for logarithms to simplify the equation and solve for the unknown.
- Sometimes an exponential equation can be solved algebraically by writing both sides of the equation with the same base (if possible), setting the exponents equal to each other, and solving for the unknown.
- Exponential equations can also be solved with graphing technology, using the same strategies that are used for other kinds of equations.

CHECK Your Understanding

1. Solve.

a) $5^x = 625$
b) $4^{2x} = 2^{5-x}$
c) $9^{x+1} = 27^{2x-3}$
d) $8^{x-1} = \sqrt[3]{16}$
e) $2^{3x} = \frac{1}{2}$
f) $4^{2x} = \frac{1}{16}$

2. Solve. Round your answers to three decimal places.

a) $2^x = 17$
b) $6^x = 231$
c) $30(5^x) = 150$
d) $210 = 40(1.5)^x$
e) $5^{1-x} = 10$
f) $6^{\frac{x}{3}} = 30$

3. Solve by rewriting in exponential form.

a) $x = \log_3 243$
b) $x = \log_6 216$
c) $x = \log_5 5\sqrt{5}$
d) $x = \log_2 \sqrt[5]{8}$
e) $x = \log_2\left(\frac{1}{4}\right)$
f) $x = \log_3\left(\frac{1}{\sqrt{3}}\right)$

PRACTISING

4. The formula to calculate the mass, $M(t)$, remaining from an original sample of radioactive material with mass P, is determined using the formula $M(t) = P\left(\frac{1}{2}\right)^{\frac{t}{h}}$, where t is time and h is the half-life of the substance. The half-life of a radioactive substance is 8 h. How long will it take for a 300 g sample to decay to each mass?

a) 200 g **b)** 100 g **c)** 75 g **d)** 20 g

5. Solve.

K **a)** $49^{x-1} = 7\sqrt{7}$
b) $2^{3x-4} = 0.25$
c) $\left(\frac{1}{4}\right)^{x+4} = \sqrt{8}$
d) $36^{2x+4} = \left(\sqrt{1296}\right)^x$
e) $2^{2x+2} + 7 = 71$
f) $9^{2x+1} = 81(27^x)$

6. A **a)** If \$500 is deposited into an account that pays 8%/a compounded annually, how long will it take for the deposit to double?

b) A \$1000 investment is made in a trust fund that pays 12%/a compounded monthly. How long will it take the investment to grow to \$5000?

c) A \$5000 investment is made in a savings account that pays 10%/a compounded quarterly. How long will it take for the investment to grow to \$7500?

d) If you invested \$500 in an account that pays 12%/a compounded weekly, how long would it take for your deposit to triple?

7. A bacteria culture doubles every 15 min. How long will it take for a culture of 20 bacteria to grow to a population of 163 840?

8. Solve for x.

a) $4^{x+1} + 4^x = 160$

b) $2^{x+2} + 2^x = 320$

c) $2^{x+2} - 2^x = 96$

d) $10^{x+1} - 10^x = 9000$

e) $3^{x+2} + 3^x = 30$

f) $4^{x+3} - 4^x = 63$

9. Choose a strategy to solve each equation, and explain your choice. (Do not solve.)

a) $225(1.05)^x = 450$

b) $3^{x+2} + 3^x = 270$

10. Solve. Round your answers to three decimal places.

a) $5^{t-1} = 3.92$

b) $x = \log_3 25$

c) $4^{2x} = 5^{2x-1}$

d) $x = \log_2 53.2$

11. A plastic sun visor allows light to pass through, but reduces the intensity of the light. The intensity is reduced by 5% if the plastic is 1 mm thick. Each additional millimetre of thickness reduces the intensity by another 5%.

a) Use an equation to model the relation between the thickness of the plastic and the intensity of the light.

b) How thick is a piece of plastic that reduces the intensity of the light to 60%?

12. T Solve $3^{2x} - 5(3^x) = -6$.

13. C If $\log_a x = y$, show that $y = \frac{\log x}{\log a}$. Explain how this relationship could be used to graph $y = \log_5 x$ on a graphing calculator.

Extending

14. Solve for x.

a) $2^{x^2} = 32(2^{4x})$

b) $3^{x^2+20} = \left(\frac{1}{27}\right)^{3x}$

c) $2 \times 3^x = 7 \times 5^x$.

15. If $\log_a 2 = \log_b 8$, show that $a^3 = b$.

16. Determine the point of intersection for the graphs of $y = 3(5^{2x})$ and $y = 6(4^{3x})$. Round your answer to three decimal places.

17. Solve for x, to two decimal places.

a) $6^{3x} = 4^{2x-3}$

b) $(1.2)^x = (2.8)^{x+4}$

c) $3(2)^x = 4^{x+1}$

18. Solve for x, to two decimal places.

$(2^x)^x = 10$

8.6 Solving Logarithmic Equations

GOAL

Solve logarithmic equations with one variable algebraically.

LEARN ABOUT the Math

The Richter scale is used to compare the intensities of earthquakes. The Richter scale magnitude, R, of an earthquake is determined using $R = \log\left(\frac{a}{T}\right) + B$, where a is the amplitude of the vertical ground motion in microns (μ), T is the period of the seismic wave in seconds, and B is a factor that accounts for the weakening of the seismic waves. (1 μ is equivalent to 10^{-6} m.)

? An earthquake measured 5.5 on the Richter scale, and the period of the seismic wave was 1.8 s. If *B* equals 3.2, what was the amplitude, *a*, of the vertical ground motion?

EXAMPLE 1 Selecting an algebraic strategy to solve a logarithmic equation

Determine the amplitude, a, of the vertical ground motion.

Solution

$$R = \log\left(\frac{a}{T}\right) + B$$

$$5.5 = \log\left(\frac{a}{1.8}\right) + 3.2$$

Substitute the given values into the equation.

$$2.3 = \log\left(\frac{a}{1.8}\right)$$

Isolate the term with the unknown, *a*, by subtracting 3.2 from both sides.

$$10^{2.3} = \frac{a}{1.8}$$

Rewrite the equation in exponential form.

$$10^{2.3} \times 1.8 = a$$

Multiply both sides by 1.8 to solve for *a*.

$$359.1\ \mu = a$$

The amplitude of the vertical ground motion was about 359.1 μ.

To get a better idea of the size of this number, change microns to metres or centimetres. 359.1 μ = 0.000 359 1 m or 0.035 91 cm.

Reflecting

A. What strategies for solving a linear equation were used to solve this logarithmic equation?

B. Why was the equation rewritten in exponential form?

C. How would the strategies have changed if the value of a had been given and the value of T had to be determined?

APPLY the Math

EXAMPLE 2 Selecting an algebraic strategy to solve a logarithmic equation

Solve.

a) $\log_x 0.04 = -2$ **b)** $\log_7(3x - 5) = \log_7 16$

Solution

a) $\log_x 0.04 = -2$

$x^{-2} = 0.04$ ← Express the equation in exponential form.

$x^{-2} = \frac{1}{25}$ ← Rewrite the decimal as a fraction. $0.04 = \frac{4}{100} = \frac{1}{25}$

$x^{-2} = 5^{-2}$ ← Express $\frac{1}{25}$ as a power with exponent -2. Since the exponents are equal, the bases must be equal.

$x = 5$

b) $\log_7(3x - 5) = \log_7 16$ ← If $\log_a M = \log_a N$, then $M = N$.

$3x - 5 = 16$ ← Since 7 is the base of both logs, the two expressions must be equal.

$3x = 21$ ← Add 5 to both sides of the equation.

$x = 7$

EXAMPLE 3 Representing sums and differences of logs as single logarithms to solve a logarithmic equation

Solve.

a) $\log_2 30x - \log_2 5 = \log_2 12$ **b)** $\log x + \log x^2 = 12$

Solution

a) $\log_2 30x - \log_2 5 = \log_2 12$

$\log_2\left(\frac{30x}{5}\right) = \log_2 12$

$\log_2(6x) = \log_2 12$

The logs are written with the same base, so the difference of the logs can be written as the log of a quotient. Simplify.

$6x = 12$

Since both sides of the equation are written with the same base, the two expressions are equal.

$x = 2$

b) $\log x + \log x^2 = 12$

$\log(x \times x^2) = 12$

The logs are written with the same base, so the sum of the logs can be written as the log of a product. Simplify.

$\log(x^3) = 12$

$x^3 = 10^{12}$

Express the equation in exponential form.

$\sqrt[3]{x^3} = \sqrt[3]{10^{12}}$

Take the cube root of both sides to solve for *x*.

$x = 10^4$

$x = 10\ 000$

EXAMPLE 4 Selecting a strategy to solve a logarithmic equation that involves quadratics

Solve $\log_2(x + 3) + \log_2(x - 3) = 4$.

Solution

$\log_2(x + 3) + \log_2(x - 3) = 4$

$\log_2(x + 3)(x - 3) = 4$

$\log_2(x^2 - 3x + 3x - 9) = 4$

$\log_2(x^2 - 9) = 4$

Since both logarithms have base 2, rewrite the left side as a single logarithm using the product law. Multiply the binomials, and simplify.

$$x^2 - 9 = 2^4$$
$$x^2 - 9 = 16$$
$$x^2 = 25$$

Rewrite the equation in exponential form and solve for x^2.

$$x = \pm\sqrt{25}$$

Take the square root of both sides. There are two possible solutions for a quadratic equation.

$$x = \pm 5$$

Check to make sure that both solutions satisfy the equation.

Check: $x = -5$

LS: $\log_2(-5 + 3) + \log_2(-5 - 3)$

$= \log_2(-2) + \log_2(-8)$

$\neq$ RS

When $x = -5$, the expression on the left side is undefined, since the logarithm of any negative number is undefined. Therefore, $x = -5$ is not a solution to the original equation. It is an inadmissible solution.

Check: $x = 5$

LS: $\log_2(5 + 3) + \log_2(5 - 3)$

$= \log_2(8) + \log_2(2)$

$= 3 + 1$

$= 4$

$=$ RS

When $x = 5$, the expression on the left side gives the value on the right side. Therefore, $x = 5$ is the solution to the original equation.

The solution is $x = 5$.

In Summary

Key Ideas

- A logarithmic equation can be solved by expressing it in exponential form and solving the resulting exponential equation.
- If $\log_a M = \log_a N$, then $M = N$, where $a, M, N > 0$.

Need to Know

- A logarithmic equation can be solved by simplifying it using the laws of logarithms.
- When solving logarithmic equations, be sure to check for inadmissible solutions. A solution is inadmissible if its substitution in the original equation results in an undefined value. Remember that the **argument** and the base of a logarithm must both be positive.

CHECK Your Understanding

1. Solve.

a) $\log_2 x = 2\log_2 5$

b) $\log_3 x = 4\log_3 3$

c) $\log x = 3\log 2$

d) $\log(x - 5) = \log 10$

e) $\log_2 8 = x$

f) $\log_2 x = \frac{1}{2}\log_2 3$

2. Solve.

a) $\log_x 625 = 4$

b) $\log_x 6 = -\frac{1}{2}$

c) $\log_5(2x - 1) = 2$

d) $\log(5x - 2) = 3$

e) $\log_x 0.04 = -2$

f) $\log_5(2x - 4) = \log_5 36$

3. Given the formula from Example 1 for the magnitude of an earthquake, $R = \log\left(\frac{a}{T}\right) + B$, determine the value of a if $R = 6.3$, $B = 4.2$, and $T = 1.6$.

PRACTISING

4. Solve.

a) $\log_x 27 = \frac{3}{2}$

b) $\log_x 5 = 2$

c) $\log_3(3x + 2) = 3$

d) $\log x = 4$

e) $\log_{\frac{1}{3}} 27 = x$

f) $\log_{\frac{1}{2}} x = -2$

5. Solve.

K **a)** $\log_2 x + \log_2 3 = 3$

b) $\log 3 + \log x = 1$

c) $\log_5 2x + \frac{1}{2}\log_5 9 = 2$

d) $\log_4 x - \log_4 2 = 2$

e) $3\log x - \log 3 = 2\log 3$

f) $\log_3 4x + \log_3 5 - \log_3 2 = 4$

6. Solve $\log_6 x + \log_6(x - 5) = 2$. Check for inadmissible roots.

7. Solve.

a) $\log_7(x + 1) + \log_7(x - 5) = 1$

b) $\log_3(x - 2) + \log_3 x = 1$

c) $\log_6 x - \log_6(x - 1) = 1$

d) $\log(2x + 1) + \log(x - 1) = \log 9$

e) $\log(x + 2) + \log(x - 1) = 1$

f) $3\log_2 x - \log_2 x = 8$

8. Describe the strategy that you would use to solve each of the following equations. (Do not solve.)

a) $\log_9 x = \log_9 4 + \log_9 5$

b) $\log x - \log 2 = 3$

c) $\log x = 2\log 8$

9. The loudness, L, of a sound in decibels (dB) can be calculated using
A
the formula $L = 10 \log \left(\frac{I}{I_0}\right)$, where I is the intensity of the sound in watts per square metre (W/m^2) and $I_0 = 10^{-12}\ \text{W/m}^2$.

a) A teacher is speaking to a class. Determine the intensity of the teacher's voice if the sound level is 50 dB.

b) Determine the intensity of the music in the earpiece of an MP3 player if the sound level is 84 dB.

10. Solve $\log_a(x + 2) + \log_a(x - 1) = \log_a(8 - 2x)$.
T

11. Use graphing technology to solve each equation to two decimal places.

a) $\log(x + 3) = \log(7 - 4x)$

b) $5^x = 3^{x+1}$

c) $2 \log x = 1$

d) $\log(4x) = \log(x + 1)$

12. Solve $\log_5(x - 1) + \log_5(x - 2) - \log_5(x + 6) = 0$.

13. Explain why there are no solutions to the equations $\log_3(-8) = x$ and $\log_{-3}9 = x$.

14. **a)** Without solving the equation, state the restrictions on the variable x in the following: $\log(2x - 5) - \log(x - 3) = 5$

b) Why do these restrictions exist?

15. If $\log\left(\frac{x + y}{5}\right) = \frac{1}{2}(\log x + \log y)$, where $x > 0$, $y > 0$, show that $x^2 + y^2 = 23xy$.

16. Solve $\frac{\log(35 - x^3)}{\log(5 - x)} = 3$.

17. Given $\log_2 a + \log_2 b = 4$, calculate all the possible integer values of a and b. Explain your reasoning.
C

Extending

18. Solve the following system of equations algebraically.

$y = \log_2(5x + 4)$

$y = 3 + \log_2(x - 1)$

19. Solve each equation.

a) $\log_5(\log_3 x) = 0$

b) $\log_2(\log_4 x) = 1$

20. If $\left(\frac{1}{2}\right)^{x+y} = 16$ and $\log_{x-y}8 = -3$, calculate the values of x and y.

8.7 Solving Problems with Exponential and Logarithmic Functions

GOAL

Pose and solve problems based on applications of exponential and logarithmic functions.

YOU WILL NEED

- graphing calculator

INVESTIGATE the Math

The following data represent the prices of IBM personal computers and the demand for these computers at a computer store in 1997.

Price ($/computer)	2300	2000	1700	1500	1300	1200	1000
Demand (number of computers)	152	159	164	171	176	180	189

? Based on the data, what do you predict the demand would have been for computers priced at $1600?

A. What is the dependent variable in this situation? Enter the data into a graphing calculator, and create a scatter plot.

B. Is it clear what type of function you could use to model this situation? Explain.

C. Try fitting a function to the scatter plot you created. Try linear, quadratic, cubic, and exponential functions.

D. Use the regression feature of the calculator to determine the equation of the curve of best fit. Try linear, quadratic, cubic, and exponential regression.

E. Which type of function gives you the best fit?

F. Use the algebraic model you found to determine the price that would have a demand of 195 computers.

G. Use your model to predict the demand for computers priced at $1600.

Tech *Support*

For help using the graphing calculator to create scatter plots or using regression to determine the equation of best fit, see Technical Appendix, T-11.

Reflecting

H. How could you use the table of values to determine what type of function the data approximates?

I. How could you have used your graph to answer parts F and G?

APPLY the Math

EXAMPLE 1 Solving a problem using a logarithmic equation

In chemistry, the pH (the measure of acidity or alkalinity of a substance) is based on a logarithmic scale. A logarithmic scale uses powers of 10 to compare numbers that vary greatly in size. For example, very small and very large concentrations of the hydrogen ion in a solution influence its classification as either a base or an acid.

Concentration of hydrogen ions compared to distilled water		Examples of solutions at this pH
10 000 000	pH = 0	battery acid, strong hydrofluoric acid
1 000 000	pH = 1	hydrochloric acid secreted by stomach lining
100 000	pH = 2	lemon juice, gastric acid, vinegar
10 000	pH = 3	grapefruit, orange juice, soda
1000	pH = 4	tomato juice, acid rain
100	pH = 5	soft drinking water, black coffee
10	pH = 6	urine, saliva
1	pH = 7	"pure" water
$\frac{1}{10}$	pH = 8	seawater
$\frac{1}{100}$	pH = 9	baking soda
$\frac{1}{1000}$	pH = 10	Great Salt Lake, milk of magnesia
$\frac{1}{10\,000}$	pH = 11	ammonia solution
$\frac{1}{100\,000}$	pH = 12	soapy water
$\frac{1}{1\,000\,000}$	pH = 13	bleaches, oven cleaner
$\frac{1}{10\,000\,000}$	pH = 14	liquid drain cleaner

A difference of one pH unit represents a tenfold (10 times) change in the concentration of hydrogen ions in the solution. For example, the acidity of a sample with a pH of 5 is 10 times greater than the acidity of a sample with a pH of 6. A difference of 2 units, from 6 to 4, would mean that the acidity is 100 times greater, and so on.

- A liquid with a pH less than 7 is considered *acidic*.
- A liquid with a pH greater than 7 is considered *alkaline*.
- A liquid with a pH of 7 is considered *neutral*. Pure distilled water has a pH value of 7.

The relationship between pH and hydrogen ion concentration is given by the formula $\text{pH} = -\log[\text{H}^+]$, where $[\text{H}^+]$ is the concentration of hydrogen ions in moles per litre (mol/L).

a) Calculate the pH if the concentration of hydrogen ions is 0.0001 mol/L.

b) The pH of lemon juice is 2. Calculate the hydrogen ion concentration.

c) If the hydrogen ion concentration is a measure of the strength of an acid, how much stronger is an acid with pH 1.6 than an acid with pH 2.5?

Solution

a) $pH = -\log[H^+]$
$pH = -\log(0.0001)$
$pH = -(-4)$
$pH = 4$
The pH of the liquid is 4.

Substitute the value for $[H^+]$ into the equation. Evaluate log (0.0001).

b) $pH = -\log[H^+]$
$2 = -\log[H^+]$

Substitute the value 2 for the pH.

$-2 = \log[H^+]$

Divide both sides of the equation by -1.

$10^{-2} = [H^+]$

Rewrite the equation in exponential form.

$0.01 = [H^+]$

Evaluate the negative exponent to determine $[H^+]$.

The concentration of hydrogen ions is 0.01 mol/L.

c) $pH = -\log[H^+]$

$1.6 = -\log[H^+]$ $2.5 = -\log[H^+]$

To calculate the hydrogen ion concentration of both solutions, substitute the given pH values into the equation.

$10^{-1.6} = [H^+]$ $10^{-2.5} = [H^+]$
$0.0251 \doteq [H^+]$ $0.0032 \doteq [H^+]$

Express both equations in exponential form, and evaluate.

$$\frac{0.0251}{0.0032} = 7.84$$

Divide the concentration of the first acid by the concentration of the second acid to find the relative strength of the acids.

An acid with pH 1.6 is about 7.8 times stronger than an acid with pH 2.5.

EXAMPLE 2 Representing exponential values using the Richter scale

The Richter magnitude scale uses logarithms to compare intensity of earthquakes.

True Intensity	Richter Scale Magnitude
10^1	$\log_{10}10^1 = 1$
10^4	$\log_{10}10^4 = 4$
$10^{5.8}$	$\log_{10}10^{5.8} = 5.8$

An earthquake of magnitude 2 is actually 10 times more intense than an earthquake of magnitude 1. The difference between the magnitudes of two earthquakes can be used to determine the difference in intensity. If the average earthquake measures 4.5 on the Richter scale, how much more intense is an earthquake that measures 8?

Solution

$$\frac{10^8}{10^{4.5}} = 10^{8-4.5}$$

Calculate the quotient between the intensities.

$$= 10^{3.5}$$
$$\doteq 3162.3$$

Since the Richter scale is logarithmic, each step on the scale is a power of 10. The difference in intensity is calculated by evaluating 10 to the power of 3.5.

An earthquake that measures 8 on the Richter scale is about 3162 times more intense than an earthquake that measures 4.5.

Evaluate the power to compare the intensities of the two earthquakes.

EXAMPLE 3 Solving a problem using an exponential equation and logarithms

Blue jeans fade when washed due to the loss of blue dye from the fabric. If each washing removes about 2.2% of the original dye from the fabric, how many washings are required to give a pair of jeans a well-worn look? (For a well-worn look, jeans should contain, at most, 30% of the original dye.)

Solution

$$D(n) = (1 - 0.022)^n$$

Write an exponential model, using $D(n)$ to represent the percent of dye remaining as a decimal and n to represent the number of washings.

$$D(n) = (0.978)^n$$

Since the jeans are losing 2.2% of the dye each time, the ratio of decline is 0.978.

$$0.30 = (0.978)^n$$

Replace $D(n)$ with 0.30 since the well-worn look requires no more than 30% of the original dye remaining.

$$\log(0.30) = \log(0.978)^n$$

To solve for n, take the log of both sides of the equation.

$$\log(0.3) = n\log(0.978)$$

Rewrite the equation with the power as a coefficient.

$$\frac{\log(0.3)}{\log(0.978)} = n$$

Divide both sides of the equation by log (0.978) to solve for n.

$$54.12 \doteq n$$

It would take about 54 washings to give the jeans a well-worn look.

EXAMPLE 4 Solving a problem about sound intensity using logarithms

The dynamic range of human hearing and sound intensity spans from 10^{-12} W/m^2 to about 10 W/m^2. The highest sound intensity that can be heard is 10 000 000 000 000 times as loud as the quietest! This span of sound intensity is impractical for normal use. A more convenient way to express loudness is a relative logarithmic scale, with the lowest sound that can be heard by the human ear, $I_0 = 10^{-12}$ W/m^2, given the measure of loudness of 0 dB.

Recall that the formula that is used to measure sound is $L = 10\log\left(\frac{I}{I_0}\right)$, where L is the loudness measured in decibels, I is the intensity of the sound being measured, and I_0 is the intensity of sound at the threshold of hearing. The following table shows the loudness of a selection of sounds measured in decibels.

Sound	Loudness (dB)
soft whisper	30
normal conversation	60
shouting	80
subway	90
screaming	100
rock concert	120
jet engine	140
space-shuttle launch	180

How many times more intense is the sound of a rock concert than the sound of a subway?

Solution

$$L = 10 \log\left(\frac{I}{I_0}\right)$$

$$120 = 10 \log\left(\frac{I_{RC}}{I_0}\right) \qquad 90 = 10 \log\left(\frac{I_S}{I_0}\right)$$

Let the intensity of sound for a rock concert be I_{RC} and for a subway be I_S. Find the values for the loudness of these sounds in the table, and substitute into the formula.

$$12 = \log\left(\frac{I_{RC}}{I_0}\right) \qquad 9 = \log\left(\frac{I_S}{I_0}\right)$$

Divide both sides of the equations by 10.

$$10^{12} = \frac{I_{RC}}{I_0} \qquad 10^9 = \frac{I_S}{I_0}$$

Express both equations in exponential form.

$$10^{12} I_0 = I_{RC} \qquad 10^9 I_0 = I_S$$

Isolate the variables for comparison.

$$\frac{I_{RC}}{I_S} = \frac{10^{12} I_0}{10^9 I_0} = 10^3 = 1000$$

Divide the results to compare the sound of a rock concert with the sound of a subway.

The sound of a rock concert is 1000 times more intense than the sound of a subway.

In Summary

Key Ideas

- When a range of values can vary greatly, using a logarithmic scale with powers of 10 makes comparisons between the large and small values more manageable.
- Growth and decay situations can be modelled by exponential functions of the form $f(x) = ab^x$. Note that
  - $f(x)$ is the final amount or number
  - a is the initial amount or number
  - for exponential growth, $b = 1 +$ growth rate
  - for exponential decay, $b = 1 -$ decay rate
  - x is the number of growth or decay periods

Need to Know

- Scales that measure a wide range of values, such as the pH scale, Richter scale, and decibel scale, are logarithmic scales.
- To compare concentrations on the pH scale, intensity on the Richter scale, or sound intensities, determine the quotient between the values being compared.
- Data from a table of values can be graphed and a curve of best fit determined. If the curve of best fit appears to be exponential, use the regression feature of the graphing calculator to determine an equation that models the data.

CHECK *Your Understanding*

1. If one earthquake has a magnitude of 5.2 on the Richter scale and a second earthquake has a magnitude of 6, compare the intensities of the two earthquakes.

2. Calculate the pH of a swimming pool with a hydrogen ion concentration of 6.21×10^{-8} mol/L.

3. A particular sound is 1 000 000 times more intense than a sound you can just barely hear. What is the loudness of the sound in decibels?

PRACTISING

4. K The loudness of a heavy snore is 69 dB. How many times as loud as a normal conversation of 60 dB is a heavy snore?

5. Calculate the hydrogen ion concentration of each substance.

a) baking soda, with a pH of 9
b) milk, with a pH of 6.6
c) an egg, with a pH of 7.8
d) oven cleaner, with a pH of 13

6. Calculate to two decimal places the pH of a solution with each concentration of H^+.

a) concentration of $H^+ = 0.000\,32$

b) concentration of $H^+ = 0.000\,3$

c) concentration of $H^+ = 0.000\,045$

d) concentration of $H^+ = 0.005$

7. a) Distilled water has an H^+ concentration of 10^{-7} mol/L. Calculate the pH of distilled water.

b) Drinking water from a particular tap has a pH between 6.3 and 6.6. Is this tap water more or less acidic than distilled water? Explain your answer.

8. A The sound level of a moving power lawn mower is 109 dB. The noise level in front of the amplifiers at a concert is about 118 dB. How many times louder is the noise at the front of the amplifiers than the noise of a moving power lawn mower?

9. The following data represent the amount of an investment over 10 years.

Year	0	1	2	5	7	9	10
Amount ($)	5000	5321	5662.61	6824.74	7729.17	8753.45	9315.42

a) Create a scatter plot, and determine the equation that models this situation.

b) Determine the average annual interest rate.

c) Use your equation to determine how long it took for the investment to double.

10. The intensity, I, of light passing through water can be modelled by the equation $I = 10^{1-0.13x}$, where x is the depth of the water in metres. Most aquatic plants require a light intensity of 4.2 units for strong growth. Determine the depth of water at which most aquatic plants receive the required light.

11. The following data represent the growth of a bacteria population over time.

Number of Hours	0	7	12	20	42
Number of Bacteria	850	2250	4500	13 500	287 200

a) Create both a graphical model and an algebraic model for the data.

b) Determine the length of time it took for the population to double.

12. **T** The amount of water vapour in the air is a function of temperature, as shown in the following table.

Temperature (°C)	0	5	10	15	20	25	30	35
Saturation (mL/m^3 of air)	4.847	6.797	9.399	12.830	17.300	23.050	30.380	39.630

a) Calculate the growth factors for the saturation row of the table, to the nearest tenth.
b) Determine the average growth factor.
c) Write an exponential model for the amount of water vapour as a function of the temperature.
d) Determine the exponential function with a graphing calculator, using exponential regression.
e) What temperature change will double the amount of water in 1 m^3 of air?

13. Dry cleaners use a cleaning fluid that is purified by evaporation and condensation after each cleaning cycle. Every time the fluid is purified, 2.1% of it is lost. The fluid has to be topped up when half of the original fluid remains. After how many cycles will the fluid need to be topped up?

14. How long will it take for \$2500 to accumulate to \$4000 if it is invested at an interest rate of 6.5%/a, compounded annually?

15. A wound, initially with an area of 80 cm^2, heals according to the formula $A(t) = 80(10^{-0.023t})$, where $A(t)$ is the area of the wound in square centimetres after t days of healing. In how many days will 75% of the wound be healed?

16. **C** Create a problem that could be solved using logarithms and another problem that could be solved without using logarithms. Explain how the two problems are different.

Extending

17. A new car has an interior sound level of 70 dB at 50 km/h. A second car, at the same speed, has an interior sound level that is two times more intense than that of the new car. Calculate the sound level inside the second car.

18. Assume that the annual rate of inflation will average 3.8% over the next 10 years.
a) Write an equation to model the approximate cost, C, of goods and services during any year in the next decade.
b) If the price of a brake job for a car is presently \$400, estimate the price 10 years from now.
c) If the price of an oil change 10 years from now will be \$47.95, estimate the price of an oil change today.

8.8 Rates of Change in Exponential and Logarithmic Functions

YOU WILL NEED

- graphing calculator

GOAL

Solve problems that involve average and instantaneous rates of change of exponential and logarithmic functions.

INVESTIGATE the Math

The following data from the U.S. Census Bureau represent the population of the United States, to the nearest million, every 10 years from 1900 to 2000.

Year	1900	1910	1920	1930	1940	1950	1960	1970	1980	1990	2000
Population (millions)	76	92	106	123	132	151	179	203	227	249	281

? At what rate was the population changing in the United States at the start of 1950?

A. Calculate the average rate of change in population over the entire 100 years.

B. Calculate the average rate of change in population over the first 50 years and over the last 50 years. Is the average rate of change in each 50-year period the same, less than, or greater than the average rate of change for the entire time period? Suggest reasons.

C. Estimate the instantaneous rate of change in population at the start of 1950 using an average rate of change calculation and a centred interval.

D. Use a graphing calculator to create a scatter plot using years since 1900 as the independent variable.

E. Determine an exponential equation that models the data.

F. Estimate the instantaneous rate of change in population at the start of 1950 using the model you found and a very small interval after 1950.

G. Estimate the instantaneous rate of change in population at the start of 1950 by drawing the appropriate tangent on your graph.

H. Compare your estimates from parts C, F, and G. Which estimate better represents the rate at which the U.S. population was changing in 1950? Explain.

Tech | *Support*

For help using a graphing calculator to create scatter plots, and using regression to determine the equation of best fit, see Technical Appendix, T-11.

Reflecting

I. How could you use your graph to determine the year that had the least or greatest instantaneous change in population?

J. Describe how the rate at which the U.S. population grew changed during the period from 1900 to 2000.

APPLY the Math

EXAMPLE 1 — Selecting a numerical strategy to calculate the average rate of change

The average number of students per computer in public schools is given in the table. Year 1 is 1983.

a) Calculate the average rate of change in students per computer during the entire time period and during the middle five years of the data.

b) What conclusions can you draw?

Year	Students per Computer
1	125
2	75
3	50
4	37
5	32
6	25
7	22
8	20
9	18
10	16
11	14
12	10.5
13	10

Solution

a) $$\text{Average rate of change} = \frac{10 - 125}{13 - 1}$$

To calculate the average rate of change during the entire time period, use the values for the number of students per computer for year 13 and year 1.

$$= \frac{-115}{12}$$

Average the difference over 12 years.

$$\doteq -9.58$$

The average rate of change in students per computer decreased by about 10 students per computer.

The middle five years are years 5 to 9.

$$\text{Average rate of change} = \frac{18 - 32}{9 - 5}$$

Calculate the difference in the number of students per computer for years 9 and 5.

$$= \frac{-14}{4}$$

$$= -3.5$$

Divide the difference by 4, since years 5 to 9 are a 4-year span.

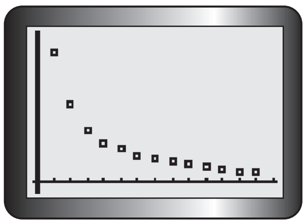
The average rate of change in students per computer decreased by 3.5 students per computer.

b) Since the rate of decline was faster over the entire period than during the middle period, the greatest change was either in the first four years or the last four years. The data show that there was a greater change in the number of students per computer during the first four years, so the decline was faster during this period.

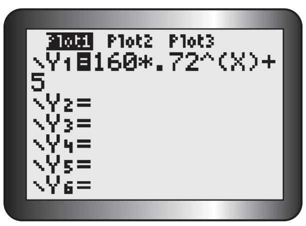
EXAMPLE 2 Selecting a strategy for calculating the instantaneous rate of change

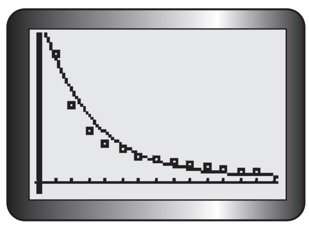
Using the data from Example 1, determine the instantaneous rate of change in students per computer for year 8.

Solution A: Calculating numerically

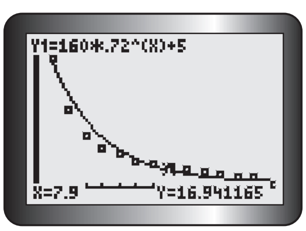


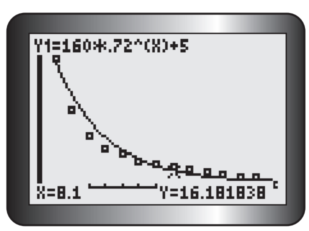
Plot the data on a graphing calculator.





Use an exponential function of the form $y = ab^x + c$ to fit a curve of the data.





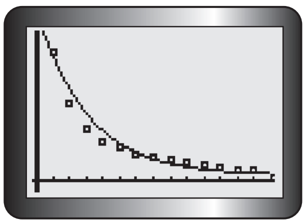
Use the VALUE feature to find values on either side of year 8.

$$\text{Instantaneous rate of change} = \frac{16.181\,838 - 16.941\,165}{8.1 - 7.9}$$
$$\doteq -3.4$$

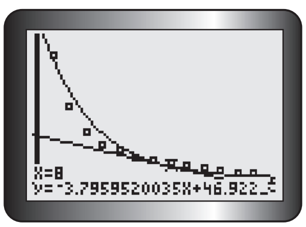
Use these values to calculate the slope of the tangent numerically.

At year 8, the instantaneous rate of change in students per computer was decreasing by about 3 students per computer.

Solution B: Calculating graphically



Plot the data, and then fit an exponential curve to the data.



Position the cursor at year 8, and draw the tangent.

The equation of the tangent is $y = -3.8x + 46.9$.

The calculator provides the equation of the tangent.

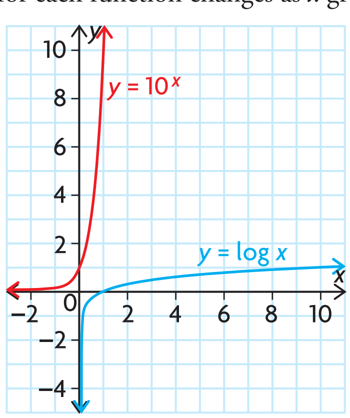
The slope of the tangent is -3.8.

At year 8, the instantaneous rate of change in students per computer decreased by about 4 students per computer.

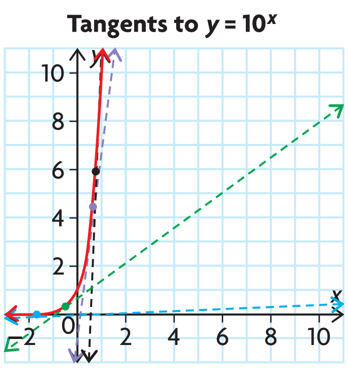
The slope of the tangent at year 8 gives the instantaneous rate of decline.

EXAMPLE 3 Comparing instantaneous rates of change in exponential and logarithmic functions

The graphs of $y = 10^x$ and $y = \log x$ are shown below. Discuss how the instantaneous rate of change in the y-values for each function changes as x grows larger.

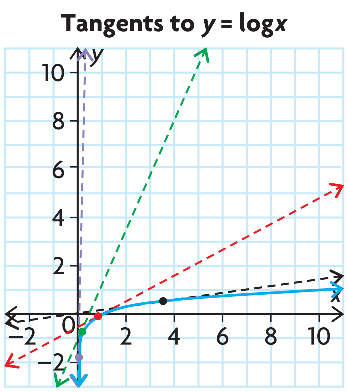


Solution



The graph of $y = 10^x$ has a horizontal asymptote. This means that the tangent lines at points with large negative values of x have a small slope, because the tangent lines are almost horizontal. As x increases, the slopes of the tangent lines also increase.

The instantaneous rate of change in the y-values is close to 0 for small values of x. As x increases, the instantaneous rate of change gets large very quickly. As $x \rightarrow \infty$, the instantaneous rate of change $\rightarrow \infty$.



The graph of $y = \log x$ has a vertical asymptote. This means that the tangent lines at points with very small values of x have very large slopes, because the tangent lines are almost vertical. As x gets larger, the tangent lines become less steep. When x is relatively small, small increases in x result in large changes in the slope of the tangent line. As x grows larger, however, the changes in the slope of the tangent line become smaller and the tangent slopes approach zero.

The instantaneous rate of change in the y-values is very large for small values of x. As x gets larger, the instantaneous rate of change gets smaller very quickly. As $x \rightarrow \infty$, the instantaneous rate of change $\rightarrow 0$.

In Summary

Key Ideas

- The average rate of change is not constant for exponential and logarithmic functions.
- The instantaneous rate of change at a particular point can be estimated by using the same strategies used with polynomial, rational, and trigonometric functions.

Need to Know

- The instantaneous rate of change for an exponential or logarithmic function can be determined numerically or graphically.
- The graph of an exponential or logarithmic function can be used to determine the period during which the average rate of change is least or greatest.
- The graph of an exponential or logarithmic function can be used to predict the greatest and least instantaneous rates of change and when they occur.

CHECK Your Understanding

Use the data from Example 1 for questions 1 to 3.

1. Calculate the average rate of change in number of students per computer during the following time periods.

a) years 2 to 10 **b)** years 1 to 5 **c)** years 10 to 13

2. Predict when the instantaneous rate of change in number of students per computer was the greatest. Give a reason for your answer.

3. Estimate the instantaneous rate of change in number of students per computer for the following years.

a) year 2 **b)** year 7 **c)** year 12

PRACTISING

4. Jerry invests \$6000 at 7.5%/a, compounded annually.

a) Determine the equation of the amount, A, after t years.

b) Estimate the instantaneous rate of change in the value at 10 years.

c) Suppose that the interest rate was compounded semi-annually instead of annually. What would the instantaneous rate of change be at 10 years?

5. K You invest \$1000 in a savings account that pays 6%/a, compounded annually.

a) Calculate the rate at which the amount is growing over the first

i) 2 years **ii)** 5 years **iii)** 10 years

b) Why is the rate of change not constant?

6. For 500 g of a radioactive substance with a half-life of 5.2 h, the amount remaining is given by the formula $M(t) = 500(0.5)^{\frac{t}{5.2}}$, where M is the mass remaining and t is the time in hours.

a) Calculate the amount remaining after 1 day.

b) Estimate the instantaneous rate of change in mass at 1 day.

7. The table shows how the mass of a chicken embryo inside an egg changes over the first 20 days after the egg is laid.

a) Calculate the average rate of change in the mass of the embryo from day 1 to day 20.

b) Determine an exponential equation that models the data.

c) Estimate the instantaneous rate of change in mass for the following days.

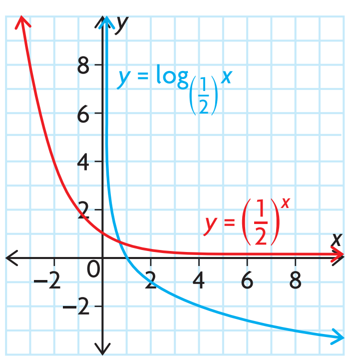
i) day 4 **ii)** day 12 **iii)** day 20

d) According to your model, when will the mass be 6.0000 g?

Days after Egg is Laid	Mass of Embryo (g)
1	0.0002
4	0.0500
8	1.1500
12	5.0700
16	15.9800
20	30.2100

8. A certain radioactive substance decays exponentially. The percent, P, of the substance left after t years is given by the formula $P(t) = 100(1.2)^{-t}$.
   a) Determine the half-life of the substance.
   b) Estimate the instantaneous rate of decay at the end of the first half-life period.

9. **A** The population of a town is decreasing at a rate of 1.8%/a. The current population of the town is 12 000.
   a) Write an equation that models the population of the town.
   b) Estimate the instantaneous rate of change in the population 10 years from now.
   c) Determine the instantaneous rate of change when the population is half its current population.



10. **T** The graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = \log_{\frac{1}{2}} x$ are given. Discuss how the instantaneous rate of change for each function changes as x grows larger.

11. As a tornado moves, its speed increases. The function $S(d) = 93 \log d + 65$ relates the speed of the wind, S, in miles per hour, near the centre of a tornado to the distance that the tornado has travelled, d, in miles.
   a) Graph this function.
   b) Calculate the average rate of change for the speed of the wind at the centre of a tornado from mile 10 to mile 100.
   c) Estimate the rate at which the speed of the wind at the centre of a tornado is changing at the moment it has travelled its 10th mile and its 100th mile.
   d) Use your graph to discuss how the rate at which the speed of the wind at the centre of a tornado changes as the distance that the tornado travels increases.

12. **C** Explain how you could estimate the instantaneous rate of change for an exponential function if you did not have access to a graphing calculator.

Extending

13. How is the instantaneous rate of change affected by changes in the parameters of the function?
   a) $y = a \log[k(x - d)] + c$
   b) $y = ab^{[k(x-d)]} + c$

8 Chapter Review

FREQUENTLY ASKED Questions

Q: How do you solve an exponential equation?

A1: All exponential equations can be solved using the following property:

If $\log_a M = \log_a N$, then $M = N$.

Take the logarithm of both sides of an exponential equation using a base of 10. Then use the power rule for logs to simplify the equation.

A2: Some exponential equations can be solved by using this property:

If $a^x = a^y$, then $x = y$, where $a > 0$, and $a \neq 1$.

Write both sides of an exponential equation with the same base, and set the exponents equal to each other.

A3: If graphing technology is available, treat both sides of an exponential equation as functions, and graph them simultaneously. The x-coordinate of the point of intersection of the two functions is the solution to the equation. There can be more than one solution.

Study *Aid*

- See Lesson 8.5, Examples 1 to 4.
- Try Chapter Review Questions 10 to 13.

Q: How do you solve an equation that contains logarithms?

A1: If there is a single logarithm in the equation, isolate the log term and then rewrite the equation in exponential form to solve it.

A2: If there is more than one term with a logarithm in the equation, simplify the equation using the laws of logarithms. The equation can then be expressed in exponential form to solve it. If there are terms with logs on both sides of the equation, use the following property:

If $\log_a M = \log_a N$, then $M = N$, where $a, M, N > 0$.

Study *Aid*

- See Lesson 8.6, Examples 1 to 3.
- Try Chapter Review Questions 14, 15, and 16.

Q: How do you compare two values on a logarithmic scale?

A: A logarithmic scale increases exponentially, usually by powers of 10. This means that each value on a logarithmic scale is an increase of 10 times the previous value. To compare the values, use the ratio rather than the difference.

Study *Aid*

- See Lesson 8.7, Example 1.
- Try Chapter Review Questions 17 to 20.

PRACTICE Questions

Lesson 8.1

1. Determine the inverse of each function. Express your answers in logarithmic form.

a) $y = 4^x$

b) $y = a^x$

c) $y = \left(\frac{3}{4}\right)^x$

d) $m = p^q$

Lesson 8.2

2. Describe the transformations that must be applied to the parent function $y = \log x$ to obtain each of the following functions.

a) $f(x) = -3\log(2x)$

b) $f(x) = \log(x - 5) + 2$

c) $f(x) = \frac{1}{2}\log 5x$

d) $f(x) = \log\left(-\frac{1}{3}x\right) - 3$

3. For each sequence of transformations of the parent function $y = \log x$, write the equation of the resulting function.

a) vertical compression by a factor of $\frac{2}{5}$, followed by a vertical translation 3 units down

b) reflection in the x-axis, followed by a horizontal stretch by a factor of 2, and a horizontal translation 3 units to the right

c) vertical stretch by a factor of 5, followed by a horizontal compression by a factor of $\frac{1}{2}$, and a reflection in the y-axis

d) a reflection of the y-axis, a horizontal translation 4 units to the left, followed by a vertical translation 2 units down

4. Describe how the graphs of $f(x) = \log x$ and $g(x) = 3\log(x - 1) + 2$ are similar yet different.

Lesson 8.3

5. Evaluate.

a) $\log_7 343$

b) $\log_{\frac{1}{5}} 25$

c) $\log_{19} 1$

d) $\log_4\left(\frac{1}{256}\right)$

6. Estimate the value to three decimal places.

a) $\log_3 53$

b) $\log_4 \frac{1}{10}$

c) $\log_6 159$

d) $\log_{15} 1456$

Lesson 8.4

7. Express as a single logarithm.

a) $\log 5 + \log 11$

b) $\log 20 - \log 4$

c) $\log_5 6 + \log_5 8 - \log_5 12$

d) $2\log 3 + 4\log 2$

8. Use the laws of logarithms to evaluate.

a) $\log_6 42 - \log_6 7$

b) $\log_3 5 + \log_3 18 - \log_3 10$

c) $\log_7 \sqrt[3]{49}$

d) $2\log_4 8$

9. Describe how the graph of $y = \log(10\,000x)$ is related to the graph of $y = \log x$.

Lesson 8.5

10. Solve.

a) $5^x = 3125$

b) $4^x = 16\sqrt{128}$

c) $4^{5x} = 16^{2x-1}$

d) $3^{5x}9^{x^2} = 27$

11. Solve. Express each answer to three decimal places.

a) $6^x = 78$

b) $(5.4)^x = 234$

c) $8(3^x) = 132$

d) $200(1.23)^x = 540$

12. Solve.

a) $4^x + 6(4^{-x}) = 5$

b) $8(5^{2x}) + 8(5^x) = 6$

13. The half-life of a certain substance is 3.6 days. How long will it take for 20 g of the substance to decay to 7 g?

Lesson 8.6

14. Solve.

a) $\log_5(2x - 1) = 3$
b) $\log 3x = 4$
c) $\log_4(3x - 5) = \log_4 11 + \log_4 2$
d) $\log(4x - 1) = \log(x + 1) + \log 2$

15. Solve.

a) $\log(x + 9) - \log x = 1$
b) $\log x + \log(x - 3) = 1$
c) $\log(x - 1) + \log(x + 2) = 1$
d) $\log\sqrt{x^2 - 1} = 2$

16. Recall that $L = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of sound in watts per square metre (W/m^2) and $I_0 = 10^{-12}\ \text{W/m}^2$. Determine the intensity of a baby screaming if the noise level is 100 dB.

Lesson 8.7

17. What is the sound intensity in watts per square metre (W/m^2) of an engine that is rated at 82 dB?

18. How many times more intense is an earthquake of magnitude 6.2 than an earthquake of magnitude 5.5?

19. Pure water has a pH value of 7.0. How many times more acidic is milk, with a pH value of 6.4, than pure water?

20. Does an increase in acidity from pH 4.7 to pH 2.3 result in the same change in hydrogen ion concentration as a decrease in alkalinity from 12.5 to 10.1? Explain.

21. Is an exponential model appropriate for the data in the following table? If it is, determine the equation that models the data.

x	0	2	4	6	8	10
y	3.0	15.2	76.9	389.2	1975.5	9975.8

22. The population of a town is decreasing at the rate of 1.6%/a. If the population today is 20 000, how long will it take for the population to decline to 15 000?

Lesson 8.8

23. The following table gives the population of a city over time.

Year	1950	1970	1980	1990	1994
Population	132 459	253 539	345 890	465 648	514 013

a) Calculate the average rate of growth over the entire time period.
b) Calculate the average rate of growth for the first 30 years. How does it compare with the rate of growth for the entire time period?
c) Determine an exponential model for the data.
d) Estimate the instantaneous rate of growth in
i) 1970 ii) 1990

24. The following data show the number of people (in thousands) who own a DVD player in a large city or linear is best for over a period of years.

Year	1998	1999	2000	2001	2002
Number of DVD Owners (thousands)	23	27	31	37	43

a) Determine if an exponential or linear model is best for this data.
b) Use your model to predict how many people will own a DVD player in the year 2015.
c) What assumptions did you make to make your prediction in part b)? Do you think this is reasonable? Explain.
d) Determine the average rate of change in the number of DVD players in this city between 1999 and 2002.
e) Estimate the instantaneous rate of change in the number of DVD players in this city in 2000.
f) Explain why using an exponential model to answer part b) does not make sense.

8 Chapter Self-Test

1. Write the equation of the inverse of each function in both exponential and logarithmic form.
   a) $y = 4^x$
   b) $y = \log_6 x$
2. State the transformations that must be applied to $f(x) = \log x$ to graph $g(x)$.
   a) $g(x) = \log[2(x - 4)] + 3$
   b) $g(x) = -\frac{1}{2}\log(x + 5) - 1$
3. Evaluate.
   a) $\log_3 \frac{1}{9}$
   b) $\log_5 100 - \log_5 4$
4. Evaluate.
   a) $\log 15 + \log 40 - \log 6$
   b) $\log_7 343 + 2\log_7 49$
5. Express $\log_4 x^2 + 3\log_4 y^{\frac{1}{3}} - \log_4 x$ as a single logarithm. Assume that x and y represent positive numbers.
6. Solve $5^{x+2} = 6^{x+1}$. Round your answer to three decimal places.
7. Solve.
   a) $\log_4(x + 2) + \log_4(x - 1) = 1$
   b) $\log_3(8x - 2) + \log_3(x - 1) = 2$
8. Carbon-14 is used by scientists to estimate how long ago a plant or animal lived. The half-life of carbon-14 is 5730 years. A particular plant contained 100 g of carbon-14 at the time that it died.
   a) How much carbon-14 would remain after 5730 years?
   b) Write an equation to represent the amount of carbon-14 that remains after t years.
   c) After how many years would 80 g of carbon-14 remain?
   d) Estimate the instantaneous rate of change at 100 years.
9. The equation that models the amount of time, t, in minutes that a cup of hot chocolate has been cooling as a function of its temperature, T, in degrees Celsius is $t = \log\left(\frac{T - 22}{75}\right) \div \log(0.75)$. Calculate the following.
   a) the cooling time if the temperature is 35 °C
   b) the initial temperature of the drink