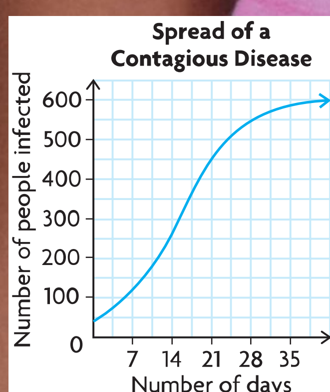
Chapter 9

Combinations of Functions

GOALS

You will be able to

- Consolidate your understanding of the characteristics of functions
- Create new functions by adding, subtracting, multiplying, and dividing functions
- Investigate the creation of composite functions numerically, graphically, and algebraically
- Determine key characteristics of these new functions
- Solve problems using a variety of function models



? *Epidemiology* is the scientific study of contagious diseases. A combination of functions is often used to model the way that a contagious disease spreads through a population. What types of functions could be combined to create an algebraic model that represents the graph shown?

9 Getting Started

SKILLS AND CONCEPTS You Need

Study Aid

- For help, see the Review of Essential Skills found at the Nelson Advanced Functions website.

Question	Appendix
3	R-8

1. Evaluate each of the following functions for $f(-1)$ and $f(4)$. Round your answers to two decimal places, if necessary.

a) $f(x) = x^3 - 3x^2 - 10x + 24$

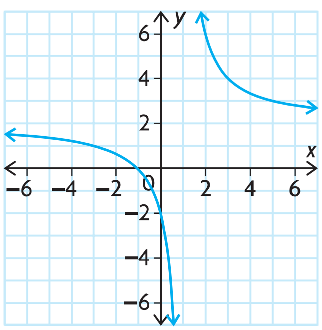
b) $f(x) = \dfrac{4x}{1 - x}$

c) $f(x) = 3\log_{10}(x)$

d) $f(x) = -5(0.5^{(x-1)})$

2. Identify the following characteristics of functions for the graph displayed.

- domain and range
- maximum or minimum values
- interval(s) where the function is increasing
- interval(s) where the function is decreasing
- end behaviour
- equations of asymptotes



3. For each parent function, apply the given transformation(s) and write the equation of the new function.

a) $y = |x|$; vertical stretch by a factor of 2, shift 3 units to the right

b) $y = \cos(x)$; reflection in the x-axis, horizontal compression by a factor of $\frac{1}{2}$

c) $y = \log_3 x$; reflection in the y-axis, shift 4 units left and 1 unit down

d) $y = \frac{1}{x}$; vertical stretch by a factor of 4, reflection in the x-axis, shift 5 units down

4. Solve each equation for x, $x \in \mathbf{R}$. State any restrictions on x, as required.

a) $2x^3 - 7x^2 - 5x + 4 = 0$

b) $\dfrac{2x + 3}{x + 3} + \dfrac{1}{2} = \dfrac{x + 1}{x - 1}$

c) $\log x + \log(x - 3) = 1$

d) $10^{-4x} - 22 = 978$

e) $5^{x+3} - 5^x = 0.992$

f) $2\cos^2 x = \sin x + 1, 0 \le x \le 2\pi$

5. Solve each inequality for x, $x \in \mathbf{R}$.

a) $x^3 - x^2 - 14x + 24 < 0$

b) $\dfrac{(2x - 3)(x - 4)}{(x + 2)} \ge 0$

6. Identify each function as even, odd, or neither.

a) $f(x) = 2\sin(x - \pi)$

b) $f(x) = \dfrac{3}{4 - x}$

c) $f(x) = 4x^4 - 3x^2$

d) $f(x) = 2^{3x-1}$

7. Classify the types of functions you have studied (polynomial, rational, exponential, logarithmic, and trigonometric) as continuous or not.

APPLYING *What You Know*

Building a Sandbox

Duncan is planning to build a rectangular sandbox in his backyard for his son to play in during the summer. He has designed the sandbox so that it will have an open top and a volume of 2 m^3. The length of the base will measure four times the height of the sandbox. The wood for the base will cost $\$5/m^2$, and the wood for the sides will cost $\$4/m^2$.

? What dimensions should Duncan use to minimize the cost of the sandbox he has designed?

A. Let h represent the height (in metres) and let w represent the width of the sandbox. Determine an expression for the width of the sandbox in terms of its height.

B. Write an expression for the cost of the wood for the base of the sandbox in terms of its height.

C. Express the cost of the wood for the two longer sides in terms of the height. Is the cost for the two shorter sides the same?

D. Let $C(h)$ represent the total cost of the wood for the sandbox as a function of its height. Determine the equation for $C(h)$.

E. What types of functions are added in your equation for $C(h)$?

F. What would be a reasonable domain and range for this cost function? Explain.

G. Using graphing technology, graph the cost function using window settings that correspond to its domain and range.

H. Determine the height of the sandbox that will minimize the total cost.

I. What dimensions would you recommend that Duncan use to build the sandbox? Justify your answer.

9.1 Exploring Combinations of Functions

YOU WILL NEED

- graphing calculator or graphing software

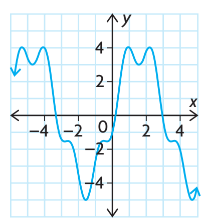
GOAL

Explore the characteristics of new functions created by combining functions.

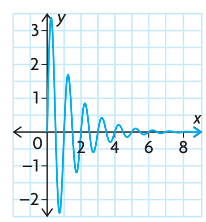
Explore the Math

Ahmad was given the graphs pictured below. They were created by combining two familiar functions.

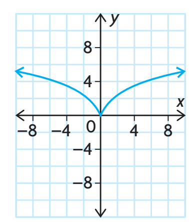
Graph 1



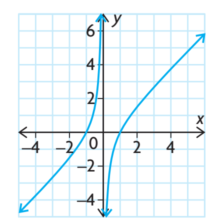
Graph 2



Graph 3



Graph 4



Ahmad does not recognize these new functions and wonders which type of functions have been combined to create them. He also wonders whether any of these graphs could model a real-life situation.

? How can two functions be combined to create a new function?

A. Compare each of the graphs above with the function equations in the table below.

$y = x\sqrt{x-1}$	$y = 4\sin x - \cos 4x$	$y = x - \frac{1}{x}$	$y = 5\log(\lvert x\rvert + 1)$
$y = (x^2)(\sin(x))$	$y = \begin{cases} -0.5(x-2)^2 + 2, x < 0 \\ 0.5(x-2)^2 - 2, x \geq 0 \end{cases}$	$y = (0.5^x)(4\sin(2\pi x))$	$y = x^3 \div (x+1)$

Predict which equations will match each graph. Copy the table on the next page, and record your predictions and your rationale for each.

Graph	Equation of Function	Rationale
1		
2		
3		
4		

B. Compare your predictions with a partner's predictions. Explain to each other why you made each prediction.

C. Using graphing technology in radian mode, graph the equation that you predicted would match graph 1. Use a domain and range in the window settings that match the scale given on each of the given graphs.

D. Does the graph of your equation match graph 1? If it does not, choose another equation from the table and try again.

E. Once you have correctly matched the equation with graph 1, repeat parts C and D until all the graphs have been correctly matched.

F. Examine the equation that matches each graph.
- List the parent functions in each equation.
- State the transformations that were applied to each parent function.
- Explain how the parent functions were combined.

Reflecting

G. Which of the four given graphs is periodic? How does it differ from other periodic functions you have seen before? What type of combination produced this effect?

H. Do any of the graphs represent an even function? Do any represent an odd function? Explain how you know.

I. Which graph contains an asymptote? Describe the functions that were combined to produce this graph. Explain how you can tell from the equation where the vertical asymptote occurs.

J. Which graph could be used to model the motion of a swaying building moments after an earthquake? Explain why.

In Summary

Key Idea

- Many interesting functions can be created by combining two or more simpler functions. This can be done by adding, subtracting, multiplying, or dividing functions to create more complex functions.

Need to Know

- The characteristics of the functions that are combined affect the properties and characteristics of the resulting function.

FURTHER *Your Understanding*

1. Using graphing technology (in radian mode) and the functions given in the chart below, experiment to create new functions by combining different types of functions. Each time, use different operations and different types of functions. You may need to experiment with the window settings to get a clear picture of what the graph looks like. Include a sketch of your new graphs and the equations that were used for the models.

$y = 2 - 0.5x$	$y = 2^x$	$y = \sin 2\pi x$	$y = \cos 2\pi x$
$y = \log x$	$y = \left(\frac{1}{2}\right)^x$	$y = x^4 - x^2$	$y = 2x$

2. Using the functions in the chart above, create a new function that has each of the characteristics given below. Include a sketch of your new graphs and the equations that were used for the models.

a) a function that has a vertical asymptote and a horizontal asymptote
b) a function that is even
c) a function that is odd
d) a function that is periodic
e) a function that resembles a periodic function with decreasing maximum values and increasing minimum values
f) a function that resembles a periodic function with increasing maximum values and decreasing minimum values

3. Select any two functions that you have studied in this course. Experiment by combining these functions in various ways and graphing them on a graphing calculator. Include a sketch of your new graphs and the equations of the functions you selected. Challenge your classmates to see who can produce the most interesting graph.

9.2 Combining Two Functions: Sums and Differences

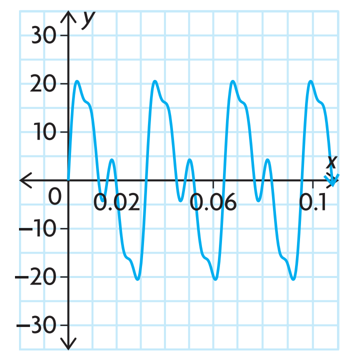
GOAL

Represent the sums and differences of two functions graphically and algebraically, and determine their properties.

YOU WILL NEED

- graphing calculator or graphing software

INVESTIGATE the Math



The sound produced when a person strums a guitar chord represents the combination of sounds made by several different strings. The sound made by each string can be represented by a sine function. The period of each function is based on the frequency of the sound, whereas the loudness of the individual sounds varies and is related to the amplitude of each function. These sine functions are literally added together to produce the desired sound.

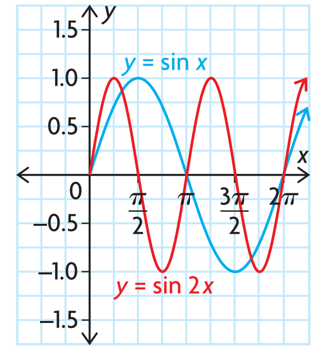
The sound of a G chord played on a six-string acoustic guitar can be approximated by the following combination of sine functions:

$$y = 16 \sin 196x + 9 \sin 392x + 4 \sin 784x$$

? When functions are added or subtracted, how do the resulting characteristics of the new function compare with those of the original functions?

A. Explore a similar but simpler combination of sine functions by examining the properties of the sum defined by $y = \sin x + \sin 2x$. Copy and complete the table of values, and use your results and the graphs shown to sketch the graph of $y = \sin x + \sin 2x$, where $0 \le x \le 2\pi$.

x	$\sin x$	$\sin 2x$	$\sin x + \sin 2x$
0	0	0	
$\frac{\pi}{4}$	0.7071	1	
$\frac{\pi}{2}$	1	0	
$\frac{3\pi}{4}$	0.7071	−1	
π	0	0	
$\frac{5\pi}{4}$	−0.7071	1	
$\frac{3\pi}{2}$	−1	0	
$\frac{7\pi}{4}$	−0.7071	−1	
2π	0	0	



B. Set the calculator to radian mode. Adjust the window settings so that $0 \le x \le 4\pi$ using an Xscl $= \frac{\pi}{4}$, and $-2 \le y \le 2$ using a Yscl $= 1$. Verify your graph in part A by graphing $y = \sin x + \sin 2x$.

C. What is the period of $y = \sin x + \sin 2x$? How does it compare with the periods of $y = \sin x$ and $y = \sin 2x$?

D. What is the amplitude of $y = \sin x + \sin 2x$? How does it compare with the amplitudes of $y = \sin x$ and $y = \sin 2x$?

E. Create a new table of values, and use your results and the graphs of $y = \sin x$ and $y = \sin 2x$ to sketch the graph of $y = \sin x - \sin 2x$, where $0 \le x \le 2\pi$. Repeat parts B to D using this difference function.

F. Do you think that the graph of $y = \sin 2x - \sin x$ will be the same as the graph you created in part E? Explain. Check your conjecture by using graphing technology to graph this function.

G. Investigate the sum of other types of functions. Use graphing technology to graph each set of functions, and describe how the characteristics of the functions are related.

i) $y_1 = -x, y_2 = x^2, y_3 = -x + x^2$
ii) $y_1 = \sqrt{x}, y_2 = \sqrt{x + 2}, y_3 = \sqrt{x} + \sqrt{x + 2}$
iii) $y_1 = 2^x,\ y_2 = 2^{-x}, y_3 = 2^x + 2^{-x}$
iv) $y_1 = \cos x, y_2 = \cos 2x, y_3 = \cos x + \cos 2x$

H. Investigate the difference of each set of functions in part G by graphing y_1 and y_2, and changing y_3 to $y_3 = y_1 - y_2$. Describe how the characteristics of the functions are related.

Reflecting

I. How does the degree of the sum or difference of two polynomial functions compare with the degree of the individual functions?

J. How does the period of the sum or difference of two trigonometric functions compare with the periods of the individual functions?

K. When looking at the sum of two functions, does the phrase "for each *x*, add the corresponding *y*-values together" describe the result you observed for every pair of functions? What phrase would you use to describe finding the difference of two functions?

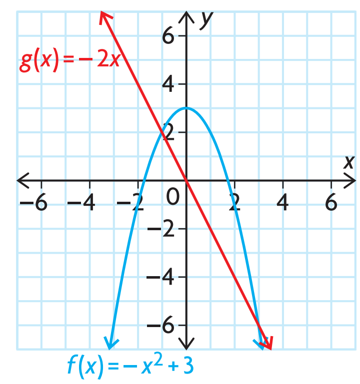
L. Looking at the graphs of the two square root functions, explain why the domain of the graph of their sum is $x \ge 0$.

M. Determine the y-intercept of y_3, where y_3 represents the difference of the two exponential functions. What does this point represent with respect to y_1 and y_2?

APPLY the Math

EXAMPLE 1 Selecting a strategy to combine functions by addition and subtraction

Given $f(x) = -x^2 + 3$ and $g(x) = -2x$, determine the graphs of $f(x) + g(x)$ and $f(x) - g(x)$. Discuss the key characteristics of the resulting graphs.

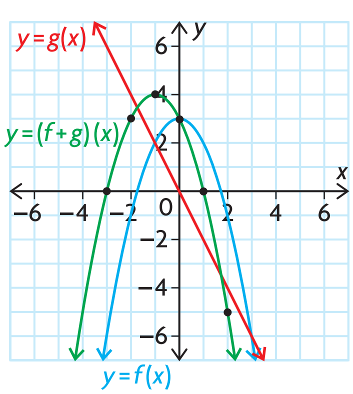


Solution A: Using a graphical strategy

x	$f(x)$	$g(x)$	$f(x) + g(x)$	$f(x) - g(x)$
-3	-6	6	$-6 + 6 = 0$	$-6 - 6 = -12$
-2	-1	4	3	-5
-1	2	2	4	0
-0.5	2.75	1	3.75	1.75
0	3	0	3	3
1	2	-2	0	4
2	-1	-4	-5	3
3	-6	-6	-12	0

Make a table of values for $f(x)$ and $g(x)$, for selected values of x. Create $f + g$ by adding the y-coordinates of f and g together. Create $f - g$ by subtracting the y-coordinates of g from f.

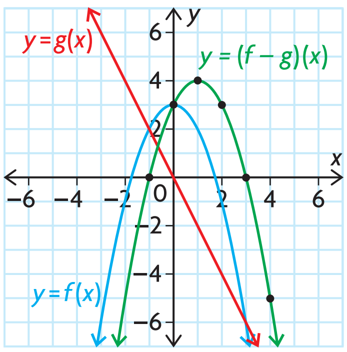
These functions can be added or subtracted over their entire domains since they both have the same domain $\{x \in \mathbf{R}\}$.



Plot the ordered pairs $(x, f(x) + g(x))$. Join the plotted points with a smooth curve.

Observe that the zeros of the new function occur when the y-values of f and g are the same distance from the x-axis, but on opposite sides. When a zero occurs for either f or g, the value of $f + g$ is the value of the other function.

At any point where f and g intersect, the value of $f + g$ is double the value of f (or g) for the corresponding x.



Plot the ordered pairs $(x, f(x) - g(x))$ from the table, and join them with a smooth curve to produce the graph of $f - g$.

Observe that the zeros of this $f - g$ graph occur when the graphs of f and g intersect.

Where g has a zero, the value of $f - g$ is the same as the value of f. Where f has a zero, the value of $f - g$ is the opposite of the value of g.

Solution B: Using an algebraic strategy

$f(x) = -x^2 + 3$ and $g(x) = -2x$

$$\begin{aligned}(f + g)(x) &= f(x) + g(x)\\ &= (-x^2 + 3) + (-2x)\\ &= -x^2 - 2x + 3\end{aligned}$$

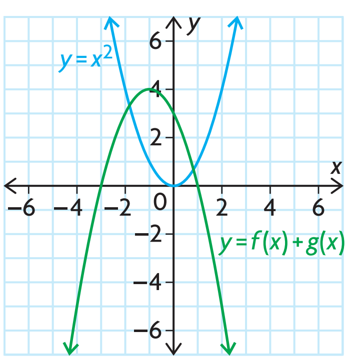
Remember that adding two functions means adding their y-values for a given value of x.

Since the expressions for $f(x)$ and $g(x)$ represent the y-values for each function, we determine an expression for $f + g$ by adding the two expressions.

$$\begin{aligned}(f + g)(x) &= -[x^2 + 2x] + 3\\ &= -[x^2 + 2x + 1 - 1] + 3\\ &= -[(x + 1)^2 - 1] + 3\\ &= -(x + 1)^2 + 4\end{aligned}$$

Recognizing that $f + g$ is a quadratic function, we can complete the square to change the expression into vertex form.

The graph of $f + g$ can be sketched by starting with the graph of $y = x^2$ and applying the following transformations: reflection in the x-axis, followed by a shift of 1 unit to the left and 4 units up.



The graph of $y = (f + g)(x)$ has the following characteristics: it is neither odd nor even; it is increasing on the interval $(-\infty, -1)$ and decreasing on the interval $(-1, \infty)$; it has zeros at $(-3, 0)$ and $(1, 0)$; it has a maximum value of $y = 4$ when $x = -1$; its domain is $\{x \in \mathbf{R}\}$; its range is $\{y \in \mathbf{R} \mid y \leq 4\}$.

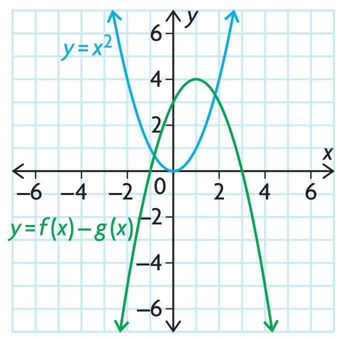
$$\begin{aligned}(f - g)(x) &= f(x) - g(x)\\ &= (-x^2 + 3) - (-2x)\\ &= -x^2 + 2x + 3\end{aligned}$$

Similarly, we obtain the expression for $f - g$ by subtracting $g(x)$ from $f(x)$.

In vertex form,

$$\begin{aligned}(f - g)(x) &= -[x^2 - 2x] + 3\\ &= -[x^2 - 2x + 1 - 1] + 3\\ &= -(x - 1)^2 + 4\end{aligned}$$

Again, we can rewrite the quadratic expression in vertex form to graph it.



The graph of $f - g$ resembles the graph of $f + g$, except it has been shifted 1 unit to the right instead of 1 unit left.

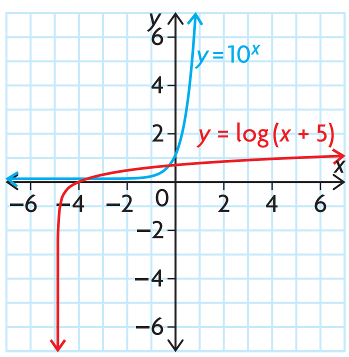
The graph of $y = (f - g)(x)$ has the following characteristics: it is neither odd nor even; it is increasing on the interval $(-\infty, 1)$ and decreasing on the interval $(1, \infty)$; it has zeros at $(-1, 0)$ and $(3, 0)$; it has a maximum value of $y = 4$ when $x = 1$; its domain is $\{x \in \mathbf{R}\}$; its range is $\{y \in \mathbf{R} | y \leq 4\}$.

EXAMPLE 2 Connecting the domains of the sum and difference of two functions

Determine the domain and range of $(f - g)(x)$ and $(f + g)(x)$ if $f(x) = 10^x$ and $g(x) = \log(x + 5)$.

Solution

Sketch the graphs of f and g.



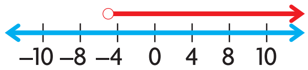
$f(x) = 10^x$ is an exponential function that has the x-axis as its horizontal asymptote. Exponential functions are defined for all real numbers, so its domain is $\{x \in \mathbf{R}\}$.

$g(x) = \log(x + 5)$ is a logarithmic function in base 10. Logarithmic functions are only defined for positive values: $x + 5 > 0$, so $x > -5$. This function has a vertical asymptote defined by $x = -5$. Its domain is $\{x \in \mathbf{R} | x > -5\}$.

$$\begin{aligned}(f - g)(x) &= f(x) - g(x)\\ &= 10^x - \log(x + 5)\\ (f + g)(x) &= f(x) + g(x)\\ &= 10^x + \log(x + 5)\end{aligned}$$

Values for the functions $f - g$ and $f + g$ can only be determined when functions f and g are both defined. This occurs for all values of x that are common to the domains of both f and g.

The domain of the functions $(f - g)(x)$ and $(f + g)(x)$ is $\{x \in \mathbf{R} | x > -5\}$.



This is the **intersection** of the domains of f and g.
$\{x \in \mathbf{R}\} \cap \{x \in \mathbf{R} | x > -5\}$
$= \{x \in \mathbf{R} | x > -5\}$

intersection

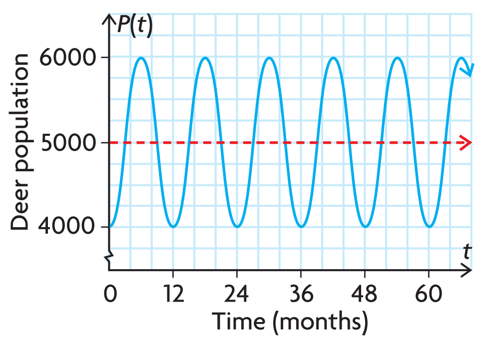
a set that contains the elements that are common to both sets; the symbol for intersection is $\cap$

EXAMPLE 3 Modelling a situation using a sum of two functions

In the past, biologists have found that the function $P(t) = 5000 - 1000\cos\left(\frac{\pi}{6}t\right)$ models the deer population in a provincial park, which undergoes a seasonal fluctuation. In this case, $P(t)$ is the size of the deer population t months after January. A disease in the wolf population has caused its population to decline, and the biologists have discovered that the deer population is increasing by 50 deer each month. Assuming that this pattern continues, determine the new function that will model the deer population over time and discuss its characteristics.

Solution

Graph $P(t)$.



In the past, the deer population varied around its average yearly size of 5000. This is represented by the horizontal axis, or midline, of the graph shown.

This suggests that the function $P(t)$ can be thought of in the following way:

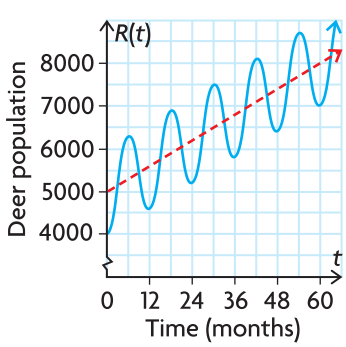
$$P(t) = \underbrace{5000}_{\text{average yearly population}} - \underbrace{1000\cos\left(\frac{\pi}{6}t\right)}_{\text{seasonal variation}}$$

$$R(t) = \underbrace{5000 + 50t}_{\text{average yearly population increasing by 50 every month}} - \underbrace{1000\cos\left(\frac{\pi}{6}t\right)}_{\text{seasonal variation}}$$

If the population of deer is increasing by 50 per month, then the average population could be represented by the expression $5000 + 50t$.

The new function $R(t)$ has been created by adding the function $f(t) = 50t$ to $P(t)$.

Therefore, $R(t) = P(t) + f(t)$.



The new population model has the following characteristics: it is neither odd nor even; it is increasing during the first six months of each year and decreasing during the last six months of each year; it has no zeros; it has no maximum or minimum value and its domain is $\{t \in \mathbf{R} \mid t \geq 0\}$; its range is $\{R(t) \in \mathbf{R} \mid R(t) \geq 4000\}$.

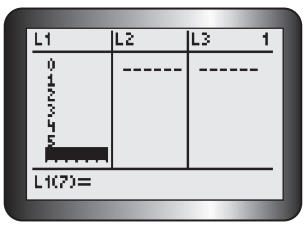
EXAMPLE 4 Reasoning about families of functions

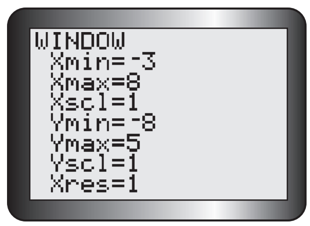
Use graphing technology to explore the graph of $f - g$, where $f(x) = x^2$ and $g(x) = nx$, and $n \in \mathbf{W}$. Discuss your results with respect to the type of function, its shape and symmetry, zeros, maximum and minimum values, intervals of increase/decrease, and domain and range.

Solution

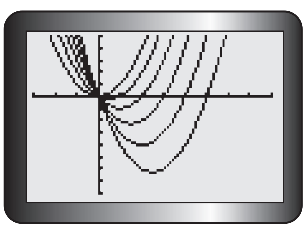
$(f - g)(x) = f(x) - g(x)$,
then for $f(x) = x^2$ and $g(x) = nx$,
$(f - g)(x) = x^2 - nx$, where $n \in \mathbf{W}$

$f - g$ will always be a quadratic function, regardless of the value of n.





Enter several values of $n \in \{0, 1, 2, 3, 4, 5\}$ into list 1 (L1) on a graphing calculator. Enter the equation $X^2 - L_1X$ into the equation editor. Then graph using the window settings shown.



The series of parabolas that $f - g$ produces will have identical shapes, since $a = 1$. It appears that, as n increases, the parabola is shifted to the right and down.

Most of these functions are neither odd nor even since their graphs are not symmetrical about the origin or y-axis.

The exception is when $n = 0$, which produces the even function $(f - g)(x) = x^2$.

The zeros of each parabola occur at $x = 0$ and $x = n$.

In factored form, $(f - g)(x) = x(x - n)$
$x = 0$ or $x - n = 0$
$x = n$

Since the parabola opens upward, the minimum value is $-\frac{n^2}{4}$, and the axis of symmetry is $x = \frac{n}{2}$.

The vertex of each parabola will occur at $\left(\frac{n}{2}, -\frac{n^2}{4}\right)$.

These functions are decreasing when $x \in \left(-\infty, \frac{n}{2}\right)$ and increasing when $x \in \left(\frac{n}{2}, \infty\right)$. The domain of $f - g$ is $\{x \in \mathbf{R}\}$, and the range is $y \in \left[-\frac{n^2}{4}, \infty\right)$.

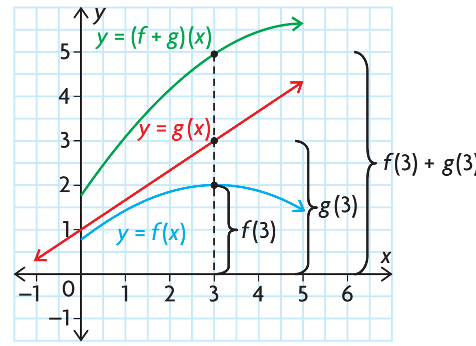
Complete the square to put the function into vertex form.

$$(f - g)(x) = x^2 - nx + \left(\frac{n}{2}\right)^2 - \left(\frac{n}{2}\right)^2$$
$$= \left(x - \frac{n}{2}\right)^2 - \frac{n^2}{4}$$

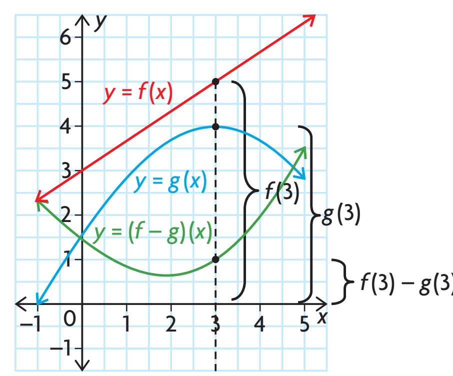
In Summary

Key Ideas

- When two functions $f(x)$ and $g(x)$ are combined to form the function $(f + g)(x)$, the new function is called the sum of f and g. For any given value of x, the value of the function is represented by $f(x) + g(x)$. The graph of $f + g$ can be obtained from the graphs of functions f and g by adding corresponding y-coordinates.



- Similarly, the difference of two functions, $f - g$, is $(f - g)(x) = f(x) - g(x)$. The graph of $f - g$ can be obtained by subtracting the y-coordinate of g from the y-coordinate of f for every pair of corresponding x-values.



Need to Know

- Algebraically, $(f + g)(x) = f(x) + g(x)$ and $(f - g)(x) = f(x) - g(x)$.
- The domain of $f + g$ or $f - g$ is the intersection of the domains of f and g. This means that the functions $f + g$ and $f - g$ are only defined where the domains of both f and g overlap.

CHECK *Your Understanding*

1. Let $f = \{(-4, 4), (-2, 4), (1, 3), (3, 5), (4, 6)\}$ and $g = \{(-4, 2), (-2, 1), (0, 2), (1, 2), (2, 2), (4, 4)\}$. Determine:

a) $f + g$ **c)** $f - g$ **e)** $f + f$
b) $g + f$ **d)** $g - f$ **f)** $g - g$

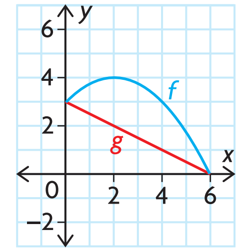
2. a) Determine $(f + g)(4)$ when $f(x) = x^2 - 3$ and $g(x) = -\dfrac{6}{x - 2}$.

b) For which value of x is $(f + g)(x)$ undefined? Explain why.

c) What is the domain of $(f + g)(x)$ and $(f - g)(x)$?

3. What is the domain of $f - g$, where $f(x) = \sqrt{x + 1}$ and $g(x) = 2\log[-(x + 1)]$?

4. Make a reasonable sketch of the graph of $f + g$ and $f - g$, where $0 \leq x \leq 6$, for the functions shown.



5. **a)** Given the function $f(x) = |x|$ (which is even) and $g(x) = x$ (which is odd), determine $f + g$.

b) Is $f + g$ even, odd, or neither?

PRACTISING

6. **K** $f = \{(-9, -2), (-8, 5), (-6, 1), (-3, 7), (-1, -2), (0, -10)\}$ and $g = \{(-7, 7), (-6, 6), (-5, 5), (-4, 4), (-3, 3)\}$. Calculate:

a) $f + g$
b) $g + f$
c) $f - g$
d) $g - f$
e) $f - f$
f) $g + g$

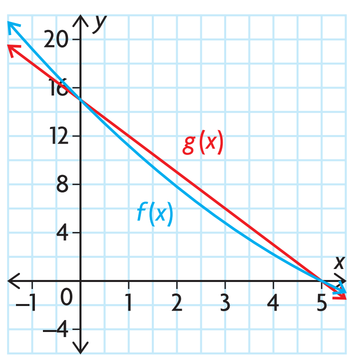
7. **a)** If $f(x) = \frac{1}{3x + 4}$ and $g(x) = \frac{1}{x - 2}$, what is $f + g$?

b) What is the domain of $f + g$?

c) What is $(f + g)(8)$?

d) What is $(f - g)(8)$?

8. The graphs of $f(x)$ and $g(x)$, where $0 \leq x \leq 5$, are shown. Sketch the graphs of $(f + g)(x)$ and $(f - g)(x)$.



9. For each pair of functions, determine the equations of $f(x) + g(x)$ and $f(x) - g(x)$. Using graphing technology, graph these new functions and discuss each of the following characteristics of the resulting graphs: symmetry, intervals of increase/decrease, zeros, maximum and minimum values, period (where applicable), and domain and range.

a) $f(x) = 2^x, g(x) = x^3$

b) $f(x) = \cos(2\pi x), g(x) = x^4$

c) $f(x) = \log(x), g(x) = 2x$

d) $f(x) = \sin(2\pi x), g(x) = 2\sin(\pi x)$

e) $f(x) = \sin(2\pi x) + 2, g(x) = \frac{1}{x}$

f) $f(x) = \sqrt{x - 2}, g(x) = \frac{1}{x - 2}$

10. a) Is the sum of two even functions even, odd, or neither? Explain.
b) Is the sum of two odd functions even, odd, or neither? Explain.
c) Is the sum of an even function and an odd function even, odd, or neither? Explain.

11. **A** Recall, from Example 3, the function $P(t) = 5000 - 1000 \cos\left(\frac{\pi}{6}t\right)$, which models the deer population in a provincial park. A disease in the deer population has caused it to decline. Biologists have discovered that the deer population is decreasing by 25 deer each month.
a) Assuming that this pattern continues, determine the new function that will model the deer population over time and discuss its characteristics.
b) Estimate when the deer population in this park will be extinct.

12. When the driver of a vehicle observes an obstacle in the vehicle's path, the driver reacts to apply the brakes and bring the vehicle to a complete stop. The distance that the vehicle travels while coming to a stop is a combination of the reaction distance, r, in metres, given by $r(x) = 0.21x$, and the braking distance, b, also in metres, given by $b(x) = 0.006x^2$. The speed of the vehicle is x km/h. Determine the stopping distance of the vehicle as a function of its speed, and calculate the stopping distance if the vehicle is travelling at 90 km/h.

13. **T** Determine a sine function, f, and a cosine function, g, such that $y = \sqrt{2}\sin(\pi(x - 2.25))$ can be written in the form of $f - g$.

14. Use graphing technology to explore the graph of $f + g$, where $f(x) = x^3$, $g(x) = nx^2$, and $n \in \mathbf{W}$. Discuss your results with respect to the type of function, its shape and symmetry, zeros, maximum and minimum values, intervals of increase/decrease, and domain and range.

15. **C** Describe or give an example of
a) two odd functions whose sum is an even function
b) two functions whose sum represents a vertical stretch applied to one of the functions
c) two rational functions whose difference is a constant function

Extending

16. Let $f(x) = x^2 - nx + 5$ and $g(x) = mx^2 + x - 3$. The functions are combined to form the new function $h(x) = f(x) + g(x)$. Points $(1, 3)$ and $(-2, 18)$ satisfy the new function. Determine the values of m and n.

9.3 Combining Two Functions: Products

GOAL

Represent the product of two functions graphically and algebraically, and determine the characteristics of the product.

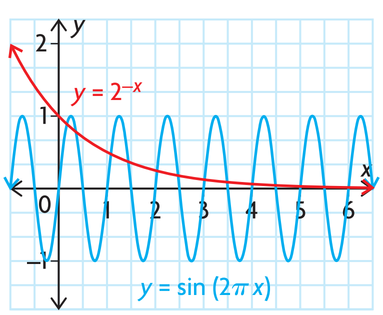
YOU WILL NEED

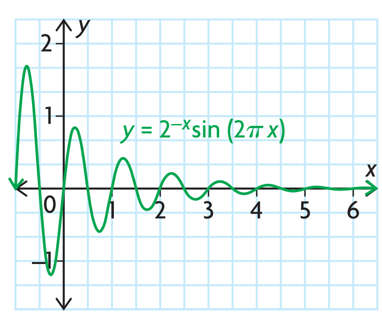
- graphing calculator or graphing software

LEARN ABOUT the Math

In the previous section, you learned that music is made up of combinations of sine waves. Have you ever wondered how sound engineers cause the music to fade out, gradually, at the end of a song? The music fades out because the sine waves that represent the music are being squashed or **damped**. Mathematically, this can be done by multiplying a sine function by another function.

The functions defined by $g(x) = \sin(2\pi x)$ and $f(x) = 2^{-x}$, where $\{x \in \mathbf{R} | x \geq 0\}$, are shown below. Observe what happens when these functions are multiplied to produce the graph of $(f \times g)(x) = 2^{-x} \sin(2\pi x)$.





? Can the product of two functions be constructed using the same strategies that are used to create the sum or difference of two functions?

EXAMPLE 1 Connecting the values of a product function to the values of each function

Investigate the product of the functions $f(x) = 2^{-x}$ and $g(x) = \sin(2\pi x)$.

Solution

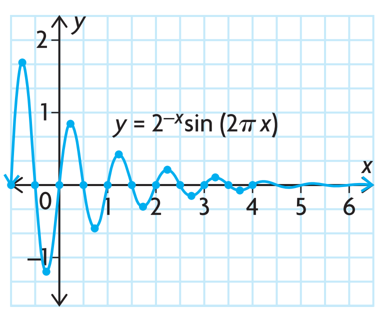
	A	B	C	D
1	x	f(x)=2^−x	g(x)=sin(2πx)	(fxg)(x)=(2^−x)sin(2πx)
2	0.00	1.00	0.00	0.00
3	0.25	0.84	1.00	0.84
4	0.50	0.71	0.00	0.00
5	0.75	0.59	−1.00	−0.59
6	1.00	0.50	0.00	0.00
7	1.25	0.42	1.00	0.42
8	1.50	0.35	0.00	0.00
9	1.75	0.30	−1.00	−0.30
10	2.00	0.25	0.00	0.00
11	2.25	0.21	1.00	0.21
12	2.50	0.18	0.00	0.00
13	2.75	0.15	−1.00	−0.15
14	3.00	0.13	0.00	0.00
15	3.25	0.11	1.00	0.11
16	3.50	0.09	0.00	0.00
17	3.75	0.07	−1.00	−0.07
18	4.00	0.06	0.00	0.00

In a spreadsheet, enter some values of x in column A, and enter the formulas for f, g, and $f \times g$ in columns B, C, and D, respectively.

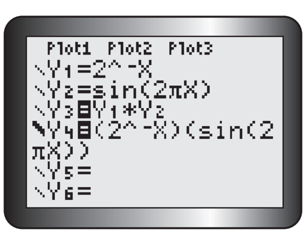
The values in the table have been rounded to two decimal places.

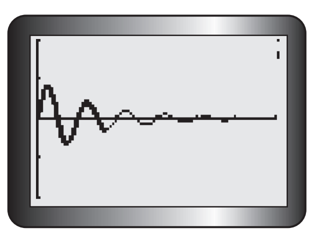
Looking at each row of the table, for any given value of x, the function value of $(f \times g)(x)$ is represented by $f(x) \times g(x)$.

This makes sense since the new function is created by multiplying the original functions together.

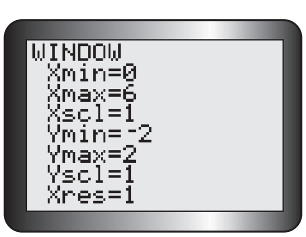


Plotting the ordered pairs $(x, (f \times g)(x))$ results in the graph of the dampened sine wave. This means that the graph of $f \times g$ can be obtained from the graphs of functions f and g by multiplying corresponding y-coordinates.





Use a graphing calculator to verify the results. Enter the functions into the equation editor as shown. Turn off the first two functions, and choose a bold line to graph the third function.



Use window settings that match the given graph of $(f \times g)(x)$.

The graph of Y4 traces over the graph of the product function Y3. This confirms that the product function is identical to, and obtained by, multiplying the expressions of the two functions together.

The graph of Y3 shows the graph produced by multiplying the corresponding *y*-values of the functions stored in Y1 and Y2.

Reflecting

A. If $(0.4, 0.76) \in f(x)$ and $(0.4, 0.59) \in g(x)$, what ordered pair belongs to $(f \times g)(x)$?

B. If $f(1) = 0.5$ and $(f \times g)(1) = 0$, what do you know about the value of $g(1)$? Explain.

C. Look at the original graphs of $f(x)$ and $g(x)$. How can you predict the locations of the zeros of $(f \times g)(x)$ before you construct a table of values or a graph? Explain.

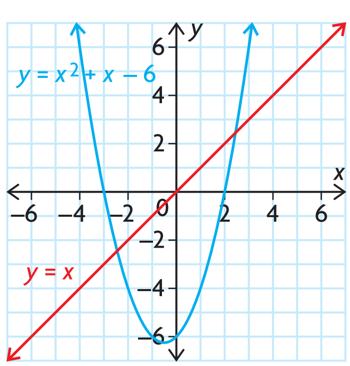
D. What is the domain of $f \times g$? How does it compare with the domains of f and g?

E. If function $f(x)$ was replaced by $f(x) = \sqrt{x}$, explain how this would change the domain of $(f \times g)(x)$.

APPLY the Math

EXAMPLE 2 Constructing the product of two functions graphically

Determine the graph of $y = (f \times g)(x)$, given the graphs of $f(x) = x^2 + x - 6$ and $g(x) = x$.



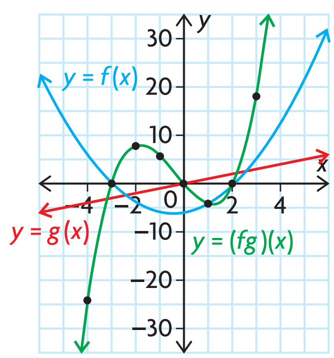
Solution

x	$f(x)$	$g(x)$	$(f \times g)(x)$
−4	6	−4	−24
−3	0	−3	0
−2	−4	−2	8
−1	−6	−1	6
0	−6	0	0
1	−4	1	−4
2	0	2	0
3	6	3	18
4	14	4	56

Use the graph to determine some of the points on the graphs of f and g, and create a table of values.

The graphs indicate that both functions have the same domain, $\{x \in \mathbf{R}\}$.

Determine the values of $(f \times g)(x)$ by multiplying the y-coordinates of f and g together for the same value of x.



The domain of the product function is the intersection of the domains of f and g, $\{x \in \mathbf{R}\}$.

Plot some of the ordered pairs $(x, (f \times g)(x))$, and use these to sketch the graph of the product function.

Notice that the zeros of the two functions, f and g, result in points that are also zeros of $f \times g$. This makes sense since the product of zero and any number is still zero.

Also notice that $(f \times g)(1) = f(1)$ because $g(1) = 1$. As a result, $(f \times g)(1) = f(1) \times 1 = -4 \times 1 = -4$. Similarly, $(f \times g)(-1) = -f(-1)$ because $g(-1) = -1$, so $(f \times g)(-1) = f(-1) \times (-1) = -6 \times -1 = 6$.

Functions f and g are second and first degree polynomial functions, so the product function fg is a third degree polynomial function (also called a cubic function).

EXAMPLE 3 Constructing the product of two functions algebraically

Let $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{2}x - 2$.

a) Find the equation of the function $(f \times g)(x)$.
b) Determine $(f \times g)(4)$.
c) Find the domain of $y = (f \times g)(x)$.
d) Use graphing technology to graph $y = (f \times g)(x)$, and discuss the key characteristics of the graph.

Solution

a) $(f \times g)(x) = f(x) \times g(x)$
$= \sqrt{x}\left(\frac{1}{2}x - 2\right)$

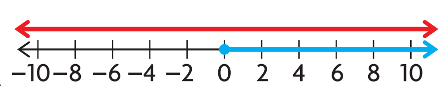
To find the formula for the product of the functions, take the expression for $f(x)$ and multiply it by the expression for $g(x)$.

b) $(f \times g)(4) = \sqrt{4}\left(\frac{1}{2}(4) - 2\right)$
$= 2(0)$
$= 0$

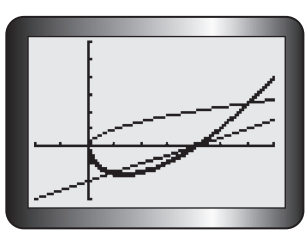
Calculate the value of $(f \times g)(4)$ by substituting $x = 4$ into the expression $(f \times g)(x)$.

c) The domain of g is $\{x \in \mathbf{R}\}$, but the domain of f is $\{x \in \mathbf{R} \mid x \geq 0\}$. So, the domain of $f \times g$ is $\{x \in \mathbf{R} \mid x \geq 0\}$.

The domain of $f \times g$ can only consist of x-values that exist in the domains of both f and g.



d)



The graph of $f \times g$ is the bold line.

The graph of $f \times g$
- lies below the x-axis when $x \in (0, 4)$, since $f(x) > 0$ and $g(x) < 0$ in that interval
- has zeros occurring at $x = 0$ when $f(x) = 0$ and at $x = 4$ when $g(x) = 0$; no other zeros will occur, since both functions are positive
- is neither odd nor even since it has no symmetry about the origin or the y-axis

EXAMPLE 4 Modelling a situation using a product function

The rate at which a contaminant leaves a storm sewer and enters a lake depends on two factors: the concentration of the contaminant in the water from the sewer and the rate at which the water leaves the sewer. Both of these factors vary with time. The concentration of the contaminant, in kilograms per cubic metre of water, is given by $c(t) = t^2$, where t is in seconds. The rate at which water leaves the sewer, in cubic metres per second, is given by $w(t) = \frac{1}{t^4 + 20}$. Determine the time at which the contaminant leaves the sewer and enters the lake at the maximum rate.

Solution

$c(t)$ is in $\frac{\text{kg}}{\text{m}^3}$ and $w(t)$ is in $\frac{\text{m}^3}{\text{s}}$

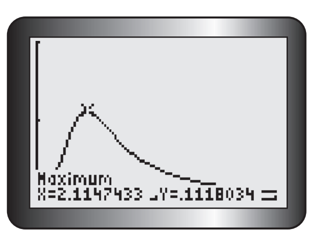
$$c(t) \times w(t) \rightarrow \left(\frac{\text{kg}}{\cancel{\text{m}^3}}\right)\left(\frac{\cancel{\text{m}^3}}{\text{s}}\right) = \frac{\text{kg}}{\text{s}}$$

The product of the concentration function and the water rate function results in a function that describes the rate of contaminant flow into the lake.

Analyze the units of both functions to help you determine the relationship between the functions that can be used to determine a function for the rate at which the contaminant flows into the lake.

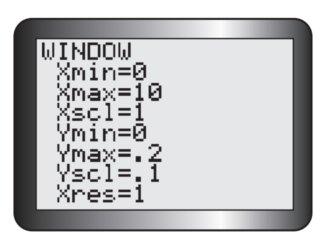
$$c(t) \times w(t) = (t^2)\left(\frac{1}{t^4 + 20}\right)$$
$$= \frac{t^2}{t^4 + 20}$$

In this context, the domain of both functions is $\{t \in \mathbf{R} | t \geq 0\}$ since both functions have time as the independent variable. Thus, $\{t \in \mathbf{R} | t \geq 0\}$ is also the domain of $c(t) \times w(t)$.



The contaminant is flowing into the lake at a maximum rate of about 0.11 kg/s. This occurs at about 2 s after the water begins to flow into the lake.

Use the maximum operation on a graphing calculator to graph $c(t) \times w(t)$ on its domain and estimate when its maximum value occurs.



Tech *Support*

For help determining the maximum value of a function using a graphing calculator, see Technical Appendix, T-9.

In Summary

Key Idea

- When two functions, $f(x)$ and $g(x)$, are combined to form the function $(f \times g)(x)$, the new function is called the product of f and g. For any given value of x, the function value is represented by $f(x) \times g(x)$. The graph of $f \times g$ can be obtained from the graphs of functions f and g by multiplying each y-coordinate of f by the corresponding y-coordinate of g.

Need to Know

- Algebraically, $f \times g$ is defined as $(f \times g)(x) = f(x) \cdot g(x)$.
- The domain of $f \times g$ is the intersection of the domains of f and g.
- If $f(x) = 0$ or $g(x) = 0$, then $(f \times g)(x) = 0$.
- If $f(x) = \pm 1$, then $(f \times g)(x) = \pm g(x)$. Similarly, if $g(x) = \pm 1$, then $(f \times g)(x) = \pm f(x)$.

CHECK Your Understanding

1. For each of the following pairs of functions, determine $(f \times g)(x)$.

a) $f(x) = \{(0, 2), (1, 5), (2, 7), (3, 12)\}$,
$g(x) = \{(0, -1), (1, -2), (2, 3), (3, 5)\}$

b) $f(x) = \{(0, 3), (1, 6), (2, 10), (3, -5)\}$,
$g(x) = \{(0, 4), (2, -2), (4, 1), (6, 3)\}$

c) $f(x) = x, g(x) = 4$

d) $f(x) = x, g(x) = 2x$

e) $f(x) = x + 2, g(x) = x^2 - 2x + 1$

f) $f(x) = 2^x, g(x) = \sqrt{x - 2}$

2. a) Graph each pair of functions in question 1, parts c) to f), on the same grid.

b) State the domains of f and g.

c) Use your graph to make an accurate sketch of $y = (f \times g)(x)$.

d) State the domain of $f \times g$.

3. If $f(x) = \sqrt{1 + x}$ and $g(x) = \sqrt{1 - x}$, determine the domain of $y = (f \times g)(x)$.

PRACTISING

4. Determine $(f \times g)(x)$ for each of the following pairs of functions.

K **a)** $f(x) = x - 7, g(x) = x + 7$

b) $f(x) = \sqrt{x + 10}, g(x) = \sqrt{x + 10}$

c) $f(x) = 7x^2, g(x) = x - 9$

d) $f(x) = -4x - 7, g(x) = 4x + 7$

e) $f(x) = 2 \sin x, g(x) = \dfrac{1}{x - 1}$

f) $f(x) = \log(x + 4), g(x) = 2^x$

5. For each of the problems in question 4, state the domain and range of $(f \times g)(x)$.

6. For each of the problems in question 4, use graphing technology to graph $(f \times g)(x)$ and then discuss each of the following characteristics of the graphs: symmetry, intervals of increase/decrease, zeros, maximum and minimum values, and period (where applicable).

7. The graph of the function $f(x)$ is a line passing through the origin with a slope of -4, whereas the graph of the function $g(x)$ is a line with a y-intercept of 1 and a slope of 6. Sketch the graph of $(f \times g)(x)$.

8. For each of the following pairs of functions, state the domain of $(f \times g)(x)$.
   a) $f(x) = \dfrac{1}{x^2 - 5x - 14}, g(x) = \sec x$
   b) $f(x) = 99^x, g(x) = \log(x - 8)$
   c) $f(x) = \sqrt{x + 81}, g(x) = \csc x$
   d) $f(x) = \log(x^2 + 6x + 9), g(x) = \sqrt{x^2 - 1}$

9. If the function $f(t)$ describes the per capita energy consumption in a particular country at time t, and the function $p(t)$ describes the population of the country at time t, then explain what the product function $(f \times p)(t)$ represents.

10. A An average of 20 000 people visit the Lakeside Amusement Park each day in the summer. The admission fee is \$25.00. Consultants predict that, for each \$1.00 increase in the admission fee, the park will lose an average of 750 customers each day.
    a) Determine the function that represents the projected daily revenue if the admission fee is increased.
    b) Is the revenue function a product function? Explain.
    c) Estimate the ticket price that will maximize revenue.

11. A water purification company has patented a unique process to remove contaminants from a container of water at the same time that more contaminated water is added for purification. The percent of contaminated material in the container of water being purified can be modelled by the function $c(t) = (0.9)^t$, where t is the time in seconds. The number of litres of water in the container can be modelled by the function $l(t) = 650 + 300t$. Write a function that represents the number of litres of contaminated material in the container at any time t, and estimate when the amount of contaminated material is at its greatest.

12. Is the following statement true or false? "If $f(x) \times g(x)$ is an odd function, then both $f(x)$ and $g(x)$ are odd functions." Justify your answer.

13. Let $f(x) = mx^2 + 2x + 5$ and $g(x) = 2x^2 - nx - 2$. The functions are combined to form the new function $h(x) = f(x) \times g(x)$. Points $(1, -40)$ and $(-1, 24)$ satisfy the new function. Determine $f(x)$ and $g(x)$.

14. Let $f(x) = \sqrt{-x}$ and $g(x) = \log(x + 10)$.

a) Determine the equation of the function $y = (f \times g)(x)$, and state its domain.

b) Provide two different strategies for sketching $y = (f \times g)(x)$. Discuss the merits of each strategy.

c) Choose one of the strategies you discussed in part b), and make an accurate sketch.

15. a) If $f(x) = x^2 - 25$, determine the equation of the product function $f(x) \times \frac{1}{f(x)}$.

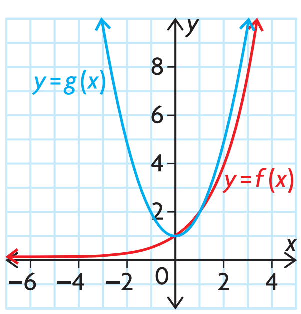
b) Determine the domain, and sketch the graph of the product function you found in part a).

c) If $f(x)$ is a polynomial function, explain how the domain and range of $f(x) \times \frac{1}{f(x)}$ changes as the degree of $f(x)$ changes.

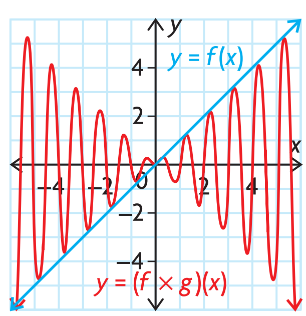
Extending

16. Given the following graphs, determine the equations of $y = f(x)$, $y = g(x)$, and $y = (f \times g)(x)$.

a)



b)



17. Determine two functions, f and g, whose product would result in each of the following functions.

a) $(f \times g)(x) = 4x^2 - 81$

b) $(f \times g)(x) = 8\sin^3 x + 27$

c) $(f \times g)(x) = 4x^{\frac{5}{2}} - 3x^{\frac{3}{2}} + x^{\frac{1}{2}}$

d) $(f \times g)(x) = \dfrac{6x - 5}{2x + 1}$

9.4 Exploring Quotients of Functions

YOU WILL NEED

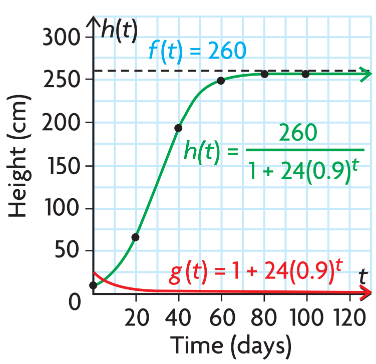
- graph paper
- graphing calculator or graphing software

GOAL

Represent the quotient of two functions graphically and algebraically, and determine the characteristics of the quotient.

EXPLORE the Math

The logistic function is often used to model growth. This function has the general equation $P(t) = \frac{c}{1 + ab^t}$, where $a > 0$, $0 < b < 1$, and $c > 0$. In this function, t is time. For example, the height of a sunflower plant can be modelled using the function $h(t) = \frac{260}{1 + 24(0.9)^t}$, where $h(t)$ is the height in centimetres and t is the time in days. The function $h(t) = \frac{f(t)}{g(t)}$ is the quotient of two functions, where $f(t) = 260$ (a constant function) and $g(t) = 1 + 24(0.9)^t$ (an exponential function). The table and graphs show that the values of a quotient function can be determined by dividing the values of the two functions.



t (days)	$f(t) = 260$	$g(t) = 1 + 24(0.9)^t$	$h(t) = \frac{260}{1 + 24(0.9)^t}$
0	260	25	$\frac{260}{25} = 10.4$
20	260	3.92	66.3
40	260	1.35	192.6
60	260	1.04	250.0
80	260	1.01	257.4
100	260	1.00	260.0

This function shows slow growth for small values of t, then rapid growth, and then slow growth again when the height of the sunflower approaches its maximum height of 260 cm.

The logistic function is an example of a quotient function. In function notation, we can express this as $(f \div g)(x) = f(x) \div g(x)$.

? What are the characteristics of functions that are produced by quotients of other types of functions?

A. Consider the function defined by $y = \frac{4}{x + 2}$ in the form $y = \frac{f(x)}{g(x)}$. Write the expressions for functions f and g.

B. On graph paper, draw and label the graphs of $y = f(x)$ and $y = g(x)$, and state their domains.

C. Locate any points on your graph of g where $g(x) = 0$. What will happen when you calculate the value of $f \div g$ for these x-coordinates? How would this appear on a graph?

D. Locate any points on your graph where $g(x) = \pm 1$. What values of x produced these results? Explain how you could determine these x-values algebraically.

E. Determine the value of $f \div g$ for each of the x's in part D. How do your answers compare with the corresponding values of f? Explain.

F. Over what interval(s) is $g(x) > 0$? Over what interval(s) is $f(x) > 0$?

G. Determine all the intervals where both f and g are positive or where both are negative. Will the function $f \div g$ be positive in the same intervals? Justify your answer.

H. Determine any intervals where either f or g is positive and the other is negative. Discuss the behaviour of $f \div g$ over these intervals. If no such intervals exist, what implication would this have for $f \div g$? Explain.

I. For what values of x is $(f \div g)(x) = f(x)$? For what values of x is $(f \div g)(x) = -f(x)$?

J. Using all the information about $f \div g$ that you have determined, make an accurate sketch of $y = (f \div g)(x)$ and state its domain.

K. Verify your results by graphing f, g, and $f \div g$ using graphing technology.

L. Repeat parts A to K using the following functions.

i) $y = \dfrac{x + 1}{(x + 3)(x - 1)}$

ii) $y = \dfrac{4}{x^2 + 1}$

iii) $y = \dfrac{\sin x}{x}$

iv) $y = \dfrac{2^x}{\sqrt{x}}$

Reflecting

M. The graphs of $y = \frac{4}{x + 2}$, $y = \frac{x + 1}{(x + 3)(x - 1)}$, and $y = \frac{2x}{\sqrt{x}}$ have vertical asymptotes, but the graphs of $h(t) = \frac{260}{1 + 24(0.9)^t}$, $y = \frac{4}{x^2 + 1}$, and $y = \frac{\sin x}{x}$ do not. Explain.

N. The graph of $y = \frac{x + 1}{(x + 3)(x - 1)}$ lies above the x-axis in the interval $x \in (-3, -1)$. By examining the behaviour of functions f and g, explain how you can reach this conclusion.

In Summary

Key Idea

- When two functions, $f(x)$ and $g(x)$, are combined to form the function $(f \div g)(x)$, the new function is called the quotient of f and g. For any given value of x, the value of the function is represented by $f(x) \div g(x)$. The graph of $f \div g$ can be obtained from the graphs of functions f and g by dividing each y-coordinate of f by the corresponding y-coordinate of g.

Need to Know

- Algebraically, $(f \div g)(x) = f(x) \div g(x)$.
- $f \div g$ will be defined for all x-values that are in the intersection of the domains of f and g, except in the case where $g(x) = 0$. If the domain of f is A, and the domain of g is B, then the domain of $f \div g$ is $\{x \in \mathbf{R} \mid x \in A \cap B, g(x) \neq 0\}$.
- If $f(x) = 0$ when $g(x) \neq 0$, then $(f \div g)(x) = 0$.
- If $f(x) = \pm 1$, then $(f \div g)(x) = \pm \frac{1}{g(x)}$. Similarly, if $g(x) = \pm 1$, then $(f \div g)(x) = \pm f(x)$. Also, if $f(x) = \pm g(x)$, then $(f \div g)(x) = \pm 1$

Further Your Understanding

1. For each of the following pairs of functions, write the equation of $y = (f \div g)(x)$.

a) $f(x) = 5, g(x) = x$

b) $f(x) = 4x, g(x) = 2x - 1$

c) $f(x) = 4x, g(x) = x^2 + 4$

d) $f(x) = x + 2, g(x) = \sqrt{x - 2}$

e) $f(x) = 8, g(x) = 1 + \left(\frac{1}{2}\right)^x$

f) $f(x) = x^2, g(x) = \log(x)$

2. **a)** Graph each pair of functions in question 1 on the same grid.
b) State the domains of f and g.
c) Use your graphs to make an accurate sketch of $y = (f \div g)(x)$.
d) State the domain of $f \div g$.

3. Recall that the function $h(t) = \frac{260}{1 + 24(0.9)^t}$ models the growth of a sunflower, where $h(t)$ is the height in centimetres and t is the time in days.

a) Calculate the average rate of growth of the sunflower over the first 20 days.

b) Determine when the sunflower has grown to half of its maximum height.

c) Estimate the instantaneous rate of change in height at the time you found in part b).

d) What happens to the instantaneous rate of change in height as the sunflower approaches its maximum height? How does this relate to the shape of the graph?

9 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: If you are given the graphs of two functions, f and g, how can you determine the location of a point that would appear on the graphs of $f + g$, $f - g$, $f \times g$, and $f \div g$?

A: For any particular x-value, determine the y-value on each graph, separately. For $f + g$, add these two y-values together. For $f - g$, subtract the y-value of g from the y-value of f. For $f \times g$, multiply these two y-values together. For $f \div g$, divide the y-value of f by the y-value of g. Each of these points has, as its coordinates, the same x-value and the new y-value.

Study *Aid*

- See Lessons 9.1 to 9.4.
- Try Mid-Chapter Review Question 2.

Q: If you are given the equations of two functions, f and g, how can you determine the equations of the functions $f + g$, $f - g$, $f \times g$, and $f \div g$?

A: Every time you combine two functions in one of these ways, you are simply performing a different arithmetic operation on every pair of y-values, one from each of the functions being combined, provided that the x-values are the same. Since the equation of each function defines the y-values of each function, the new equation can be determined by adding, subtracting, multiplying, or dividing the y-value expressions as required.

For example, if $f(x) = x^2 + 8$ and $g(x) = 5^x$, then

$$\begin{aligned}(f + g)(x) &= f(x) + g(x)\\ &= x^2 + 8 + 5^x\end{aligned} \qquad \begin{aligned}(f \times g)(x) &= f(x) \times g(x)\\ &= (x^2 + 8)(5^x)\end{aligned}$$

$$\begin{aligned}(f - g)(x) &= f(x) - g(x)\\ &= x^2 + 8 - 5^x\end{aligned} \qquad \begin{aligned}(f \div g)(x) &= f(x) \div g(x)\\ &= \frac{x^2 + 8}{5^x}\end{aligned}$$

Study *Aid*

- See Lessons 9.1 to 9.4.
- Try Mid-Chapter Review Questions 5 and 7.

Q: How can you determine the domain of the combined functions $f + g$, $f - g$, $f \times g$, and $f \div g$?

A: Since you can only combine points from two functions when they share the same x-value, the domain of the combined function must consist of the set of x-values where the domains of the two given functions intersect. The only exception occurs when you are dividing two functions. The function $f \div g$ is not defined when its denominator is equal to zero, since division by zero is undefined. As a result, x-values that cause $g(x)$ to equal zero must be excluded from the domain.

Study *Aid*

- See Lessons 9.1 to 9.4.
- Try Mid-Chapter Review Questions 5 and 7.

PRACTICE Questions

Lesson 9.1

1. Given the functions $f(x) = \cos x$ and $g(x) = \sin x$, which operations can be used to combine the two functions to create a new function with an amplitude that is less than 1?

Lesson 9.2

2. Let $f(x) = \{(-9, -2), (-6, -3), (-3, 0), (0, 2), (3, 7)\}$ and $g(x) = \{(-12, 9), (-9, 4), (-8, 1), (-7, 10), (-6, -6), (0, 12)\}$. Determine
   a) $(f + g)(x)$
   b) $(g + f)(x)$
   c) $(f - g)(x)$
   d) $(g - f)(x)$

3. The cost, in thousands of dollars, for a company to produce x thousand of its product is given by the function $C(x) = 10x + 30$. The revenue from the sales of the product is given by the function $R(x) = -5x^2 + 150x$.
   a) Write the function that represents the company's profit on sales of x thousand of its product.
   b) Graph the cost, revenue, and profit functions on the same coordinate grid, where $0 \leq x \leq 40$.
   c) What is the company's profit on the sale of 7500 of its product?

4. Steve earns \$24.39/h operating an industrial plasma torch at a rail-car manufacturing plant. He receives \$0.58/h more for working the night shift, as well as \$0.39/h more for working weekends.
   a) Write a function that describes Steve's daily earnings under regular pay.
   b) What function shows his daily earnings under the night-shift premium?
   c) What function shows his daily earnings under the weekend premium?
   d) What function represents his earnings for the night shift on Saturday?
   e) How much does Steve earn for working 11 h on Saturday night, if he earns time and a half on that day's rate for more than 8 h of work?

Lesson 9.3

5. Determine $(f \times g)(x)$ for each of the following pairs of functions, and state its domain.
   a) $f(x) = x + \frac{1}{2}, g(x) = x + \frac{1}{2}$
   b) $f(x) = \sqrt{x - 10}, g(x) = \sin(3x)$
   c) $f(x) = 11x^3, g(x) = \frac{2}{x + 5}$
   d) $f(x) = 90x - 1, g(x) = 90x + 1$

6. A diner is open from 6 a.m. to 6 p.m., and the average number of customers in the diner at any time can be modelled by the function $C(h) = -30 \cos\left(\frac{\pi}{6}h\right) + 34$, where h is the number of hours after the 6 a.m. opening time. The average amount of money, in dollars, that each customer in the diner will spend can be modelled by the function $D(h) = -3 \sin\left(\frac{\pi}{6}h\right) + 7$.
   a) Write the function that represents the diner's average revenue from the customers.
   b) Graph the function you wrote in part a).
   c) What is the average revenue from the customers in the diner at 2 p.m.?

Lesson 9.4

7. Calculate $(f \div g)(x)$ for each of the following pairs of functions, and state its domain.
   a) $f(x) = 240, g(x) = 3x$
   b) $f(x) = 10x^2, g(x) = x^3 - 3x$
   c) $f(x) = x + 8, g(x) = \sqrt{x - 8}$
   d) $f(x) = 14x^2, g(x) = 2 \log x$

8. Recall that $y = \tan x$ can be written as the quotient of two functions: $f(x) = \sin x$ and $g(x) = \cos x$. List as many other trigonometric functions as possible that could be written as the quotient of two functions.

9.5 Composition of Functions

GOAL

Determine the composition of two functions numerically, graphically, and algebraically.

LEARN ABOUT the Math

Sometimes you will find a situation in which two related functions are present. Often both functions are needed to analyze the situation or solve a problem.

Forest fires often spread in a roughly circular pattern. The area burned depends on the radius of the fire. The radius, in turn, may increase at a constant rate each day.

Suppose that $A(r) = \pi r^2$ represents the area, A, of a fire as a function of its radius, r. If the radius of the fire increases by 0.5 km/day, then $r(t) = 0.5t$ represents the radius of the fire as a function of time, t. The area is measured in square kilometres, the radius is measured in kilometres, and the time is measured in days.

? How can the area burned be determined on the sixth day of the fire?

EXAMPLE 1 Reasoning numerically, graphically, and algebraically about a composition of functions

Determine the area burned by the fire on the sixth day.

Solution A: Using graphical and numerical analysis

Use the given functions to make tables of values.

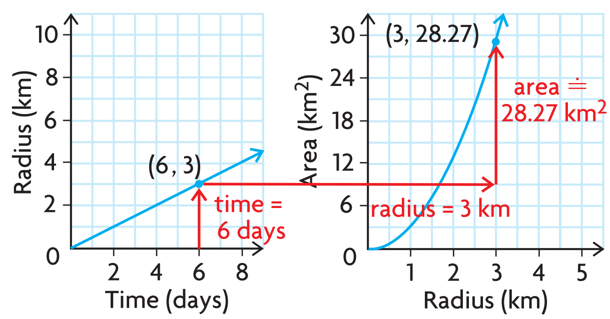
t	$r(t) = 0.5t$
0	0
2	1
4	2
6	3
8	4

r	$A(r) = \pi r^2$
0	0
1	3.14
2	12.57
3	28.27
4	50.27

Both time and radius must be positive, so $t \geq 0$ and $r \geq 0$.

$r(t)$ is a linear function, and $A(r)$ is a quadratic function.

Use the tables of values to sketch the graphs.



To find the radius of the area burned by the forest fire, the length of time that the fire has been burning must be known. Once the radius is known, the total area burned can be determined.

Reading from the first graph, the radius is 3 km when $t = 6$ days. Then reading from the second graph, a radius of 3 km indicates an area of about 28.3 km^2.

In the tables of values, time corresponds with radius, and radius corresponds with area.

$\boldsymbol{r}$: time $\rightarrow$ radius
$\boldsymbol{A}$: radius $\rightarrow$ area

The output in the first table becomes the input in the second table.

$$6 \xrightarrow[r(6)]{} 3 \xrightarrow{} 28.3$$

$$\begin{aligned} &A(r(6)) \\ &= A(3) \\ &\doteq 28.3 \end{aligned}$$

Determine the radius after six days, $r(6)$, and use it as the input for the area function, $A(r(6))$, to determine the area burned after six days.

$r(6) = 0.5(6) = 3$ and $A(3) = \pi(3)^2 \doteq 28.3$

The fire has burned about 28.3 km^2 on the sixth day.

Solution B: Using algebraic analysis

$$r = g(t) = 0.5t$$
$$A = f(r) = \pi r^2$$

The radius of the fire, r, grows at 0.5 km per day, so it is a function of time.

The area, A, of the fire increases in a circular pattern as its radius, r, increases, so it is a function of the circle's radius.

Since $r = g(t)$
$A = f(r) = f(g(t))$

To solve the problem, combine the area function with the radius function by using the output for the radius function as the input for the area function.

$$A = f(0.5t)$$
$$= \pi(0.5t)^2$$
$$= 0.25\pi t^2$$

This process yields the **composite function** for the area of the fire as a function of time.

$$A(6) = 0.25\pi(6)^2$$
$$= 0.25\pi(36)$$
$$= 9\pi$$
$$\doteq 28.3$$

To determine the size of the burn area after six days, substitute $t = 6$ into this new function.

The fire has burned an area of about 28.3 km^2 after six days.

composite function
a function that is the composite of two other functions; the function $f(g(t))$ is called the composition of f with g; the function $f(g(t))$ is denoted by $(f \circ g)(t)$ and is defined by using the output of the function g as the input for the function f

Reflecting

A. A point on the second graph was used to solve the problem. Explain how the x-coordinate of this point was determined.

B. What connection was observed between the tables of values for the two functions? Why does it make sense that there is a function that combines the two functions to solve the forest fire problem?

C. Explain how the two functions were combined algebraically to determine a single function that predicts the area burned for a given time. How is the range of r related to the domain of A in this combination?

Communication *Tip*

$f \circ g$ is read as "f operates on g" while $f(g(x))$ is read as "f of g of x."

APPLY the Math

EXAMPLE 2 Reasoning about the order in which two functions are composed

Given the functions $f(x) = 2x + 3$ and $g(x) = \sqrt{x}$, determine whether $(f \circ g)(x) = (g \circ f)(x)$.

Solution

$$(f \circ g)(x) = f(\underbrace{g(x)})$$

outer function   inner function

When f is composed with g, take the output for the inner function g and use it as the input for the outer function f.

$$x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x))$$

$$g(x) = \sqrt{x}$$

$$\begin{aligned} f(g(x)) &= f(\sqrt{x}) \\ &= 2(\sqrt{x}) + 3 \\ &= 2\sqrt{x} + 3 \end{aligned}$$

The output for g is the expression $\sqrt{x}$. Use this as the input for f, replacing x everywhere it occurs with $\sqrt{x}$.

$f(g(x)) = 2\sqrt{x} + 3$

Algebraically, the composition of f with g is the function $y = 2\sqrt{x} + 3$.

In terms of transformations, $f \circ g$ represents the function $y = g(x)$ stretched vertically by a factor of 2 and translated 3 units up. Its domain is $\{x \in \mathbf{R} \mid x \geq 0\}$.

$$\underset{\text{outer function}}{(g \circ f)(x) = g}(\underbrace{f(x)}_{\text{inner function}})$$

When g is composed with f, take the output from the inner function f and use it as the input for the outer function g.

$$x \xrightarrow{f} f(x) \xrightarrow{g} g(f(x))$$

$$f(x) = 2x + 3$$

$$\begin{aligned} g(f(x)) &= g(2x + 3) \\ &= \sqrt{(2x + 3)} \end{aligned}$$

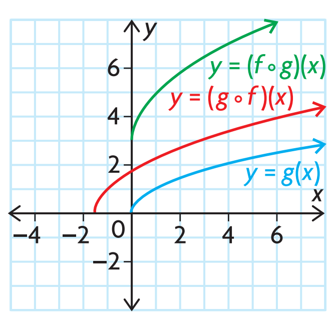
The output from f is the expression $2x + 3$. Use this as the input for g, and replace x everywhere it occurs with $2x + 3$.

$g(f(x)) = \sqrt{2x + 3}$

Algebraically, the composition of g with f is the function

$g(f(x)) = \sqrt{2x + 3} = \sqrt{2(x + 1.5)}$

In terms of transformations, $y = g(x)$ is compressed horizontally by a factor of $\frac{1}{2}$ and translated 1.5 units to the left. Its domain is $\left\{x \in \mathbf{R} \mid x \geq -\frac{3}{2}\right\}$.



Clearly, the expressions for $y = (f \circ g)(x)$ and $y = (g \circ f)(x)$ are different. Comparing their graphs illustrates the result of applying different sequences of transformations to $y = g(x)$.

$(f \circ g)(x) \neq (g \circ f)(x)$. The compositions of these two functions generate different answers depending on the order of the composition.

EXAMPLE 3 Reasoning about the domain of a composite function

Let $f(x) = \log_2 x$ and $g(x) = x + 4$.

a) Determine $f \circ g$, and find its domain.

b) What is the relationship between the domain of $f \circ g$ and the domain and range of f and g?

Solution

a) $$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x + 4) \\ &= \log_2(x + 4)\end{aligned}$$

Use the output for g as the input for f.

Since $x + 4 > 0 \Rightarrow x > -4$.

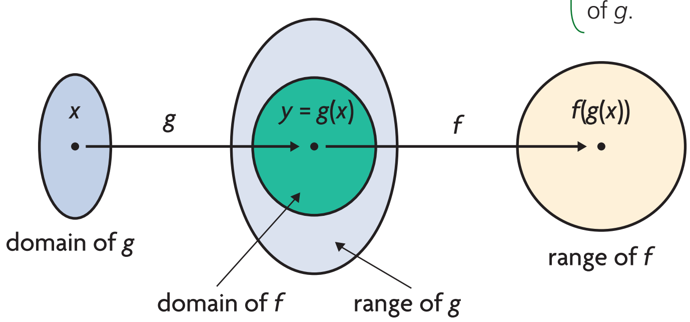
The domain of $f \circ g$ is $x \in (-4, \infty)$.

The domain of a logarithmic function with base a contains only positive real numbers, so the expression $x + 4$ must be greater than zero.

b) Domain of f: $x \in (0, \infty)$ Range of f: $y \in \mathbf{R}$

Domain of g: $x \in \mathbf{R}$ Range of g: $y \in \mathbf{R}$

Looking at the domain of $f \circ g$, we can see that it is not equal to either the domain of f or the domain of g.



Recall that the output values (range of g) for $y = g(x)$, are used as the input values (domain) for f.

In this example, the domain of f is $x > 0$ and the domain of g is $x \in \mathbf{R}$, so the only y-values of g that can be used occur when $g(x) > 0$.

Since $g(x) = x + 4$, $x + 4 > 0$

$$x > -4$$

The domain of $f \circ g$ is the set of values, x, in the domain of g for which $g(x)$ is in the domain of f.

EXAMPLE 4 Reasoning about a function composed with its inverse

Show that, if $f(x) = \frac{1}{x-2}$ then $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x)$.

Solution

$$
\begin{aligned}
x &= \frac{1}{y-2} \\
x(y-2) &= 1 \\
y-2 &= \frac{1}{x} \\
y &= \frac{1}{x} + 2 \text{ or } f^{-1}(x) = \frac{1}{x} + 2
\end{aligned}
$$

To find the inverse of *f*, switch *x* and *y* and then solve for *y*.

$$
\begin{aligned}
(f \circ f^{-1})(x) &= f(f^{-1}(x)) \\
&= f\left(\frac{1}{x} + 2\right) \\
&= \frac{1}{\left(\frac{1}{x} + 2\right) - 2} \\
&= \frac{1}{\left(\frac{1}{x}\right)} \\
&= x
\end{aligned}
$$

So, $(f \circ f^{-1})(x) = x$

The composition of *f* with its inverse maps a number in the domain of *f* onto itself. In other words, the result of this composition is the line $y = x$.

$$
\begin{aligned}
(f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\
&= f^{-1}\left(\frac{1}{x-2}\right) \\
&= \frac{1}{\left(\frac{1}{x-2}\right)} + 2 \\
&= x - 2 + 2 \\
&= x
\end{aligned}
$$

The composition of f^{-1} with *f* maps a number in the domain of f^{-1} onto itself. In other words, the result of this composition is also the line $y = x$.

So, $(f^{-1} \circ f)(x) = x$

Therefore, $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x)$

Even though the order of the functions in the composition is reversed, the results are the same.

EXAMPLE 5 Working backward to decompose a composite function

Given $h(x) = |x^3 - 1|$, find two functions, f and g, such that $h = f \circ g$.

Solution

To evaluate h for any value of x, take that value, cube it, and subtract 1. This defines a sequence of operations for the inner function. Then, take the absolute value. This defines the outer function.

Let $g(x) = x^3 - 1$ and $f(x) = |x|$.

When evaluating the composition of f with g, you start by evaluating g for some value of x. So, it makes sense to define the inner function g that h performs on any input value. Then define the outer function f to represent the remaining operation(s) required by h.

$$\begin{aligned} \text{Then } (f \circ g)(x) &= f(g(x)) \\ &= f(x^3 - 1) \\ &= |x^3 - 1| \\ &= h(x) \\ h(x) &= (f \circ g)(x) \end{aligned}$$

Another solution would be to let $g(x) = x^3$ and $f(x) = |x - 1|$.

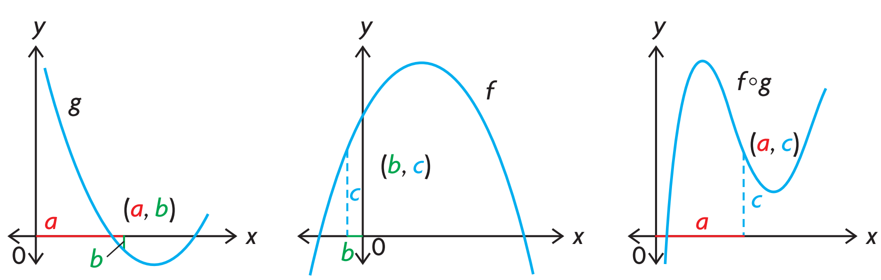
In Summary

Key Idea

- Two functions, f and g, can be combined using a process called composition, which can be represented by $f(g(x))$. The output for the inner function g is used as the input for the outer function f. The function $f(g(x))$ can be denoted by $(f \circ g)(x)$.

Need to Know

- Algebraically, the composition of f with g is denoted by $(f \circ g)(x)$, whereas the composition of g with f is denoted by $(g \circ f)(x)$. In most cases, $(f \circ g)(x) \neq (g \circ f)(x)$ because the order in which the functions are composed matters.
- Let $(a, b) \in g$ and $(b, c) \in f$. Then $(a, c) \in f \circ g$. A point in $f \circ g$ exists where an element in the range of g is also in the domain of f. The function $f \circ g$ exists only when the range of g overlaps the domain of f.



- The domain of $(f \circ g)(x)$ is a subset of the domain of g. It is the set of values, x, in the domain of g for which $g(x)$ is in the domain of f.
- If both f and f^{-1} are functions, then $(f^{-1} \circ f)(x) = x$ for all x in the domain of f, and $(f \circ f^{-1})(x) = x$ for all x in the domain of f^{-1}.

CHECK Your Understanding

1. Use $f(x) = 2x - 3$ and $g(x) = 1 - x^2$ to evaluate the following expressions.

a) $f(g(0))$

b) $g(f(4))$

c) $(f \circ g)(-8)$

d) $(g \circ g)\left(\frac{1}{2}\right)$

e) $(f \circ f^{-1})(1)$

f) $(g \circ g)(2)$

2. Given $f = \{(0, 1), (1, 2), (2, 5), (3, 10)\}$ and $g = \{(2, 0), (3, 1), (4, 2), (5, 3), (6, 4)\}$, determine the following values.

a) $(g \circ f)(2)$

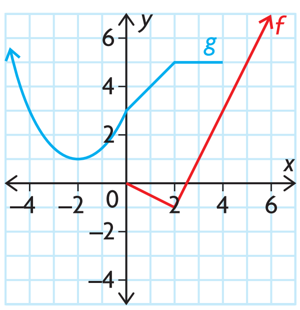
b) $(f \circ f)(1)$

c) $(f \circ g)(5)$

d) $(f \circ g)(0)$

e) $(f \circ f^{-1})(2)$

f) $(g^{-1} \circ f)(1)$



3. Use the graphs of f and g to evaluate each expression.

a) $f(g(2))$

b) $g(f(4))$

c) $(g \circ g)(-2)$

d) $(f \circ f)(2)$

4. For a car travelling at a constant speed of 80 km/h, the distance driven, d kilometres, is represented by $d(t) = 80t$, where t is the time in hours. The cost of gasoline, in dollars, for the drive is represented by $C(d) = 0.09d$.

a) Determine $C(d(5))$ numerically, and interpret your result.

b) Describe the relationship represented by $C(d(t))$.

PRACTISING

5. **K** In each case, functions f and g are defined for $x \in \mathbf{R}$. For each pair of functions, determine the expression and the domain of $f(g(x))$ and $g(f(x))$. Graph each result.

a) $f(x) = 3x^2, g(x) = x - 1$

b) $f(x) = 2x^2 + x, g(x) = x^2 + 1$

c) $f(x) = 2x^3 - 3x^2 + x - 1, g(x) = 2x - 1$

d) $f(x) = x^4 - x^2, g(x) = x + 1$

e) $f(x) = \sin x, g(x) = 4x$

f) $f(x) = |x| - 2, g(x) = x + 5$

6. For each of the following,

- determine the defining equation for $f \circ g$ and $g \circ f$
- determine the domain and range of $f \circ g$ and $g \circ f$

a) $f(x) = 3x, g(x) = \sqrt{x - 4}$

b) $f(x) = \sqrt{x}, g(x) = 3x + 1$

c) $f(x) = \sqrt{4 - x^2}, g(x) = x^2$

d) $f(x) = 2^x, g(x) = \sqrt{x - 1}$

e) $f(x) = 10^x, g(x) = \log x$

f) $f(x) = \sin x, g(x) = 5^{2x} + 1$

7. For each function h, find two functions, f and g, such that $h(x) = f(g(x))$.

a) $h(x) = \sqrt{x^2 + 6}$

b) $h(x) = (5x - 8)^6$

c) $h(x) = 2^{(6x+7)}$

d) $h(x) = \dfrac{1}{x^3 - 7x + 2}$

e) $h(x) = \sin^2(10x + 5)$

f) $h(x) = \sqrt[3]{(x + 4)^2}$

8. a) Let $f(x) = 2x - 1$ and $g(x) = x^2$. Determine $(f \circ g)(x)$.

b) Graph f, g, and $f \circ g$ on the same set of axes.

c) Describe the graph of $f \circ g$ as a transformation of the graph of $y = g(x)$.

9. Let $f(x) = 2x - 1$ and $g(x) = 3x + 2$.

a) Determine $f(g(x))$, and describe its graph as a transformation of $g(x)$.

b) Determine $g(f(x))$, and describe its graph as a transformation of $f(x)$.

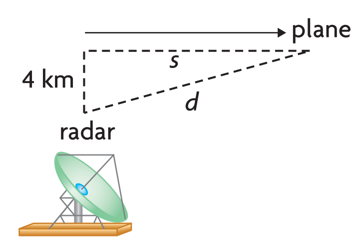
10. A A banquet hall charges \$975 to rent a reception room, plus \$39.95 per person. Next month, however, the banquet hall will be offering a 20% discount off the total bill. Express this discounted cost as a function of the number of people attending.

11. The function $f(x) = 0.08x$ represents the sales tax owed on a purchase with a selling price of x dollars, and the function $g(x) = 0.75x$ represents the sale price of an item with a price tag of x dollars during a 25% off sale. Write a function that represents the sales tax owed on an item with a price tag of x dollars during a 25% off sale.

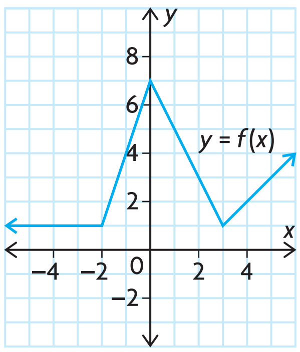
12. An airplane passes directly over a radar station at time $t = 0$. The plane maintains an altitude of 4 km and is flying at a speed of 560 km/h. Let d represent the distance from the radar station to the plane, and let s represent the horizontal distance travelled by the plane since it passed over the radar station.

a) Express d as a function of s, and s as a function of t.

b) Use composition to express the distance between the plane and the radar station as a function of time.



13. In a vehicle test lab, the speed of a car, v kilometres per hour, at a time of t hours is represented by $v(t) = 40 + 3t + t^2$. The rate of gasoline consumption of the car, c litres per kilometre, at a speed of v kilometres per hour is represented by $c(v) = \left(\frac{v}{500} - 0.1\right)^2 + 0.15$. Determine algebraically $c(v(t))$, the rate of gasoline consumption as a function of time. Determine, using technology, the time when the car is running most economically during a 4 h simulation.



14. Given the graph of $y = f(x)$ shown and the functions below, match the correct composition with each graph. Justify your choices.

T

i) $g(x) = x + 3$ **iii)** $h(x) = x - 3$ **v)** $k(x) = -x$

ii) $m(x) = 2x$ **iv)** $n(x) = -0.5x$ **vi)** $p(x) = x - 4$

a) $y = (f \circ g)(x)$ **g)** $y = (g \circ f)(x)$

b) $y = (f \circ h)(x)$ **h)** $y = (h \circ f)(x)$

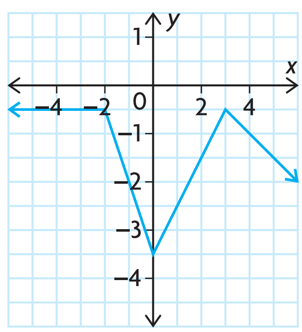
c) $y = (f \circ k)(x)$ **i)** $y = (k \circ f)(x)$

d) $y = (f \circ m)(x)$ **j)** $y = (m \circ f)(x)$

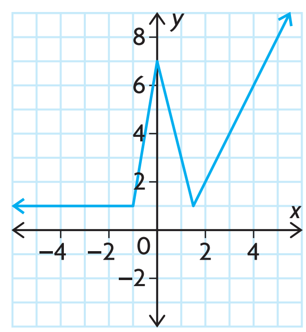
e) $y = (f \circ n)(x)$ **k)** $y = (n \circ f)(x)$

f) $y = (f \circ p)(x)$ **l)** $y = (p \circ f)(x)$

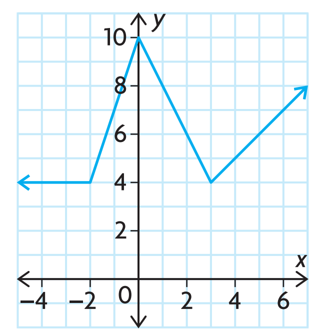
A



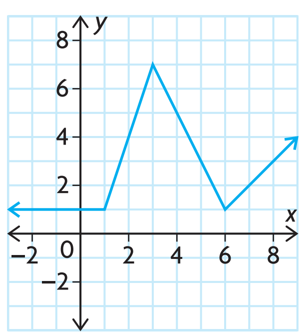
C



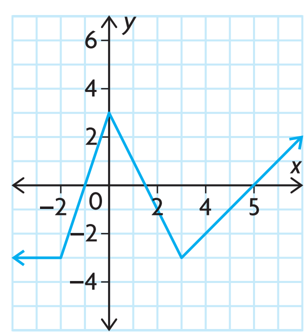
E



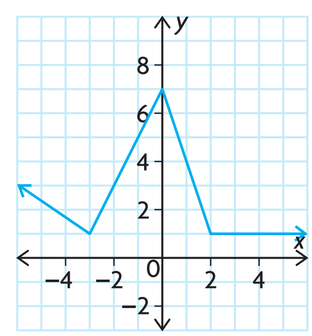
B



D



F



Sum	Product
$y = \frac{4}{x-3} + 1$	
Quotient	Composition

15. Find two functions, f and g, to express the given function in the centre box of the chart in each way shown.

C

Extending

16. **a)** If $y = 3x - 2$, $x = 3t + 2$, and $t = 3k - 2$, find an expression for $y = f(k)$.

b) Express y as a function of k if $y = 2x + 5$, $x = \sqrt{3t - 1}$, and $t = 3k - 5$.

9 Chapter Review

FREQUENTLY ASKED Questions

Q: How can you determine the composition of two functions, *f* and *g*?

A1: The composition of f with g can be determined numerically by evaluating g for some input value, x, and then evaluating f using $g(x)$ as the input value.

A2: The composition of f with g can be determined graphically by interpolating on the graph of g to determine its output for some input value, x, and then interpolating on the graph of f using the input value $g(x)$.

A3: The composition of f with g can be determined algebraically by taking the expression for g and then substituting this into the function f.

Study *Aid*

- See Lesson 9.5, Examples 1 and 2.
- Try Chapter Review Questions 8, 9, and 10.

Q: How do you solve an equation or inequality when an algebraic strategy is difficult or not possible?

A1: If you have access to graphing technology, there are two different strategies you can use to solve an equation:

- Represent the two sides of the equation/inequality as separate functions. Then graph the functions together using a graphing calculator or graphing software, and apply the intersection operation to determine the solution(s).
- Rewrite the equation/inequality so that one side is zero. Graph the nonzero side as a function. Use the zero operation to determine each of the zeros of the function.

A2: If you do not have access to graphing technology, you can use a guess and improvement strategy to solve an equation. Estimate where the intersection of $f(x)$ and $g(x)$ will occur, and substitute this value into both sides of the equation. Based on the outcome, adjust your estimate. Repeat this process until the desired degree of accuracy is found.

A3: Solving an inequality requires using either of the three previous strategies to find solutions to either $f(x) - g(x) = 0$ or $f(x) = g(x)$. Use these values to construct intervals. Test each interval to see whether it satisfies the inequality.

Study *Aid*

- See Lesson 9.6, Example 3.
- Try Chapter Review Question 12.

PRACTICE Questions

Lesson 9.1

1. Given the functions $f(x) = x + 5$ and $g(x) = x^2 - 6x - 55$, determine which of the following operations can be used to combine the two functions into one function that has both a vertical asymptote and a horizontal asymptote: addition, subtraction, multiplication, division.

Lesson 9.2

2. A franchise owner operates two coffee shops. The sales, S_1, in thousands of dollars, for shop 1 are represented by $S_1(t) = 700 - 1.4t^2$, where $t = 0$ corresponds to the year 2000. Similarly, the sales for shop 2 are represented by $S_2(t) = t^3 + 3t^2 + 500$.

a) Which shop is showing an increase in sales after the year 2000?

b) Determine a function that represents the total sales for the two coffee shops.

c) What are the expected total sales for the year 2006?

d) If sales continue according to the individual functions, what would you recommend that the owner do? Explain.

3. A company produces a product for \$9.45 per unit, plus a fixed operating cost of \$52 000. The company sells the product for \$15.80 per unit.

a) Determine a function, $C(x)$, to represent the cost of producing x units.

b) Determine a function, $I(x)$, to represent income from sales of x units.

c) Determine a function that represents profit.

Lesson 9.3

4. Calculate $(f \times g)(x)$ for each of the following pairs of functions.

a) $f(x) = 3\tan(7x)$, $g(x) = 4\cos(7x)$

b) $f(x) = \sqrt{3x^2}$, $g(x) = 3\sqrt{3x^2}$

c) $f(x) = 11x - 7$, $g(x) = 11x + 7$

d) $f(x) = ab^x$, $g(x) = 2ab^{2x}$

5. A country projects that the average amount of money, in dollars, that it will collect in taxes from each taxpayer over the next 50 years can be modelled by the function $A(t) = 2850 + 200t$, where t is the number of years from now. It also projects that the number of taxpayers over the next 50 years can be modelled by the function $C(t) = 15\,000\,000(1.01)^t$.

a) Write the function that represents the amount of money, in dollars, that the country expects to collect in taxes over the next 50 years.

b) Graph the function you wrote in part a).

c) How much does the country expect to collect in taxes 26 years from now?

Lesson 9.4

6. Calculate $(f \div g)(x)$ for each of the following pairs of functions.

a) $f(x) = 105x^3$, $g(x) = 5x^4$

b) $f(x) = x - 4$, $g(x) = 2x^2 + x - 36$

c) $f(x) = \sqrt{x + 15}$, $g(x) = x + 15$

d) $f(x) = 11x^5$, $g(x) = 22x^2 \log x$

7. State the domain of $(f \div g)(x)$ for each of your answers in the previous question.

Lesson 9.5

8. Let $f(x) = \dfrac{1}{\sqrt{x + 1}}$ and $g(x) = x^2 + 3$.

a) What are the domain and range of $f(x)$ and $g(x)$?

b) Find $f(g(x))$.

c) Find $g(f(x))$.

d) Find $f(g(0))$.

e) Find $g(f(0))$.

f) State the domain of each of the functions you found in parts b) and c).

9. Let $f(x) = x - 3$. Determine each of the following functions:

a) $(f \circ f)(x)$

b) $(f \circ f \circ f)(x)$

c) $(f \circ f \circ f \circ f)(x)$

d) f composed with itself n times

10. A circle has radius r.

a) Write a function for the circle's area in terms of r.

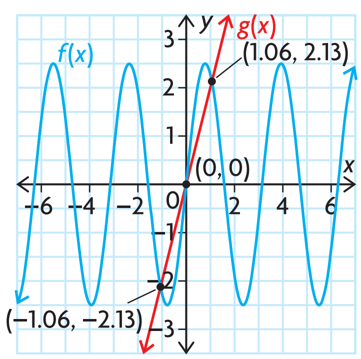
b) Write a function for the radius in terms of the circumference, C.

c) Determine $A(r(C))$.

d) A tree's circumference is 3.6 m. What is the area of the cross-section?

Lesson 9.6

11. In the graph shown below, $f(x) = 5 \sin x \cos x$ and $g(x) = 2x$. State the values of x in which $f(x) < g(x)$, $f(x) = g(x)$, and $f(x) > g(x)$. Express the values to the nearest tenth.



12. Solve each of the following equations for x in the given interval, using a guess and improvement strategy. Express your answers to the nearest tenth, and verify them using graphing technology.

a) $-3 \csc x = x,\ \pi \le x \le \dfrac{3\pi}{2}$

b) $\cos^2 x = 3 - 2\sqrt{x},\ 0 \le x \le \pi$

c) $8^x = x^8,\ -1 \le x \le 1$

d) $7 \sin x = \dfrac{3}{x},\ 0 \le x \le 2$

Lesson 9.7

13. Let P represent the size of the frog population in a marsh at time t, in years. At $t = 0$, a species of frog is released into a marsh. When $t = 5$, biologists estimate that there are 2000 frogs in the marsh. Two years later, the biologists estimate that there are 3200 frogs.

a) Find a formula for $P = f(t)$, assuming linear growth. Interpret the slope and the P-intercept of your formula in terms of the frog population.

b) Find a formula for $P = g(t)$, assuming exponential growth. Interpret the parameters of your formula in terms of the frog population.

14. The population of the world from 1950 to 2000 is shown. Create a scatter plot of the data, and determine an algebraic model for this situation. Use your model to estimate the world's population in 1963, 1983, and 2040.

Year	Population (millions)
1950	2555
1955	2780
1960	3039
1965	3346
1970	3708
1975	4088
1980	4457
1985	4855
1990	5284
1995	5691
2000	6080

Source: U.S. Census Bureau

9 Chapter Self-Test

1. A sphere has radius r.
   a) Write a function for the sphere's surface area in terms of r.
   b) Write a function for the radius in terms of the volume, V.
   c) Determine $A(r(V))$.
   d) A mother wrapped a ball in wrapping paper and gave it to her son on his birthday. The volume of the ball was 0.75 m^3. Assuming that she used the minimum amount of wrapping paper possible to cover the ball, how much wrapping paper did she use?

2. Solve $x \sin x \geq x^2 - 1$. Use any strategy.

3. Let $f(x) = (2x + 3)^7$. Find at least two different pairs of functions, $g(x)$ and $h(x)$, such that $f(x) = (g \circ h)(x)$.

n	$N(n)$
0	400
2	520
4	752
6	1144
8	1744
10	2600
15	6175

4. In the table at the left, $N(n)$ is the number, in thousands, of Canadian home computers sold, where n is the number of years since 1990.
   a) Determine the equation that best models this relationship.
   b) How many home computers were sold in June 1993?

5. The graph of the function $f(x)$ is a line passing through the point $(2, -3)$ with a slope of 6. The graph of the function $g(x)$ is the graph of the function $h(x) = x^2$ vertically stretched by a factor of 5, horizontally translated 8 units to the left, and vertically translated 1 unit down. Find $(f \times g)(x)$.

6. The height of a species of dwarf evergreen tree, in centimetres, as a function of time, in months, can be modelled by the logistic function $h(t) = \frac{275}{1 + 26(0.85)^t}$.
   a) If this function is graphed, are there any asymptotes? If so, name each asymptote and describe what it means.
   b) Determine when this tree will reach a height of 150 cm.

7. The cost, in dollars, to produce a product can be modelled by the function $C(x) = 5x + 18$, where x is the number of the product produced, in thousands. The revenue generated by producing and selling x units of this product can be modelled by the function $R(x) = 2x^2$. How much of the product must the company produce in order to break even?

8. Solve $\frac{\cot x}{x} = x^3 + 3$. Use any strategy. Round your answer(s) to the nearest tenth, if necessary.

9. Given $f(x) = \sin x$ and $g(x) = \cos x$, which of the following operations make it possible to combine the two functions into one function that is not sinusoidal: addition, subtraction, multiplication, or division?