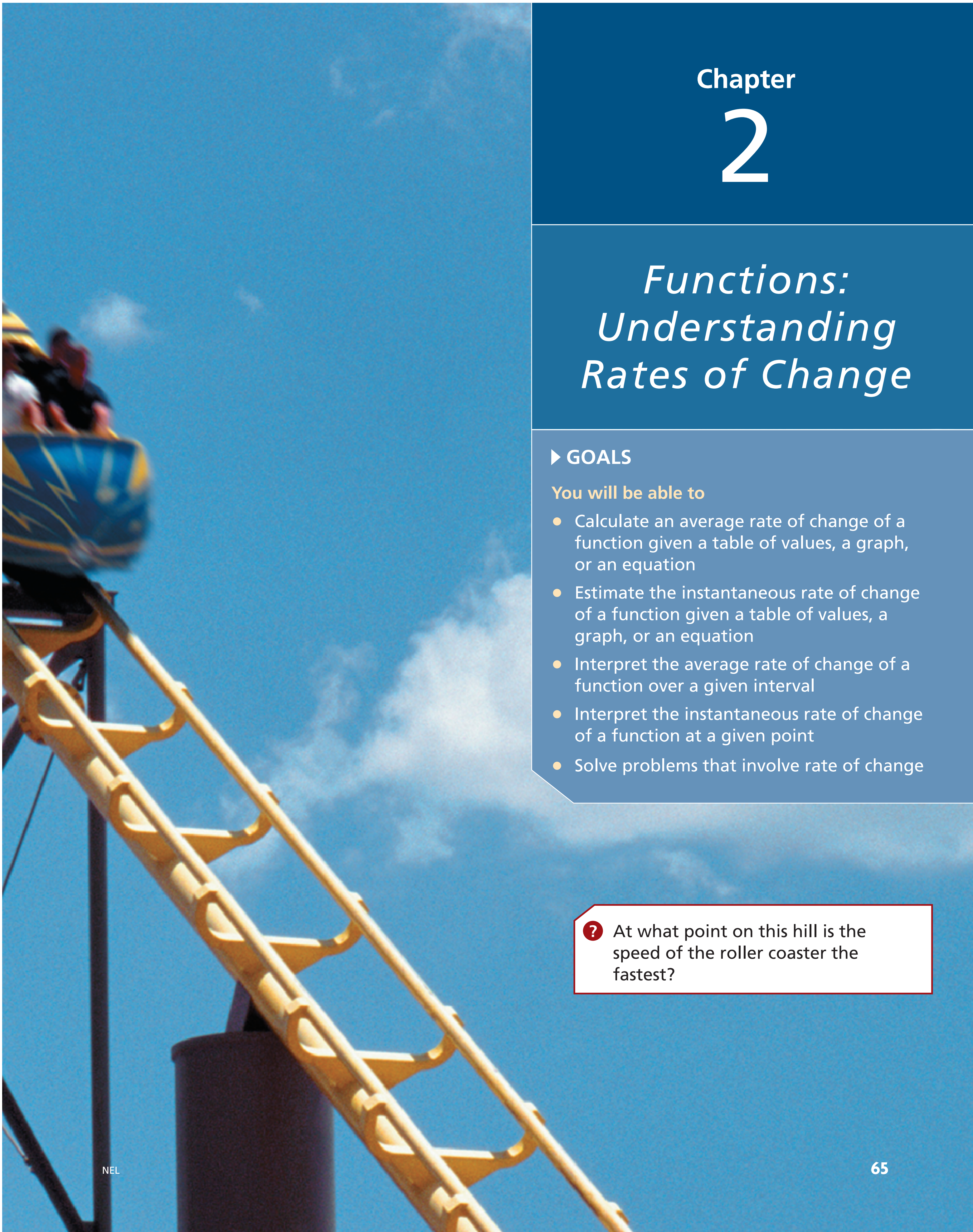


Chapter

2

Functions: Understanding Rates of Change

▶ GOALS

You will be able to

- Calculate an average rate of change of a function given a table of values, a graph, or an equation
- Estimate the instantaneous rate of change of a function given a table of values, a graph, or an equation
- Interpret the average rate of change of a function over a given interval
- Interpret the instantaneous rate of change of a function at a given point
- Solve problems that involve rate of change

? At what point on this hill is the speed of the roller coaster the fastest?

2 Getting Started

Study **Aid**

- For help, see the Review of Essential Skills found at the Nelson Advanced Functions website.

Question	Appendix
1	R–5
3	R–6
4	R–8
6	R–12

SKILLS AND CONCEPTS *You Need*

1. Determine the slope of the line through each pair of points.

a) A $(2, 3)$ and B $(5, 7)$ **b)** C $(3, -1)$ and D $(-4, 5)$

2. Calculate the finite differences for each table, identify the type of function that each table represents, and provide a reason for your choice.

a)

x	1	2	3	4	5	6
y	1	−1	−5	−13	−29	−61

b)

x	1	2	3	4	5	6
y	0	11	28	51	80	115

3. Determine the zeros for each of the following functions.

a) $g(x) = 2x^2 - x - 6$

b) $h(x) = 2^x - 1$

c) $j(x) = \sin(x - 45°), 0° \leq x \leq 360°$

d) $k(x) = 2\cos x, -360° \leq x \leq 0°$

4. Given $y = f(x)$, describe how the graph of $f(x)$ is transformed in each of the following functions.

a) $y = \frac{1}{2}f(x)$

b) $y = 2f(x - 4)$

c) $y = -3f(x) + 7$

d) $y = 5f(x - 3) - 2$

5. Suppose you invest $1000 in a savings account that pays 8%/a compounded annually.

a) Write an equation for the amount of money you will have after t years.

b) How much money will you have after three years?

c) Does the amount of money in your account increase at a constant rate each year? Explain.

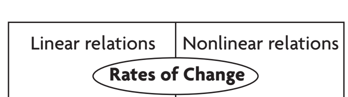
6. The height above the ground of one of the seats of a Ferris wheel, in metres, can be modelled by the function $h(t) = 8 + 7\sin(15°t)$, where t is measured in seconds.

a) What is the maximum and minimum height reached by any seat?

b) How long does one seat on this ride take to rotate back to its starting point?

c) After 30 s, what will the height of the seat be?

7. Create a chart to show what you know about rates of change in **linear** and **nonlinear relations**.



APPLYING What You Know

Safe Driving

It is important for drivers to know how much time they need to come to a safe stop. The time needed to stop depends on the speed of the vehicle.

The following table gives safe stopping distances on dry pavement.

Speed (km/h)	Reaction-Time Distance (m)	Braking Distance (m)	Overall Stopping Distance (m)
0	0.00	0.00	0.00
20	8.33	1.77	10.10
40	16.67	7.09	23.77
60	25.00	15.96	40.96
80	33.33	28.38	61.71
100	41.67	44.35	86.02

? What might be realistic reaction-time distances, braking distances, and overall stopping distances for speeds of 70 km/h and 120 km/h?

A. What type of function best models the reaction-time distances for the given speeds? Explain how you know.

B. Sketch a graph and determine an equation for the type of function you chose in part A, using the data in the table.

C. What type of function best models the braking distances for the given speeds? Explain how you know.

D. Sketch a graph and determine an equation for the type of function you chose in part C, using the data in the table.

E. Do any of the three distances in the table increase at a constant rate? Explain.

F. What other factors, in addition to the speed of the vehicle, may affect the overall stopping distance?

G. Use the graphs and equations you found to predict the reaction times, braking distances, and overall stopping distances, in metres, for speeds of 70 km/h and 120 km/h.

2.1 Determining Average Rate of Change

average rate of change

in a relation, the change in the quantity given by the dependent variable (Δy) divided by the corresponding change in the quantity represented by the independent variable (Δx); for a function $y = f(x)$, the average rate of change in the internal $x_1 \leq x \leq x_2$ is $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

GOAL

Calculate and interpret the average rate of change on an interval of the independent variable.

LEARN ABOUT the Math

The following table represents the growth of a bacteria population over a 10 h period.

Time (h)	0	2	4	6	8	10
Number of Bacteria	850	1122	1481	1954	2577	3400

? During which 2 h interval did the bacteria population grow the fastest?

EXAMPLE 1 Reasoning about rate of change

Use the data in the table of values to determine the 2 h interval in which the bacteria population grew the fastest.

Solution A: Using a table

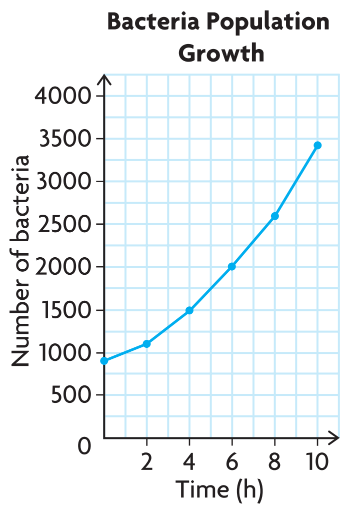
Time Interval (h)	Δb = Change in Number of Bacteria	Δt = Change in Time (h)	$\frac{\Delta b}{\Delta t}$ = Average Rate of Change (bacteria/h)
$0 \leq t \leq 2$	$1122 - 850 = 272$	$2 - 0 = 2$	$\frac{272}{2} = 136$
$2 \leq t \leq 4$	$1481 - 1122 = 359$	$4 - 2 = 2$	$\frac{359}{2} = 179.5$
$4 \leq t \leq 6$	$1954 - 1481 = 473$	$6 - 4 = 2$	$\frac{473}{2} = 236.5$
$6 \leq t \leq 8$	$2577 - 1954 = 623$	$8 - 6 = 2$	$\frac{623}{2} = 311.5$
$8 \leq t \leq 10$	$3400 - 2577 = 823$	$10 - 8 = 2$	$\frac{823}{2} = 411.5$

Calculate the **average rate of change** in the dependent variable (bacteria population) for each 2 h interval. Divide the change in the number of bacteria by the corresponding change in time (2 h for each interval). Identify the interval with the greatest change in population.

The greatest change in the bacteria population occurred during the last 2 h, when the population increased by an average of 412 bacteria per hour.

The average rate of change is expressed using the units of the two related quantities.

Solution B: Using points on a graph

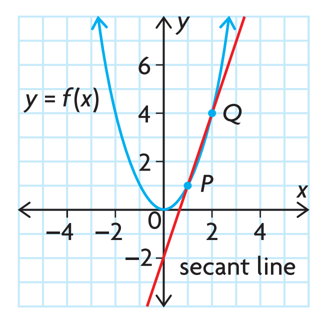


Create a scatter plot using the data in the table of values. Draw a **secant line** that passes through each pair of the endpoints for each 2 h interval. The slope of each secant line is equivalent to the average rate of change in the number of bacteria over each interval.

From the graph, it appears that the secant line with the greatest slope occurs during the last interval, from 8 to 10 h.

secant line

a line that passes through two points on the graph of a relation



Calculate the slope of each secant line to verify.

Recall that $m = \frac{y_2 - y_1}{x_2 - x_1}$, where (x_1, y_1) and (x_2, y_2) are points on the line.

In the first interval, the secant line passes through $(0, 850)$ and $(2, 1122)$.

$$m_1 = \frac{1122 - 850}{2 - 0} = \frac{272}{2}$$
$$= 136$$

Perform the same calculations for the other intervals.

$$m_2 = \frac{1481 - 1122}{4 - 2}$$
$$= 179.5$$
$$m_3 = \frac{1954 - 1481}{6 - 4}$$
$$= 236.5$$
$$m_4 = \frac{2577 - 1954}{8 - 6}$$
$$= 311.5$$
$$m_5 = \frac{3400 - 2577}{10 - 8}$$
$$= 411.5$$

The greatest change in the bacteria population occurred when the secant line is the steepest, during the last 2 h. The bacteria population increased by an average of 412 bacteria per hour in this interval.

Slope has no units, but average rate of change does.

Reflecting

A. Why is the average rate of change of the bacteria population positive on each interval, and what does this mean? How is this represented by the secant lines on the graph of the data?

B. How is calculating an average rate of change like calculating the slope of a secant line?

C. Why does rate of change have units, even though slope does not?

D. Why is the rate of change in the bacteria population not a constant?

APPLY the Math

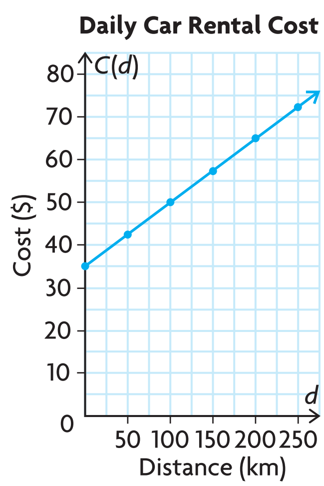
EXAMPLE 2 Reasoning about average rates of change in linear relationships

Sarah rents a car from a rental company. She is charged \$35 a day, plus a fee of \$0.15/km for the distances she drives each day. The equation $C(d) = 0.15d + 35$ can be used to calculate her daily cost to rent the car, where $C(d)$ is her daily cost in dollars and d is the daily distance she drives in kilometres.

Discuss the average rate of change of her daily costs in relation to the distance she drives.

Solution

Graph the equation $C(d) = 0.15d + 35$.



The relationship is linear, so the rate of change in the daily cost is constant. This means that the average rate of change between any two points on the graph is always constant. The secant lines that are drawn between any two pairs of points on the graph have the same slope.

Using the distance interval $0 \le d \le 100$,

$$\frac{\Delta C}{\Delta d} = \frac{C(100) - C(0)}{100 - 0}$$

$$= \frac{50 - 35}{100} = \$0.15/\text{km}$$

Calculate some average rates of change in the daily cost, using different distance intervals to verify.

Using the distance interval $100 \le d \le 250$,

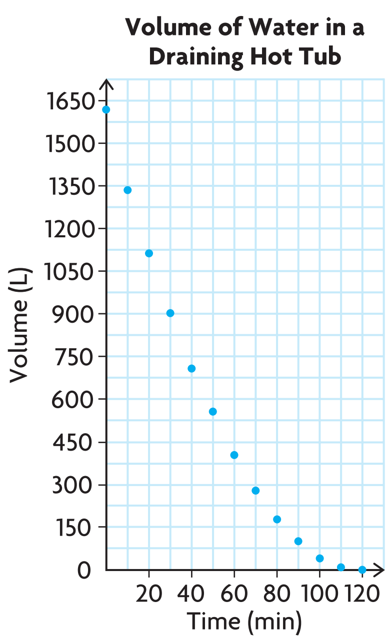
$$\frac{\Delta C}{\Delta d} = \frac{C(250) - C(100)}{250 - 100}$$

$$= \frac{72.50 - 50}{150} = \$0.15/\text{km}$$

The farther she drives each day, the more she will pay to rent the car. However, the rate at which the daily cost increases does not change. For every additional kilometre she drives, her daily cost increases by $0.15.

EXAMPLE 3 Using a graph to determine the average rate of change

Andrew drains the water from a hot tub. The tub holds 1600 L of water. It takes 2 h for the water to drain completely. The volume V, in litres, of water remaining in the tub at various times t, in minutes, is shown in the table and graph.



Time (min)	Volume (L)
0	1600
10	1344
20	1111
30	900
40	711
50	544
60	400
70	278
80	178
90	100
100	44
110	10
120	0

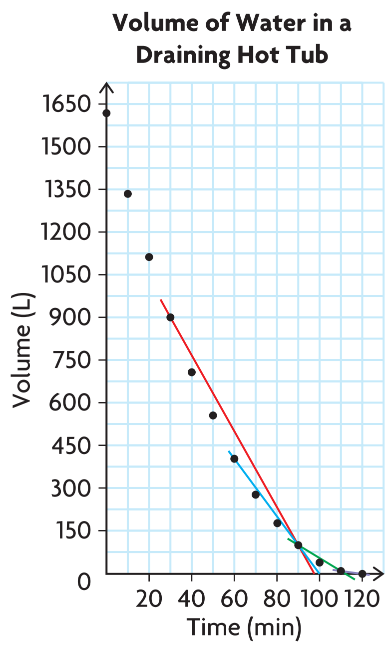
a) Calculate the average rate of change in volume during each of the following time intervals.

i) $30 \leq t \leq 90$
ii) $60 \leq t \leq 90$
iii) $90 \leq t \leq 110$
iv) $110 \leq t \leq 120$

b) Why is the rate of change in volume negative during each of these time intervals?

c) Does the hot tub drain at a constant rate? Explain.

Solution



Draw secant lines that pass through the endpoints for each given interval.

a) i) $m = \frac{\Delta V}{\Delta t} = \frac{V(90) - V(30)}{90 - 30}$

$= \frac{100 - 900}{60}$

$\doteq -13.3$

Use the points (30, 900) and (90, 100) on the graph for the red secant line to calculate the slope of the line. The average rate of change in volume that corresponds to this change in time is equivalent to the slope of the secant line.

The volume of water is decreasing, on average, at the rate of 13.3 L/min between 30 min and 90 min.

ii) $m = \frac{\Delta V}{\Delta t} = \frac{V(90) - V(60)}{90 - 60}$

$= \frac{100 - 400}{30}$

$= -10$

Use the points (60, 400) and (90, 100) on the graph for the blue secant line to calculate its slope.

The volume of water is decreasing, on average, at the rate of 10 L/min between 60 min and 90 min.

iii) $m = \frac{\Delta V}{\Delta t} = \frac{V(110) - V(90)}{110 - 90}$

$= \frac{10 - 100}{20}$

$= -4.5$

Use the points (90, 100) and (110, 10) on the graph for the green secant line to calculate its slope.

The volume of water is decreasing, on average, at the rate of 4.5 L/min between 90 min and 110 min.

Use the points (110, 10) and (120, 0) on the graph for the purple secant line to calculate its slope.

iv) $m = \frac{\Delta V}{\Delta t} = \frac{V(120) - V(110)}{120 - 110}$

$= \frac{0 - 10}{10}$

$= -1$

The volume of water is decreasing, on average, at the rate of 1 L/min between 110 min and 120 min.

b) The volume decreases as the time increases. So the numerator in each slope calculation is negative, while the denominator is positive. This makes the **rational number** that represents the rate of change negative.

The water is flowing out of the tub, so the volume that remains in the tub decreases with time.

c) The water is not draining at a constant rate over the 2 h period. This can be seen from the graph, because it is a non-linear relationship. The water is draining from the tub faster over time intervals at the beginning of the 2 h period. As the volume of water decreases, the pressure also decreases, causing the water to flow out of the tub more slowly.

Looking at the slope calculations, the slopes of the red and blue secant lines have a greater **magnitude** than the slopes of the green and purple secant lines. Also, the slopes of the secant lines between points over each 10 min interval are smaller in magnitude as time increases.

magnitude
the absolute value of a quantity

If you are given the equation of a relation or function, the average rate of change on a given interval can be calculated.

EXAMPLE 4 Using an equation to determine the average rate of change

A rock is tossed upward from a cliff that is 120 m above the water. The height of the rock above the water is modelled by $h(t) = -5t^2 + 10t + 120$, where $h(t)$ is the height in metres and t is time in seconds.

a) Calculate the average rate of change in height during each of the following time intervals.
i) $0 \leq t \leq 1$ **ii)** $1 \leq t \leq 2$ **iii)** $2 \leq t \leq 3$ **iv)** $3 \leq t \leq 4$

b) As the time increases, what do you notice about the average rate of change in height during each 1 s interval? What does this mean?

c) Describe what the average rate of change means in this situation.

Solution

a) i)
$$\frac{\Delta h}{\Delta t} = \frac{h(1) - h(0)}{1 - 0} = \frac{125 - 120}{1} = 5 \text{ m/s}$$

ii)
$$\frac{\Delta h}{\Delta t} = \frac{h(2) - h(1)}{2 - 1} = \frac{120 - 125}{1} = -5 \text{ m/s}$$

iii)
$$\frac{\Delta h}{\Delta t} = \frac{h(3) - h(2)}{3 - 2} = \frac{105 - 120}{1} = -15 \text{ m/s}$$

iv)
$$\frac{\Delta h}{\Delta t} = \frac{h(4) - h(3)}{4 - 3} = \frac{80 - 105}{1} = -25 \text{ m/s}$$

Substitute $t = 1$ and $t = 0$ into the equation to determine $h(1)$ and $h(0)$.

Calculate the change in height. Divide by the corresponding change in time to determine the average rate of change.

Repeat this process for the other three intervals.

b) The average rates of change are positive and then negative because the rock's height increases and then decreases. The average rates of change in height are also changing for each 1 s interval. After 1 s, as time increases, the rock is dropping a greater distance. The magnitude of the average rates of change are increasing. The rock is not falling at a constant rate.

Between 0 s and 1 s, the rock rises 5 m.
Between 1 s and 2 s, the rock drops 5 m.
Between 2 s and 3 s, the rock drops 15 m.
Between 3 s and 4 s, the rock drops 25 m.

c) Since the rate of change compares a change in distance over an interval of time, the rate of change represents the speed of the rock over the interval.

In this situation, as time increases, the rock picks up speed once it has passed its maximum height, because the distance it drops increases with each second.

In Summary

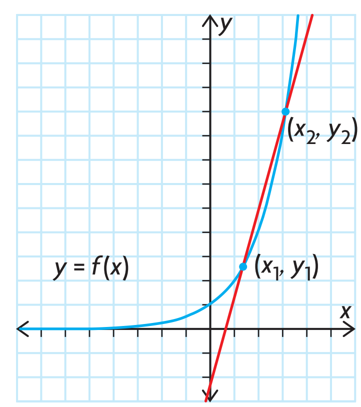
Key Ideas

- The average rate of change is the change in the quantity represented by the dependent variable (Δy) divided by the corresponding change in the quantity represented by the independent variable (Δx) over an interval. Algebraically, the average rate of change for any function $y = f(x)$ over the interval $x_1 \leq x \leq x_2$ can be determined by

$$\text{Average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- Graphically, the average rate of change for any function $y = f(x)$ over the interval $x_1 \leq x \leq x_2$ is equivalent to the slope of the secant line passing through two points (x_1, y_1) and (x_2, y_2).

$$\text{Average rate of change} = m_{\text{secant}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



Need to Know

- Average rate of change is expressed using the units of the two quantities that are related to each other.
- A positive average rate of change indicates that the quantity represented by the dependent variable is increasing on the specified interval, compared with the quantity represented by the independent variable. Graphically, this is indicated by a secant line that has a positive slope (the secant line rises from left to right).
- A negative average rate of change indicates that the quantity represented by the dependent variable is decreasing on the specified interval, compared with the quantity represented by the independent variable. Graphically, this is indicated by a secant line that has a negative slope (the secant line falls from left to right).
- All linear relationships have a constant rate of change. Average rate of change calculations over different intervals of the independent variable give the same result.
- Nonlinear relationships do not have a constant rate of change. Average rate of change calculations over different intervals of the independent variable give different results.

CHECK Your Understanding

1. Calculate the average rate of change for the function $g(x) = 4x^2 - 5x + 1$ over each interval.

a) $2 \leq x \leq 4$

b) $2 \leq x \leq 3$

c) $2 \leq x \leq 2.5$

d) $2 \leq x \leq 2.25$

e) $2 \leq x \leq 2.1$

f) $2 \leq x \leq 2.01$

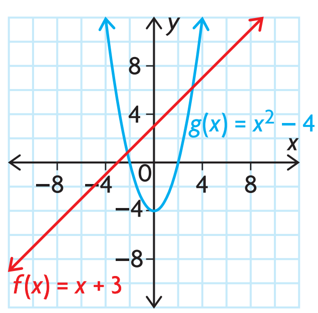
2. An emergency flare is shot into the air. Its height, in metres, above the ground at various times in its flight is given in the following table.

Time (s)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Height (m)	2.00	15.75	27.00	35.75	42.00	45.75	47.00	45.75	42.00

a) Determine the average rate of change in the height of the flare during each interval.

i) $1.0 \leq t \leq 2.0$ **ii)** $3.0 \leq t \leq 4.0$

b) Use your results from part a) to explain what is happening to the height of the flare during each interval.



3. Given the functions $f(x)$ and $g(x)$ shown on the graph, discuss how the average rates of change, $\frac{\Delta y}{\Delta x}$, differ in each relationship.

PRACTISING

4. This table shows the growth of a crowd at a rally over a 3 h period.
K

Time (h)	0.0	0.5	1.0	1.5	2.0	2.5	3.0
Number of People	0	176	245	388	402	432	415

a) Determine the average rate of change in the size of the crowd for each half hour of the rally.

b) What do these numbers represent?

c) What do positive and negative rates of change mean in this situation?

Day	**Cumulative Distance (km)**
0	0
1	203
2	396
3	561
4	739.5
5	958
6	1104

5. a) The cumulative distance travelled over several days of the 2007 Tour de France bicycle race is shown in the table to the left. Calculate the average rate of change in cumulative distance travelled between consecutive days.

b) Does the Tour de France race travel over the same distance each day? Explain.

6. What is the average rate of change in the values of the function $f(x) = 4x$ from $x = 2$ to $x = 6$? What about from $x = 2$ to $x = 26$? What do your results indicate about $f(x)$?

7. Shelly has a cell phone plan that costs \$39 per month and allows her 250 free anytime minutes. Any minutes she uses over the 250 free minutes cost \$0.10 per minute. The function

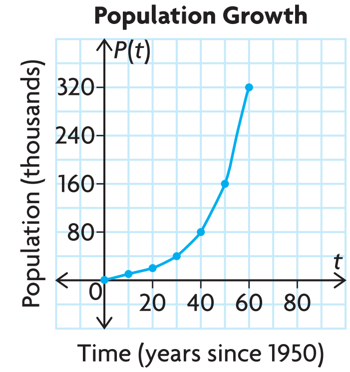
$$C(m) = \begin{cases} 39, \text{ if } 0 \leq m \leq 250 \\ 0.10(m - 250) + 39, \text{ if } m > 250 \end{cases}$$

can be used to determine her monthly cell phone bill, where $C(m)$ is her monthly cost in dollars and m is the number of minutes she talks. Discuss how the average rate of change in her monthly cost changes as the minutes she talks increases.

8. **A** The population of a city has continued to grow since 1950. The population P, in thousands, and the time t, in years, since 1950 are given in the table below and in the graph.

Time, *t* (years)	0	10	20	30	40	50	60
Population, *P* (thousands)	5	10	20	40	80	160	320

Population Growth



a) Calculate the average rate of change in population for the following intervals of time.

i) $0 \leq t \leq 20$
ii) $20 \leq t \leq 40$
iii) $40 \leq t \leq 60$
iv) $0 \leq t \leq 60$

b) Is the population growth constant?

c) To predict what the population will be in 2050, what assumptions must you make?

9. During the Apollo 14 mission, Alan Shepard hit a golf ball on the Moon. The function $h(t) = 18t - 0.8t^2$ models the height of the golf ball's trajectory on the Moon, where $h(t)$ is the height, in metres, of the ball above the surface of the Moon and t is the time in seconds. Determine the average rate of change in the height of the ball over the time interval $10 \leq t \leq 15$.

10. A company that sells sweatshirts finds that the profit can be modelled by $P(s) = -0.30s^2 + 3.5s + 11.15$, where $P(s)$ is the profit, in thousands of dollars, and s is the number of sweatshirts sold (expressed in thousands).

a) Calculate the average rate of change in profit for the following intervals.

i) $1 \leq s \leq 2$ **ii)** $2 \leq s \leq 3$ **iii)** $3 \leq s \leq 4$ **iv)** $4 \leq s \leq 5$

b) As the number of sweatshirts sold increases, what do you notice about the average rate of change in profit on each sweatshirt? What does this mean?

c) Predict if the rate of change in profit will stay positive. Explain what this means.

11. The population of a town is modelled by $P(t) = 50t^2 + 1000t + 20\ 000$, where $P(t)$ is the size of the population and t is the number of years since 2000.

a) Use graphing technology to graph $P(t)$.

b) Predict if the average rate of change in the population size will be greater closer to the year 2000 or farther in the future. Explain how you made your prediction.

c) Calculate the average rate of change in the population size for each time period.

- **i)** 2000–2010
- **ii)** 2002–2012
- **iii)** 2005–2015
- **iv)** 2010–2020

d) Evaluate your earlier prediction using the data you developed when answering part c).

12. C Your classmate was absent today and phones you for help with today's lesson. Share with your classmate

a) two real-life examples of when someone might calculate an average rate of change (one positive and one negative)

b) an explanation of when an average rate of change might be useful

c) an explanation of how an average rate of change is calculated

13. T Vehicles lose value over time. A car is purchased for $23 500, but is worth only $8750 after eight years. What is the average annual rate of change in the value of the car, as a percent?

14. Complete the following table by providing a definition in your own words, a personal example, and a visual representation of an average rate of change.

AVERAGE RATE OF CHANGE		
Definition in your own words	Personal example	Visual representation

Extending

15. The function $F(x) = -0.005x^2 + 0.8x + 12$ models the relationship between a certain vehicle's speed and fuel economy, where $F(x)$ is the fuel economy in kilometres per litre and x is the speed of the vehicle in kilometres per hour. Determine the rate of change in fuel economy for 10 km/h intervals in speed, and use your results to determine the speed that gives the best fuel economy.

2.2 Estimating Instantaneous Rates of Change from Tables of Values and Equations

GOAL

Estimate and interpret the rate of change at a particular value of the independent variable.

YOU WILL NEED

- graphing calculator or graphing software

INVESTIGATE the Math

A small pebble was dropped into a 3.0 m tall cylindrical tube filled with thick glycerine. A motion detector recorded the time and the total distance that the pebble fell after its release. The table below shows some of the measurements between 6.0 s and 7.0 s after the initial drop.

Time, t (s)	6.0	6.2	6.4	6.6	6.8	7.0
Distance, $d(t)$ (cm)	208.39	221.76	235.41	249.31	263.46	277.84

? How can you estimate the rate of change in the distance that the pebble fell at exactly $t = 6.4$ s?

A. Calculate the average rate of change in the distance that the pebble fell during each of the following time intervals.

i) $6.0 \le t \le 6.4$
ii) $6.2 \le t \le 6.4$
iii) $6.4 \le t \le 7.0$
iv) $6.4 \le t \le 6.8$
v) $6.4 \le t \le 6.6$

B. Use your results for part A to estimate the instantaneous rate of change in the distance that the pebble fell at exactly $t = 6.4$ s. Explain how you determined your estimate.

instantaneous rate of change

the exact rate of change of a function $y = f(x)$ at a specific value of the independent variable $x = a$; estimated using average rates of change for small intervals of the independent variable very close to the value $x = a$

C. Calculate the average rate of change in the distance that the pebble fell during the time interval $6.2 \le t \le 6.6$. How does your calculation compare with your estimate?

Reflecting

D. Why do you think each of the intervals you used to calculate the average rate of change in part A included 6.4 as one of its endpoints?

E. Why did it make sense to examine the average rates of change using time intervals on both sides of $t = 6.4$ s? Which of these intervals provided the best estimate for the instantaneous rate of change at $t = 6.4$ s?

F. Even though 6.4 is not an endpoint of the interval used in the average rate of change calculation in part C, explain why this calculation gave a reasonable estimate for the instantaneous rate of change at $t = 6.4$ s.

G. Using the table of values given, is it possible to get as accurate an estimate of the instantaneous rate of change for $t = 7.0$ s as you did for $t = 6.4$ s? Explain.

APPLY the Math

EXAMPLE 1 Selecting a strategy to estimate instantaneous rate of change using an equation

The population of a small town appears to be growing exponentially. Town planners think that the equation $P(t) = 35\,000\,(1.05)^t$, where $P(t)$ is the number of people in the town and t is the number of years after 2000, models the size of the population. Estimate the instantaneous rate of change in the population in 2015.

Solution A: Selecting a strategy using intervals

preceding interval
an interval of the independent variable of the form $a - h \leq x \leq a$, where h is a small positive value; used to determine an average rate of change

Using a **preceding interval** in which $14 \leq t \leq 15$,

$$\frac{\Delta P}{\Delta t} = \frac{P(15) - P(14)}{15 - 14}$$
$$\doteq \frac{72\,762 - 69\,298}{15 - 14}$$
$$= \frac{3464}{1}$$
$$= 3464 \text{ people/year}$$

Calculate average rates of change using some dates that precede the year 2015. Since 2015 is 15 years after 2000, use $t = 15$ to represent the year 2015.

Use $14 \leq t \leq 15$ and $14.5 \leq t \leq 15$ as preceding intervals (intervals on the left side of 15) to calculate the average rates of change in the population.

Using a preceding interval in which $14.5 \leq t \leq 15$,

$$\frac{\Delta P}{\Delta t} = \frac{P(15) - P(14.5)}{15 - 14.5}$$
$$\doteq \frac{72\,762 - 71\,009}{15 - 14.5}$$
$$= 3506 \text{ people/year}$$

following interval
an interval of the independent variable of the form $a \leq x \leq a + h$, where h is a small positive value; used to determine an average rate of change

Using a **following interval** in which $15 \leq t \leq 16$,

$$\frac{\Delta P}{\Delta t} = \frac{P(16) - P(15)}{16 - 15}$$
$$\doteq \frac{76\,401 - 72\,762}{16 - 15}$$
$$= 3639 \text{ people/year}$$

Calculate average rates of change using some dates that follow the year 2015. Use $15 \leq t \leq 16$ and $15 \leq t \leq 15.5$ as following intervals (intervals on the right side of 15) to calculate the average rates of change in the population.

Using a following interval in which $15 \le t \le 15.5$,

$$\frac{\Delta P}{\Delta t} = \frac{P(15.5) - P(15)}{15.5 - 15}$$
$$\doteq \frac{74\,559 - 72\,762}{15.5 - 15}$$
$$= 3594 \text{ people/year}$$

As the size of the preceding interval decreases, the average rate of change increases.

As the size of the following interval decreases, the average rate of change decreases.

Examine the average rates of change in population on both sides of $t = 15$ to find a trend.

The instantaneous rate of change in the population is somewhere between the values above.

$$\text{Estimate} = \frac{3506 + 3594}{2}$$
$$= 3550 \text{ people/year}$$

Make an estimate using the average of the two calculations for smaller intervals on either side of $t = 15$.

Solution B: Selecting a different interval strategy

Calculate some average rates of change using intervals that have the year 2015 as their midpoint.

Using a centred interval in which $14 \le t \le 16$,

$$\frac{\Delta P}{\Delta t} = \frac{P(16) - P(14)}{16 - 14}$$
$$\doteq \frac{76\,401 - 69\,298}{16 - 14}$$
$$\doteq 3552 \text{ people/year}$$

Using a centred interval in which $14.5 \le t \le 15.5$,

$$\frac{\Delta P}{\Delta t} = \frac{P(15.5) - P(14.5)}{15.5 - 14.5}$$
$$\doteq \frac{74\,559 - 71\,009}{15.5 - 14.5}$$
$$= 3550 \text{ people/year}$$

Use $14 \le t \le 16$ and $14.5 \le t \le 15.5$ as centred intervals (intervals with 15 as their midpoint) to calculate the average rates of change in the population. Examine the corresponding rates of change to find a trend. Using centred intervals allows you to move in gradually to the value that you are interested in. Sometimes this is called the *squeeze technique*.

The instantaneous rate of change in the population is about 3550 people/year.

The average rates of change are very similar. Make an estimate using the smallest centred interval.

centred interval

an interval of the independent variable of the form $a - h \le x \le a + h$, where h is a small positive value; used to determine an average rate of change

EXAMPLE 2 — Selecting a strategy to estimate the instantaneous rate of change

The volume of a cubic crystal, grown in a laboratory, can be modelled by $V(x) = x^3$, where $V(x)$ is the volume measured in cubic centimetres and x is the side length in centimetres. Estimate the instantaneous rate of change in the crystal's volume with respect to its side length when the side length is 5 cm.

Solution A: Squeezing the centred intervals

Look at the average rates of change near $x = 5$ using a series of centred intervals that get progressively smaller. By using intervals that get systematically smaller and smaller, you can make a more accurate estimate for the instantaneous rate of change than if you were to use intervals that are all the same size.

Using $4.5 \leq x \leq 5.5$,

$$\frac{\Delta V}{\Delta x} = \frac{166.375 - 91.125}{5.5 - 4.5}$$
$$= 75.25 \text{ cm}^3/\text{cm}$$

Using $4.9 \leq x \leq 5.1$,

In this case, use $4.5 \leq x \leq 5.5$, then $4.9 \leq x \leq 5.1$, and finally $4.99 \leq x \leq 5.01$.

$$\frac{\Delta V}{\Delta x} = \frac{132.651 - 117.649}{5.1 - 4.9}$$
$$= 75.01 \text{ cm}^3/\text{cm}$$

Using $4.99 \leq x \leq 5.01$,

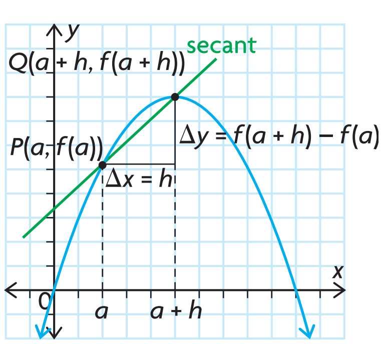
$$\frac{\Delta V}{\Delta x} = \frac{125.751\,501 - 124.251\,499}{5.01 - 4.99}$$
$$= 75.0001 \text{ cm}^3/\text{cm}$$

As the interval gets smaller, the average rate of change in the volume of the cube appears to be getting closer to 75 cm^3/cm. So it seems that the instantaneous rate of change in volume should be 75 cm^3/cm.

When the side length of the cube is exactly 5 cm, the volume of the cube is increasing at the rate of 75 cm^3/cm.

difference quotient

if $P(a, f(a))$ and $Q(a + h, f(a + h))$ are two points on the graph of $y = f(x)$, then the instantaneous rate of change of y with respect to x at P can be estimated using $\frac{\Delta y}{\Delta x} = \frac{f(a + h) - f(a)}{h}$, where h is a very small number. This expression is called the difference quotient.



Solution B: Using an algebraic approach and a general point

Write the difference quotient for the average rate of change in volume as the side length changes between 5 and any value: $(5 + h)$.

Use two points. Let one point be $(5, 5^3)$ or $(5, 125)$ because you are investigating the rate of change for $V(x) = x^3$ when $x = 5$. Let the other point be $(5 + h, (5 + h)^3)$, where h is a very small number, such as 0.01 or -0.01.

$$\frac{\Delta V}{\Delta x} = \frac{(5 + h)^3 - 125}{5 + h - 5}$$
$$= \frac{(5 + h)^3 - 125}{h}$$

Let $h = -0.01$.

$$\frac{\Delta V}{\Delta x} = \frac{(5 + (-0.01))^3 - 125}{h}$$

$$= \frac{124.251\,499 - 125}{-0.01}$$

$$= 74.8501 \text{ cm}^3/\text{cm}$$

The value $h = -0.01$ corresponds to a very small preceding interval, where $4.99 \leq x \leq 5$. This gives an estimate of the instantaneous rate of change when the side length changes from 4.99 cm to 5 cm.

Let $h = 0.01$.

$$\frac{\Delta V}{\Delta x} = \frac{(5 + 0.01)^3 - 125}{0.01}$$

$$= \frac{125.751\,501 - 125}{0.01}$$

$$= 75.1501 \text{ cm}^3/\text{cm}$$

The value $h = 0.01$ corresponds to a very small following interval, where $5 \leq x \leq 5.01$. This gives an estimate of the instantaneous rate of change when the side length changes from 5 cm to 5.01 cm.

The instantaneous rate of change in the volume of the cube is somewhere between the two values calculated.

$$\text{Estimate} = \frac{74.8501 + 75.1501}{2}$$

$$\doteq 75.0001 \text{ cm}^3/\text{cm}$$

Determine an estimate using the average of the two calculations on either side of $x = 5$.

EXAMPLE 3 Selecting a strategy to estimate an instantaneous rate of change

The following table shows the temperature of an oven as it heats from room temperature to 400°F.

Time (min)	0	1	2	3	4	5	6	7	8	9	10
Temperature (°F)	70	125	170	210	250	280	310	335	360	380	400

a) Estimate the instantaneous rate of change in temperature at exactly 5 min using the given data.

b) Estimate the instantaneous rate of change in temperature at exactly 5 min using a quadratic model.

Tech Support

For help using a graphing calculator to create scatter plots and determine an algebraic model using quadratic regression, see Technical Appendix, T-11.

Solution

a) Using the interval $2 \leq t \leq 8$,

$$\frac{\Delta T}{\Delta t} = \frac{360 - 170}{8 - 2}$$

$$\doteq 31.67°\text{F/min}$$

Choose some centred intervals around 5 min. Examine the average rates of change as the intervals of time get smaller, and find a trend.

Using the interval $3 \leq t \leq 7$,

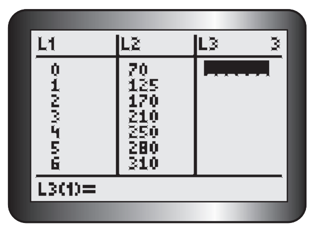
$$\frac{\Delta T}{\Delta t} = \frac{335 - 210}{7 - 3}$$
$$= 31.25°\text{F/min}$$

Using the interval $4 \leq t \leq 6$,

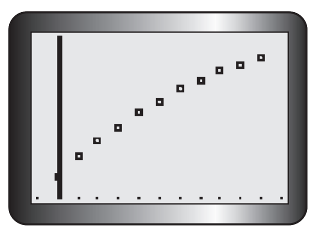
$$\frac{\Delta T}{\Delta t} = \frac{310 - 250}{6 - 4}$$
$$= 30°\text{F/min}$$

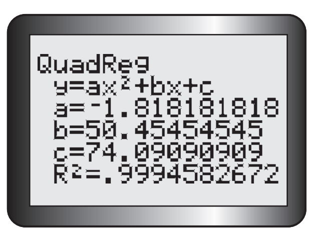
As the centred intervals around 5 min get smaller, it appears that the average rates of change in the temperature of the oven get closer to about 30°F/min.

b)



Enter the data into the lists of the graphing calculator, and create a scatter plot.





Using quadratic regression, determine an equation to represent the data. Rounding to two decimal places, the model is $f(x) = -1.82x^2 + 50.45x + 74.09$, where $f(x)$ is oven temperature and x is time.

Next, calculate the average rate of change in oven temperature using a very small centred interval near $x = 5$. For example, use $4.99 \leq x \leq 5.01$.

Interval	$\Delta f(x)$	Δx	$\frac{\Delta f(x)}{\Delta x}$
$4.99 \leq x \leq 5.01$	$f(5.01) - f(4.99)$ $\doteq 281.16 - 280.52$ $= 0.64$	$5.01 - 4.99$ $= 0.02$	$0.64/0.02$ $= 32°\text{F/min}$

The instantaneous rate of change in temperature at 5 min is about 32°F/min.

In Summary

Key Idea

- The instantaneous rate of change of the dependent variable is the rate at which the dependent variable changes at a specific value of the independent variable, $x = a$.

Need to Know

- The instantaneous rate of change of the dependent variable, in a table of values or an equation of the relationship, can be estimated using the following methods:
  - Using a series of preceding $(a - h \leq x \leq a)$ and following $(a \leq x \leq a + h)$ intervals: Calculate the average rate of change by keeping one endpoint of each interval fixed. (This is $x = a$, the location where the instantaneous rate of change occurs.) Move the other endpoint of the interval closer and closer to the fixed point from either side by making h smaller and smaller. Based on the trend for the average rates of change, make an estimate for the instantaneous rate of change at the specific value.
  - Using a series of centred intervals $(a - h \leq x \leq a + h)$: Calculate the average rate of change by picking endpoints for each interval on either side of $x = a$, where the instantaneous rate of change occurs. Choose these endpoints so that the value where the instantaneous rate of change occurs is the midpoint of the interval. Continue to calculate the average rate of change by moving both endpoints closer and closer to where the instantaneous rate of change occurs. Based on the trend, make an estimate for the instantaneous rate of change.
  - Using the difference quotient and a general point: Calculate the average rate of change using the location where the instantaneous rate of change occurs $(a, f(a))$ and a general point $(a + h, f(a + h))$, i.e., $\frac{f(a + h) - f(a)}{h}$. Choose values for h that are very small, such as ± 0.01 or ± 0.001. The smaller the value used for h, the better the estimate will be.
- The best estimate for the instantaneous rate of change occurs when the interval used to calculate the average rate of change is made as small as possible.

CHECK Your Understanding

1. **a)** Copy and complete the tables, if $f(x) = 5x^2 - 7$.

Preceding Interval	$\Delta f(x)$	Δx	Average Rate of Change, $\frac{\Delta f(x)}{\Delta x}$
$1 \leq x \leq 2$	$13 - (-2) = 15$	$2 - 1 = 1$	
$1.5 \leq x \leq 2$	8.75	0.5	
$1.9 \leq x \leq 2$			
$1.99 \leq x \leq 2$			

Following Interval	$\Delta f(x)$	Δx	Average Rate of Change, $\frac{\Delta f(x)}{\Delta x}$
$2 \leq x \leq 3$	$38 - 13 = 25$	$3 - 2 = 1$	
$2 \leq x \leq 2.5$	11.25	0.5	
$2 \leq x \leq 2.1$			
$2 \leq x \leq 2.01$			

b) Based on the trend in the average rates of change, estimate the instantaneous rate of change when $x = 2$.

2. A soccer ball is kicked into the air. The following table of values shows the height of the ball above the ground at various times during its flight.

Time (s)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Height (m)	0.5	11.78	20.6	26.98	30.9	32.38	31.4	27.98	22.1	13.78	3.0

a) Estimate the instantaneous rate of change in the height of the ball at exactly $t = 2.0$ s using the preceding and following interval method.
b) Estimate the instantaneous rate of change in the height of the ball at exactly $t = 2.0$ s using the centred interval method.
c) Which estimation method do you prefer? Explain.

3. A population of raccoons moves into a wooded area. At t months, the number of raccoons, $P(t)$, can be modelled using the equation $P(t) = 100 + 30t + 4t^2$.
a) Determine the population of raccoons at 2.5 months.
b) Determine the average rate of change in the raccoon population over the interval from 0 months to 2.5 months.
c) Estimate the rate of change in the raccoon population at exactly 2.5 months.
d) Explain why your answers for parts a), b), and c) are different.

PRACTISING

4. For the function $f(x) = 6x^2 - 4$, estimate the instantaneous rate of change for the given values of x.
a) $x = -2$ **b)** $x = 0$ **c)** $x = 4$ **d)** $x = 8$

5. An object is sent through the air. Its height is modelled by the function $h(x) = -5x^2 + 3x + 65$, where $h(x)$ is the height of the object in metres and x is the time in seconds. Estimate the instantaneous rate of change in the object's height at 3 s.

6. A family purchased a home for \$125 000. Appreciation of the home's value, in dollars, can be modelled by the equation $H(t) = 125\ 000(1.06)^t$, where $H(t)$ is the value of the home and t is the number of years that the family owns the home. Estimate the instantaneous rate of change in the home's value at the start of the eighth year of owning the home.

7. **K** The population of a town, in thousands, is described by the function $P(t) = -1.5t^2 + 36t + 6$, where t is the number of years after 2000.

a) What is the average rate of change in the population between the years 2000 and 2024?

b) Does your answer to part a) make sense? Does it mean that there was no change in the population from 2000 to 2024?

c) Explain your answer to part b) by finding the average rate of change in the population from 2000 to 2012 and from 2012 to 2024.

d) For what value of t is the instantaneous rate of change in the population 0?

8. Jacelyn purchased a new car for \$18 999. The yearly depreciation of the value of the car can be modelled by the equation $V(t) = 18\ 999(0.93)^t$, where $V(t)$ is the value of the car and t is the number of years that Jacelyn owns the car. Estimate the instantaneous rate of change in the value of the car when the car is 5 years old. What does this mean?

9. **A** A diver is on the 10 m platform, preparing to perform a dive. The diver's height above the water, in metres, at time t can be modelled using the equation $h(t) = 10 + 2t - 4.9t^2$.

a) Determine when the diver will enter the water.

b) Estimate the rate at which the diver's height above the water is changing as the diver enters the water.

10. **T** To make a snow person, snow is being rolled into the shape of a sphere. The volume of a sphere is given by the function $V(r) = \frac{4}{3}\pi r^3$, where r is the radius in centimetres. Use two different methods to estimate the instantaneous rate of change in the volume of the snowball with respect to the radius when $r = 5$ cm.

11. David plans to drive to see his grandparents during his winter break. How can he determine his average speed for a part of his journey along the way? Be as specific as possible. Describe the steps he must take and the information he must know.

12. The following table shows the temperature of an oven as it cools.

Time (min)	0	1	2	3	4	5	6
Temperature (°F)	400	390	375	350	330	305	270

a) Use the data in the table to estimate the instantaneous rate of change in the temperature of the oven at exactly 4 min.
b) Use a graphing calculator to determine a quadratic model. Use your quadratic model to estimate the instantaneous rate of change in the temperature of the oven at exactly 4 min.
c) Discuss why your answers for parts a) and b) are different.
d) Which is the better estimate? Explain.

13. **C** In a table like the one below, list all the methods that can be used to estimate the instantaneous rate of change. What are the advantages and disadvantages of each method?

Method of Estimating Instantaneous Rate of Change	**Advantage**	**Disadvantage**

Extending

14. Concentric circles form when a stone is dropped into a pool of water.
a) What is the average rate of change in the area of one circle with respect to the radius as the radius grows from 0 cm to 100 cm?
b) How fast is the area changing with respect to the radius when the radius is 120 cm?

15. A crystal in the shape of a cube is growing in a laboratory. Estimate the rate at which the surface area is changing with respect to the side length when the side length of the crystal is 3 cm.

16. A spherical balloon is being inflated. Estimate the rate at which its surface area is changing with respect to the radius when the radius measures 20 cm.

2.3 Exploring Instantaneous Rates of Change Using Graphs

GOAL

Estimate instantaneous rates of change using slopes of lines.

YOU WILL NEED

- graphing calculator or graphing software

EXPLORE the Math

In the previous lesson, you used numerical and algebraic techniques to estimate instantaneous rates of change. Graphically, you have seen that the average rate of change is equivalent to the slope of a secant line that passes through two points on the graph of a function.

? How can you use the slopes of secant lines to estimate the instantaneous rate of change?

A. Enter the function $f(x) = x^2$ into the equation editor of your graphing calculator, graph it, and draw a sketch of the graph.

B. On your sketch, draw a secant line that passes through the points $(1, f(1))$ and $(3, f(3))$.

C. Calculate the slope of the secant line. Copy the table and record the slope. Calculate and record the slopes of other secant lines using the points listed.

Points	Slope of Secant
$(1, f(1))$ and $(3, f(3))$	
$(1, f(1))$ and $(2, f(2))$	
$(1, f(1))$ and $(1.5, f(1.5))$	
$(1, f(1))$ and $(1.1, f(1.1))$	
$(1, f(1))$ and $(1.01, f(1.01))$	

D. Create a formula for calculating the slope of any secant line that passes through $(1, f(1))$ and the general point $(x, f(x))$.

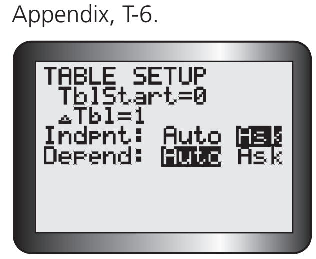
E. Enter this formula into Y1 of the equation editor.

F. Set the TBLSET feature of your graphing calculator by scrolling down and across so that the cursor is over Ask in the Indpnt: row of the screen as shown.

G. Confirm that the first slope you calculated in part C, for the secant line that passes through the points $(1, f(1))$ and $(3, f(3))$, is correct by entering $X = 3$ in the TABLE on your graphing calculator. (If there are already x-values in the table, delete them by moving the cursor over each value and pressing DEL.)

Tech *Support*

For help using the TBLSET and TABLE features of a graphing calculator, see Technical Appendix, T-6.



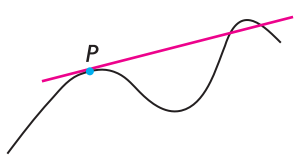
H. On your sketch, draw another secant line that passes through the points $(1, f(1))$ and $(2, f(2))$. Calculate its slope by entering X = 2 in the TABLE on your graphing calculator, and compare this to the slope in the table you created in part F.

I. Draw and calculate three other secant lines, always using $(1, f(1))$ as a fixed point and moving the other points closer to $(1, f(1))$ each time. You can do this by using the points given in the table in part C.

J. Examine your sketch and your table of secant slopes. Describe what happens to each secant line in your sketch, and compare this with the values of the slopes in your table as the points get closer and closer to the fixed point $(1, f(1))$.

tangent line

a line that touches the graph at only one point, *P*, within a small interval of a relation; it could, but does not have to, cross the graph at another point outside this interval. The tangent line goes in the same direction as the relation at point *P* (called the point of tangency).



K. Estimate the slope of the tangent line to the curve $f(x) = x^2$ at the point $(1, f(1))$ by examining the trend in the secant slopes you calculated.

L. Repeat parts B to K using the points in the table below.

Points	Slope of Secant
(1, f(1)) and (−1, f(−1))	
(1, f(1)) and (0, f(0))	
(1, f(1)) and (0.5, f(0.5))	
(1, f(1)) and (0.9, f(0.9))	
(1, f(1)) and (0.99, f(0.99))	

Tech Support

For help using the graphing calculator to draw tangent lines, see Technical Appendix, T-17.

M. Verify your estimates by drawing the tangent line to the graph of $f(x) = x^2$ at $x = 1$ using your graphing calculator.

N. Repeat parts B to L with two other functions of your choice. Use two different types of functions, such as an exponential function, a **sinusoidal function**, or a different quadratic function.

Reflecting

O. What happens to the slopes of the secant lines as the points move closer to the fixed point?

P. How do the slopes of the secant lines relate to the slope of the tangent line when $x = 1$? Explain.

Q. How is estimating the slope of a tangent like estimating the instantaneous rate of change?

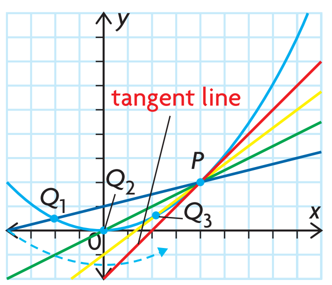
In Summary

Key Ideas

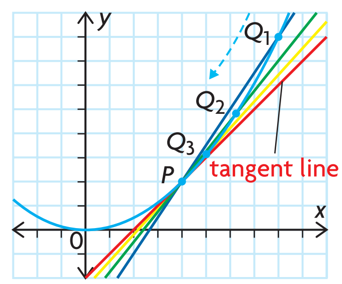
- The slope of a secant line is equivalent to the average rate of change over the interval defined by the *x*-coordinates of the two points that are used to define the secant line.
- The slope of a tangent at a point on a graph is equivalent to the instantaneous rate of change of a function at this point.

Need to Know

- The slope of a tangent cannot be calculated directly using the slope formula, because the coordinates of only one point are known. The slope can be estimated, however, by calculating the slopes of a series of secant lines that go through the fixed point of tangency *P* and points that get closer and closer to this fixed point, Q_1, Q_2, and Q_3.



As *Q* approaches *P* from the left, the slope of *QP* increases and approaches the slope of the tangent line.



As *Q* approaches *P* from the right, the slope of *QP* decreases and approaches the slope of the tangent line.

FURTHER Your Understanding

1. Graph each of the following functions using a graphing calculator, and then sketch the graph. On your sketch, draw a series of secant lines that you could use to estimate the slope of the tangent when $x = 2$. Calculate and record the slopes of these secant lines. Use the slopes to estimate the slope of the tangent line when $x = 2$.

a) $f(x) = 3x^2 - 5x + 1$

b) $f(x) = 3^x + 1$

c) $f(x) = \sqrt{x + 2}$

d) $f(x) = 2x - 7$

2. Verify your estimates for each function in question 1 by drawing the tangent line when $x = 2$ on your graphing calculator.

3. **a)** For each of the following sets of functions, estimate the slopes of the tangents at the given values of x.

b) What do all the slopes in each set of functions have in common?

Set A

$f(x) = -x^2 + 6x - 4$ when $x = 3$

$g(x) = \sin x$ when $x = 90°$

$h(x) = x^2 + 4x + 11$ when $x = -2$

$j(x) = 5$ when $x = 1$

Set B

$f(x) = 3x^2 + 2x - 1$ when $x = 2$

$g(x) = 2^x + 3$ when $x = 1$

$h(x) = 5x + 4$ when $x = 3$

$j(x) = \sin x$ when $x = 60°$

Set C

$f(x) = 3x^2 + 2x - 1$ when $x = -1$

$g(x) = -2^x + 3$ when $x = 0$

$h(x) = -3x + 5$ when $x = 2$

$j(x) = \sin x$ when $x = 120°$

4. The following table gives the temperature of an oven as it heats up.

Time (min)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Temperature (°F)	70	125	170	210	250	280	310	335	360	380	400	415	430	440	445

a) Graph the data.

b) Draw a **curve of best fit** and the tangent line at $x = 5$.

c) Determine the slope of the tangent line using the y-intercept of the tangent line and the point of tangency $(5, 280)$.

d) Estimate the instantaneous rate of change in temperature at exactly 5 min using a centred interval from the table of values.

e) Compare your answers to parts c) and d).

5. In the first two sections of this chapter, you calculated the slopes of successive secant lines to estimate the slope of a tangent line, and you calculated the average rate of change to estimate the instantaneous rate of change. How are these two calculations similar and different?

6. **a)** On graph paper, sketch the graph of $f(x) = x^2$.

b) Draw the secant line that passes through $(1, 2)$ and $(2, 4)$.

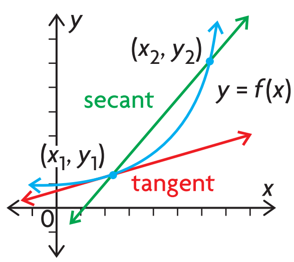
c) Estimate the location of the point of tangency on the graph of $f(x)$ whose tangent line has the same slope as the secant line you drew in part b).

2 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: What is the difference between the average rate of change and the instantaneous rate of change?

A: The average rate of change of a quantity represented by a dependent variable occurs over an interval of the independent variable. The instantaneous rate of change of a quantity represented by a dependent variable occurs at a single value of the independent variable. As a result, average rate of change can be represented graphically using secant lines, while instantaneous rate of change can be represented graphically using tangent lines.



Study *Aid*

- See Lesson 2.1, Example 1.
- Try Mid-Chapter Review Question 1.

Q: How do you determine the average rate of change?

A1: To determine the average rate of change from a table of values or from the equation of any function $y = f(x)$, over the interval between the x-coordinates of points (x_1, y_1) and (x_2, y_2), divide the change in y by the change in x.

$$\text{Average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

A2: To determine the average rate of change from the graph of a function, calculate the slope of the secant line that passes through the two points that define the interval on the graph. The slope is equivalent to the average rate of change on the defined interval.

Study *Aid*

- See Lesson 2.1, Examples 2 to 4.
- Try Mid-Chapter Review Questions 1 and 2.

Study **Aid**

- See Lesson 2.2, Examples 1, 2, and 3, and Lesson 2.3.
- Try Mid-Chapter Review Questions 2 to 5.

Q: How can you estimate the instantaneous rate of change?

A1: Calculate the average rate of change for values that are very close to the location where the instantaneous rate of change occurs. You can use preceding and following intervals, or you can use centred intervals. Use your results to find the trend and then estimate the instantaneous rate of change.

A2: Calculate the average rate of change using the difference quotient with the location where the instantaneous rate of change occurs $(a, f(a))$ and a general point $(a + h, f(a + h))$:

$$\frac{f(a + h) - f(a)}{h}$$

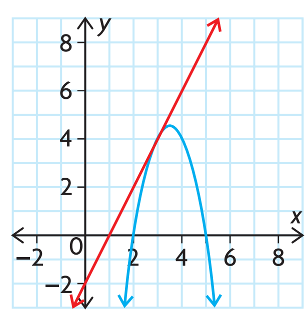
Choose values for h that are very small.

A3: Draw a tangent line at the point where the instantaneous rate of change occurs. Calculate the slope of this line.

For example, to estimate the instantaneous rate of change of $f(x) = -2x^2 + 14x - 20$ at the point $(3, 4)$, graph $f(x)$ and draw a tangent line at $(3, 4)$.

Use the points $(3, 4)$ and $(1, 0)$ on the tangent line to calculate the slope of the tangent line.

$$\text{Slope} = \frac{0 - 4}{1 - 3}$$
$$= 2$$



So the instantaneous rate of change in y with respect to x is about 2.

PRACTICE Questions

Lesson 2.1

1. The following table gives the amount of water that is used on a farm during the first six months of the year.

Month	Volume (1000 of m^3)
January	3.00
February	3.75
March	3.75
April	4.00
May	5.10
June	5.50

a) Plot the data in the table on a graph.
b) Find the rate of change in the volume of water used between consecutive months.
c) Between which two months is the change in the volume of water used the greatest?
d) Determine the average rate of change in the volume of water used between March and June.

Lesson 2.2

2. A city's population (in tens of thousands) is modelled by the function $P(t) = 1.2(1.05)^t$, where t is the number of years since 2000. Examine the equation for this function and its graph.
a) What can you conclude about the average rate of change in population between consecutive years as time increases?
b) Estimate the instantaneous rate of change in population in 2010.

3. The height of a football that has been kicked can be modelled by the function $h(t) = -5t^2 + 20t + 1$, where $h(t)$ is the height in metres and t is the time in seconds.
a) What is the average rate of change in height on the interval $0 \leq t \leq 2$ and on the interval $2 \leq t \leq 4$?
b) Use the information given in part a) to find the time for which the instantaneous rate of change in height is 0 m/s. Verify your response.

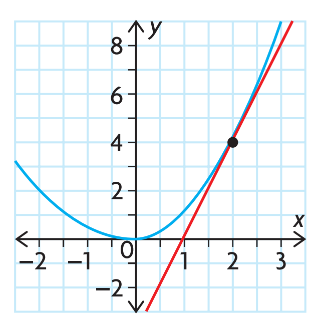
4. The movement of a certain glacier can be modelled by $d(t) = 0.01t^2 + 0.5t$, where d is the distance, in metres, that a stake on the glacier has moved, relative to a fixed position, t days after the first measurement was made. Estimate the rate at which the glacier is moving after 20 days.

5. Create a graphic organizer, such as a web diagram, mind map, or concept map, for rate of change. Include both average rate of change and instantaneous rate of change in your graphic organizer.

Lesson 2.3

6. Create a table to estimate the slope of the tangent to $y = x^3 + 1$ at $P(2, 9)$. Be sure to approach P from both directions.

7. Estimate the slope of the tangent line in the graph of this function.



8. Explain what the answer for question 7 represents.

9. Graph the function $f(x) = 0.5x^2 + 5x - 15$ using your graphing calculator. Estimate the instantaneous rate of change for each value of x.
a) $x = -5$
b) $x = -1$
c) $x = 0$
d) $x = 3$

2.4

Using Rates of Change to Create a Graphical Model

YOU WILL NEED

- graphing calculator or graphing software

GOAL

Represent verbal descriptions of rates of change using graphs.

LEARN ABOUT the Math

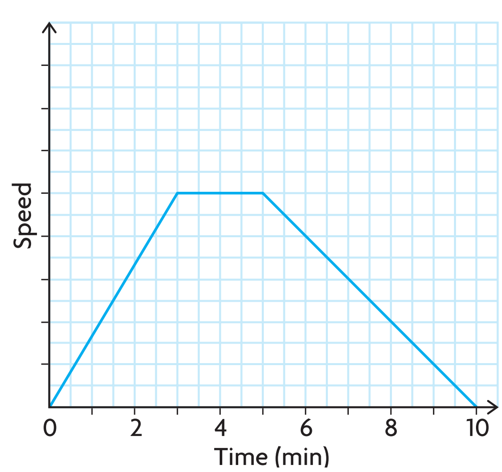
Today Steve walked to his part-time job. As he started walking, he sped up for 3 min. Then he walked at a constant pace for another 2 min. When he realized that he would be early for work, he slowed down. His walk ended and he came to a complete stop once he reached his destination 10 min after he started.

? What would the speed versus time graph of Steve's walk to work look like?

EXAMPLE 1 Representing the situation with a graph

Create a speed versus time graph for Steve's walk to work.

Solution A: Assuming that he changed speed at a constant rate

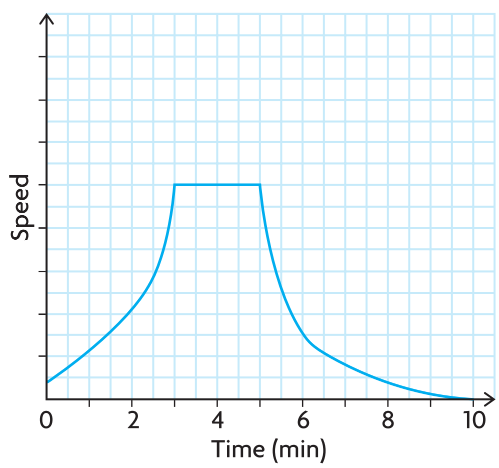


Because Steve was speeding up, his speed increased as time increased. His speed increased at a constant rate, so the graph should be a straight line with a positive slope that begins at (0, 0) and ends at $x = 3$.

Between 3 min and 5 min, Steve walked at the same rate, so his speed did not change. The graph should be a horizontal line that connects to the first line.

After 5 min, Steve slowed down at a constant rate, decreasing his speed as time increased, so you might draw a straight line with a negative slope that begins at $x = 5$ and ends at $x = 10$.

Solution B: Assuming that he walked at a variable speed



Because Steve was speeding up, his speed increased as time increased. His speed increased at a variable rate, so you might draw an increasing curve that starts at (0, 0) and ends at $x = 3$.

Between 3 min and 5 min, Steve walked at the same rate, so his speed did not change. The graph should be a horizontal line that connects to the first line.

After 5 min, Steve slowed down at a variable rate, decreasing his speed as time increased, so you might draw a decreasing curve that begins at $x = 5$ and ends at $x = 10$.

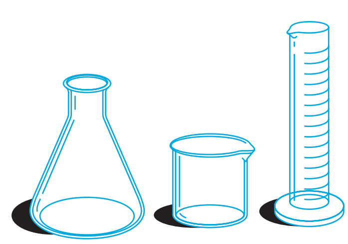
Reflecting

A. Which details in the given description were most important for determining the shape of the graph?

B. Are these the only two graphs that could represent Steve's walk to work? Explain.

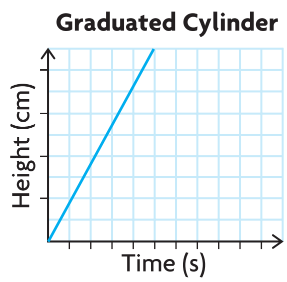
APPLY the Math

EXAMPLE 2 Representing the situation with a graph

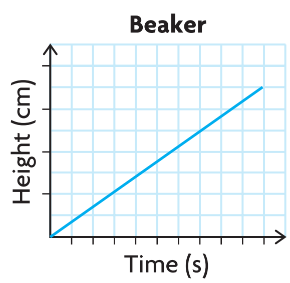


A flask, a beaker, and a graduated cylinder are being filled with water. The rate at which the water flows from the tap is the same when filling all three containers. Draw possible water level versus time graphs for the three containers.

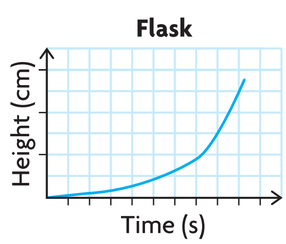
Solution



Since the containers are being filled with water, the height of the water in the containers increases as time increases. All the graphs should be increasing curves. Both the graduated cylinder and the beaker have a constant diameter so the water level increases at a constant rate. The water level will rise the fastest in the container with the smallest diameter.



The water level in the graduated cylinder increases faster than the water level in the beaker, so the slope of the line for the graduated cylinder must be greater than the slope of the line for the beaker.



The diameter of the flask varies, so the water level will increase at different rates. As the water level rises, the diameter of each cross-section gets smaller, causing the water level to increase more rapidly. So the graph must be nonlinear.

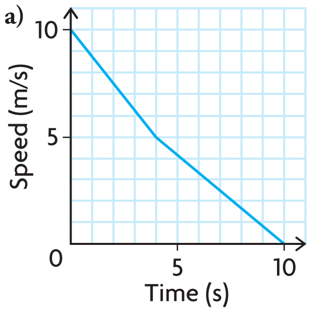
EXAMPLE 3 Using a graph to determine the rate of change

A cyclist is observed moving at a speed of 10 m/s. She begins to slow down at a constant rate and, 4 s later, is at a speed of 5 m/s. She continues to slow down at a different constant rate and finally comes to a stop 6 s later.

a) Sketch a graph of speed versus time.
b) What is the average rate of change of the cyclist's speed in the first 4 s?
c) Estimate the instantaneous rate of change in speed at 3 s.

Solution

a)



The cyclist begins at 10 m/s and slows down at a constant rate. Place a point at (0, 10) and another at (4, 5), and connect the points with a straight line.

Because the cyclist stops at 10 s, her speed is 0 m/s. Place another point at (10, 0), and connect it to the previous point with a straight line.

b) $$\text{Average rate of change} = \frac{5 - 10}{4 - 0} = -1.25$$

To determine the average rate of change of the cyclist's speed in the first 4 s, calculate the slope of the secant line between (0, 10) and (4, 5).

The cyclist's speed is decreasing at the rate of $1.25\ \text{m/s}^2$.

c) The equation of the line for the first part of the graph is $y = -\frac{5}{4}x + 10$, since the slope is $-\frac{5}{4}$ and the y-intercept is 10.

When $x = 3.1$, $y = 6.125$.
When $x = 2.9$, $y = 6.375$.

Substitute $x = 3.1$ and $x = 2.9$ into the equation of the line. These values are close to $x = 3$, but on opposite sides of it. Determine the corresponding y-values.

$$\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{6.125 - 6.375}{3.1 - 2.9} = -1.25\ \text{m/s}^2$$

Calculate the average rate of change between these points, and estimate the instantaneous rate of change based on your answer.

The instantaneous rate of change is the same as the average rate of change calculated in part b). This should not be surprising, since the tangent line at $x = 3$ has the same slope as the secant line on the interval $0 \leq x \leq 4$.

Recall that the rate of change of a linear function is constant, so the rate of change at 3 s will be same as the rate of change at any time between 0 s and 4 s.

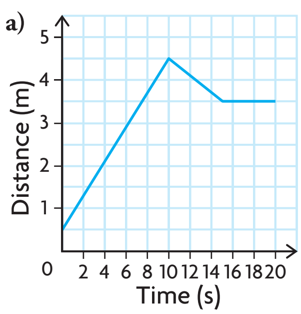
EXAMPLE 4 Using reasoning to represent and analyze a situation

Adam and his friend are testing a motion sensor. Adam stands 0.5 m in front of the sensor and then walks 4 m away from it at a constant rate for 10 s. Next, Adam walks 1 m toward the sensor for 5 s and then waits there for another 5 s.

a) Draw a distance versus time graph for Adam's motion sensor walk.
b) What is the average rate of change in his distance in the first 10 s?
c) What are the instantaneous rates of change at $t = 1$ s and $t = 7$ s?
d) What is the average rate of change in the next 5 s?
e) What are the instantaneous rates of change at $t = 12$ s and $t = 14$ s?
f) What is the instantaneous rate of change at $t = 18$ s?
g) Draw a speed versus time graph for Adam's motion sensor walk.

Solution

a)



The graph begins with a straight line since the rate at which Adam walks is constant. The graph has a positive slope since he walks away from the sensor, and his distance from the sensor increases as time increases.

Adam starts 0.5 m from the sensor. Use (0, 0.5) as the distance intercept.

Adam walks 4 m away from the sensor at a constant rate for 10 s, so use the point (10, 4.5).

Adam then walks toward the sensor. The line has a negative slope, because his distance from the sensor decreases as time increases. The line is not very steep because he is walking slowly.

The graph ends with a horizontal line that has a slope of 0 because Adam is not moving. The slope indicates that his distance from the sensor does not change.

b) Average rate of change

$= \text{slope of secant}$

$$= \frac{4.5 - 0.5}{10 - 0}$$

$= 0.4$

Calculate the slope of the secant line between the points (0, 0.5) and (10, 4.5).

Adam's distance from the sensor is increasing, on average, by 0.4 m/s.

c) Instantaneous rate of change
$= \text{slope of tangent}$
$= 0.4$
Adam's distance from the sensor is increasing. He is moving away from the sensor at a rate of 0.4 m/s.

Estimate the slopes of the tangent lines at 1 s and 7 s. Both of the tangent lines have the same slope as the secant line since the graph is linear on the interval $0 \leq t \leq 10$.

d) Average rate of change
$= \text{slope of secant}$
$= \frac{3.5 - 4.5}{15 - 10}$
$= -0.2$
Adam's distance from the sensor is decreasing. He is moving toward the sensor at a rate of 0.2 m/s.

Calculate the slope of the secant line between the points (10, 4.5) and (15, 3.5).

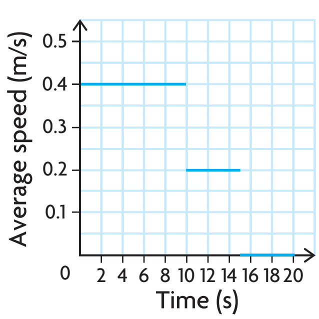
e) Instantaneous rate of change
$= \text{slope of tangent}$
$= -0.2$
Adam's distance from the sensor is decreasing by 0.2 m/s at 12 s and 14 s.

Estimate the slope of the tangent lines at 12 s and 14 s. As in part c), both of these tangent lines have the same slope as the secant line since the graph is linear on the interval $10 < t \leq 15$.

f) Instantaneous rate of change
$= \text{slope of tangent}$
$= 0$
Adam's distance from the sensor is not changing at 18 s.

Estimate the slope of the tangent line at 18 s. Again, the tangent line has the same slope as the secant line since the graph is linear on the interval $15 < t \leq 20$. Since the line is horizontal, its slope is 0.

g)



There are three different speeds at which Adam walks, over three different intervals of time. Using the previous calculations,
Speed $= 0.4$ m/s when $0 \leq t \leq 10$
Speed $= 0.2$ m/s when $10 < t \leq 15$
Speed $= 0$ m/s when $15 < t \leq 20$
Note that speed is a non-negative quantity.
Speed $= \left|\frac{\Delta d}{\Delta t}\right|$

In Summary

Key Ideas

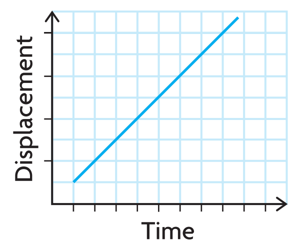
- In a problem that involves movement, a possible graph shows **displacement** (distance, height, or depth) versus time. Distance, height, or depth is the dependent variable, and time is the independent variable. The rate of change in these relationships is speed:

  $$\text{Speed} = \left|\frac{\text{change in displacement}}{\text{change in time}}\right|, \quad S = \left|\frac{\Delta d}{\Delta t}\right|$$

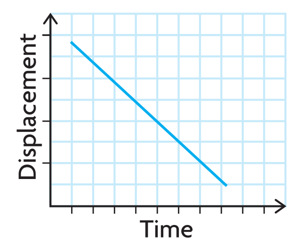
- On a displacement (distance, height, or depth) versus time graph, the magnitude of the slope of a secant line represents the average speed on the corresponding interval. The magnitude of the slope of a tangent line represents the instantaneous speed at a specific point. As a result, observing how the slopes of tangent lines change at different points on a graph gives you insight into how the speed changes over time.

Need to Know

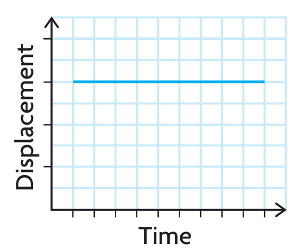
- When the rate of change of displacement (or speed) is constant:



An increasing line indicates that displacement increases as time increases.



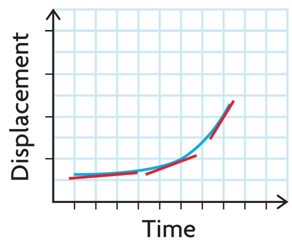
A decreasing line indicates that displacement decreases as time increases.



A horizontal line indicates that there is no change in displacement as time increases.

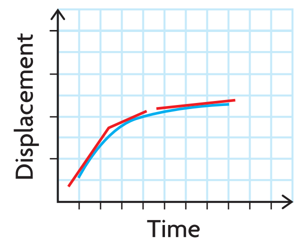
(*continued*)

- When the rate of change of displacement (or speed) is variable, an increasing curve indicates that displacement increases as time increases.



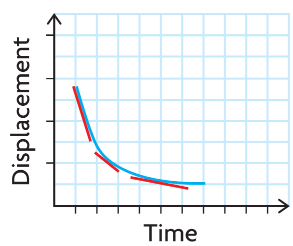
The speed is increasing as the time increases.

or



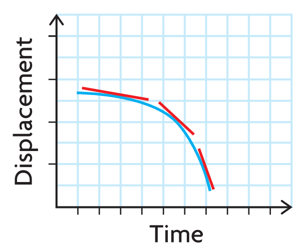
The speed is decreasing as the time increases.

- A decreasing curve indicates that displacement decreases as time increases.



The speed is decreasing as the time increases.

or

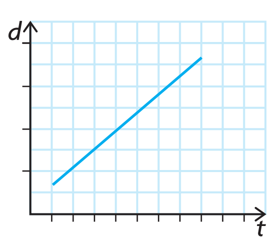


The speed is increasing as the time increases.

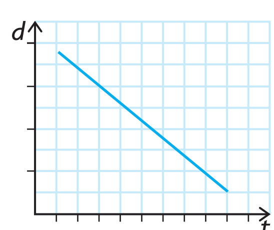
***CHECK** Your Understanding*

1. The following graphs show distance versus time. Match each graph with the description given below.

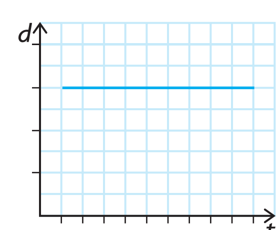
a)



b)



c)



A Distance is decreasing over time.

B There is no change in distance over time.

C Distance is increasing over time.

2. Which of the graphs in question 1 show that the speed is constant? Explain.

3. Jan stands 5 m away from a motion sensor and then walks 4 m toward it at a constant rate for 5 s. Then she walks 2 m away from the location where she changed direction at a variable rate for the next 3 s. She stops and waits at this location for 2 s. Draw a distance versus time graph to show Jan's motion sensor walk.

PRACTISING

4. Rachel climbed Mt. Fuji while in Japan. There are 10 levels to the mountain. She was able to drive to Level 5, where she began her climb.

- She walked at a constant rate for 40 min from Level 5 to Level 6.
- She slowed slightly but then continued at a constant rate for a total of 90 min from Level 6 to Level 7.
- She decided to eat and rest there, which took approximately 2 h.
- From Level 7 to Level 8, a 40 min trip, she travelled at a constant rate.
- Continuing on to Level 9, a 45 min trip, she decreased slightly to a new constant rate.
- During most of the 1 h she took to reach Level 10, the top of Mt. Fuji, she maintained a constant rate. As she neared the top, however, she accelerated.

a) Using the information given and the following table, draw an elevation versus time graph to describe Rachel's climb.

Level	5	6	7	8	9	10
Elevation (m)	2100	2400	2700	3100	3400	3740

b) Calculate Rachel's average speed over each part of her climb.

c) Draw a speed versus time graph to describe Rachel's climb.

5. K The containers shown are being filled with water at a constant rate. Draw a graph of the water level versus time for each container.

a) a 2 L plastic pop bottle **b)** a vase

6. John is riding a bicycle at a constant cruising speed along a flat road. He slows down as he climbs a hill. At the top of the hill, he speeds up, back to his constant cruising speed on a flat road. He then accelerates down the hill. He comes to another hill and coasts to a stop as he starts to climb.

a) Sketch a possible graph to show John's speed versus time, and another graph to show his distance travelled versus time.

b) Sketch a possible graph of John's elevation (in relation to his starting point) versus time.

7. **A** A swimming pool is 50 m long. Kommy swims from one end of the pool to the other end in 45 s. He rests for 10 s and then takes 55 s to swim back to his starting point.

a) Use the information given to find the average speed for Kommy's first length of the pool.

b) What is the average speed for Kommy's second length of the pool?

c) If you were to graph Kommy's distance versus time for his first and second lengths of the pool, how would the two graphs compare? How is this related to Kommy's speed?

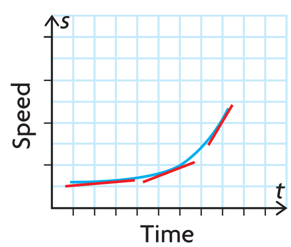
d) Draw a distance versus time graph for Kommy's swim.

e) What is Kommy's speed at time $t = 50$?

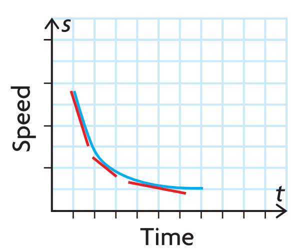
f) Draw a speed versus time graph for Kommy's swim.

8. The following graphs show speed versus time. Match each graph with the description given below.

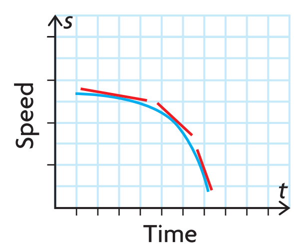
a)



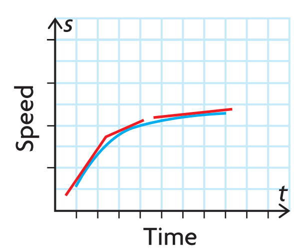
b)



c)



d)



A The rate at which the speed increases is increasing as time increases.

B The rate at which the speed increases is decreasing as time increases.

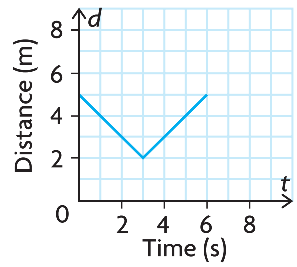
C The rate at which the speed decreases is decreasing as time increases.

D The rate at which the speed decreases is increasing as time increases.

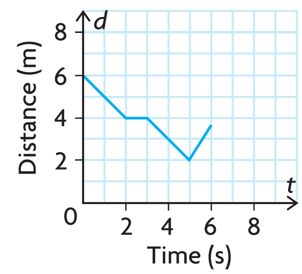
9. **T** A jockey is warming up a horse. Whenever the jockey has the horse accelerate or decelerate, she does so at a nonconstant rate—at first slowly and then more quickly. The jockey begins by having the horse trot around the track at a constant rate. She then increases the rate to a canter and allows the horse to canter at a constant rate for several laps. Next, she slowly begins to decrease the speed of the horse to a trot and then to a walk. To finish, the jockey walks the horse around the track once. Draw a speed versus time graph to represent this situation.

10. C **a)** Describe how you would walk toward or away from a motion sensor detector to give each distance versus time graph shown below.

i)



ii)



b) For each part of each graph, determine the speed at which you must walk.

11. A cross-country runner is training for a marathon. His training program requires him to run at different speeds for different lengths of time. His program also requires him to accelerate and decelerate at a constant rate. Today he begins by jogging for 10 min at a rate of 5 miles per hour. He then spends 1 min accelerating to a rate of 10 miles per hour. He stays at this rate for 5 min. He then decelerates for 1 min to a rate of 7 miles per hour. He stays at this rate for 30 min. Finally, to cool down, he decelerates for 2 min to a rate of 3 miles per hour. He stays at this rate for a final 10 min and then stops.

a) Make a speed versus time graph to represent this situation.

b) What is the instantaneous rate of change in the runner's speed at 10.5 min?

c) Calculate the runner's average rate at which he changed speeds from minute 11 to minute 49.

d) Explain why your answer for part c) does not accurately represent the runner's training schedule from minute 11 to minute 49.

12. Create a scenario that could be used to create either a distance versus time graph or a speed versus time graph. Exchange your scenario with a partner and create the corresponding graph.

Extending

13. Two women are running on the same track. One has just finished her workout and is decelerating—at first slowly and then more quickly as she comes to a complete stop. The other woman is just starting her workout and is accelerating—at first quickly and then more slowly as she reaches her target speed. Use one graph to illustrate the rates of both women.

14. A graph displays changes in rate of speed versus time. The graph has straight lines from point to point. If the graph had been drawn to display changes in distance versus time, how would it be different?

2.5 Solving Problems Involving Rates of Change

GOAL

Use rates of change to solve problems that involve functions.

YOU WILL NEED

- dynamic geometry software, spreadsheet, or graphing calculator

INVESTIGATE the Math

A theatre company's profit $P(x)$, in dollars, is described by the equation $P(x) = -60x^2 + 1800x + 16\,500$, where x is the cost of a ticket in dollars.

? What ticket price will give the maximum profit?

A. Calculate the average rate of change in profit for each interval of ticket prices:

$12 \le x \le 15$	$15 \le x \le 17$
$14 \le x \le 15$	$15 \le x \le 16$
$14.5 \le x \le 15$	$15 \le x \le 15.5$
$14.8 \le x \le 15$	$15 \le x \le 15.2$

B. What do all the values for the first four rate of change calculations have in common? The last four?

C. Use your results to estimate the instantaneous rate of change when $x = 15$.

D. Graph the profit function. Where does the maximum occur on your graph and what ticket price gives this maximum profit?

Reflecting

E. What is the relationship between the instantaneous rate of change in profit and the cost of a ticket at the point where the maximum profit occurs? How do you know?

F. How else could you use your graph and your knowledge of rates of change to verify that a maximum occurs at this point?

G. What would the tangent line look like at the point where a maximum occurs on your graph?

H. Explain how you could use tangents and rates of change to identify the value where a minimum occurs on a graph.

APPLY the Math

EXAMPLE 1 Selecting a strategy to identify the location of a minimum value

Leonard is riding a Ferris wheel. Leonard's elevation $h(t)$, in metres above the ground at time t in seconds, can be modelled by the function $h(t) = 5\cos(4(t - 10)^\circ) + 6$. Shu thinks that Leonard will be closest to the ground at 55 s. Do you agree? Support your answer.

Solution

Using $t = 54$ and $t = 55$,
Average rate of change in elevation

$$= \frac{h(55) - h(54)}{55 - 54}$$

$$\doteq \frac{1 - 1.0122}{1}$$

$$= -0.0122 \text{ m/s}$$

Estimate the instantaneous rate of change in height near $t = 55$. If it is equal to zero, then a minimum could happen there.

Using $t = 56$ and $t = 55$,
Average rate of change in elevation

$$= \frac{h(56) - h(55)}{56 - 55}$$

$$\doteq \frac{1.0122 - 1}{1}$$

$$= 0.0122 \text{ m/s}$$

A minimum could occur at $t = 55$.
$h(54) = 1.0122$, $h(55) = 1$, and $h(56) = 1.0122$

Since the rate of change in height using a point to the left of $t = 55$ is negative, and using a point to the right of $t = 55$ is positive, and since both are close to 0, the instantaneous rate of change could be zero at $t = 55$.

Since the estimate of the instantaneous rate of change at $t = 55$ is zero, and since $h(55)$ is less than both $h(54)$ and $h(56)$, Leonard is closest to the ground at $t = 55$ s, just as Shu predicted.

Check that $h(55)$ is really lower than $h(54)$ and $h(56)$ to be sure that a minimum occurs at $t = 55$.

EXAMPLE 2 Selecting an algebraic strategy to identify the location of a minimum value

Show that the minimum value for the function $f(x) = x^2 + 4x - 21$ happens when $x = -2$.

Solution

Estimate the slope of the tangent to the curve when $a = -2$ by writing an equation for the slope of any secant line on the graph of $f(x)$.

$$\begin{aligned} m &= \frac{f(-2 + h) - f(-2)}{h} \\ &= \frac{(-2 + h)^2 + 4(-2 + h) - 21 - (-25)}{h} \\ &= \frac{4 - 4h + h^2 - 8 + 4h + 4}{h} \\ &= \frac{h^2}{h} \\ &= h \end{aligned}$$

To estimate the slope of the tangent at $x = -2$, use two points and the difference quotient. For one point, use $(-2, -25)$, the point where you want the tangent line to be. For the other point, use the general point on the graph of $f(x)$, for example, $(-2 + h, f(-2 + h))$.

To estimate the slope of the tangent to the curve when $x = -2$, replace h with small values.

When $h = -0.01$, $m = -0.01$.
When $h = 0.01$, $m = 0.01$.

Since the slope of the tangent is close to 0, there could be a minimum value at $x = -2$.

Take the average of these rates of change to improve your estimate of the instantaneous rate of change at $x = -2$.

$$\begin{aligned} \text{Instantaneous rate of change} &= \frac{0.01 + (-0.01)}{2} \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(-1.9) &= -24.99 \\ f(-2) &= -25 \\ f(-2.1) &= -24.99 \end{aligned}$$

Use two values of x that are close to -2, but on opposite sides of it, to calculate values for the function. Compare these values for the function to the value at $x = -2$.

Since the slope of the tangent is equal to zero when $x = -2$, and since the values of the function when $x = -2.1$ and $x = -1.9$ are greater than the value when $x = -2$, a minimum value occurs at $x = -2$.

EXAMPLE 3 Selecting a strategy that involves instantaneous rate of change to solve a problem

Tim has a culture of 25 bacteria that is growing at a rate of 15%/h. He observes the culture for 12 h. During this time period, when is the instantaneous rate of change the greatest?

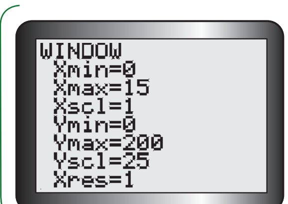
Solution

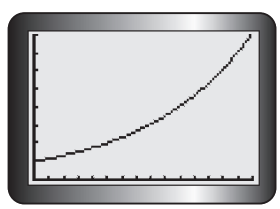
Determine an algebraic model for the situation.

$P(t) = 25(1.15)^t$

This situation involves exponential growth. The algebraic model will be in the form $y = ab^x$, where a represents the initial size of the population, 25, and b is 1 + the growth rate, which is 15%.

Graph the algebraic model over the given domain, $0 \leq t \leq 12$, using a suitable window setting.



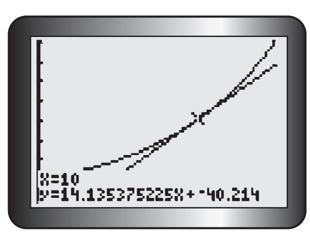


It looks like the instantaneous rate of change is the greatest near the end of the time period, because the graph is increasing faster then.

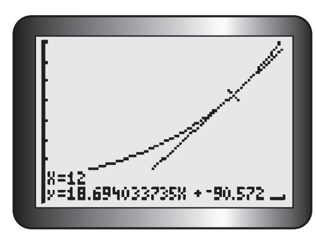
Estimate the instantaneous rate of change at $t = 10$ and $t = 12$ by drawing tangent lines at each of these points.

Tech Support

For help using the graphing calculator to draw tangent lines, see Technical Appendix, T-17.



At $x = 10$, the slope of the tangent line is about 14.1. So here the bacteria population is increasing by about 14 bacteria per hour.



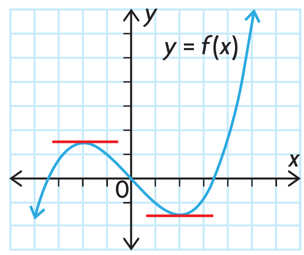
At $x = 12$, the slope of the tangent line is about 18.7. At this point, the bacteria population is increasing by about 19 bacteria per hour.

The instantaneous rate of change is the greatest at 12 h.

In Summary

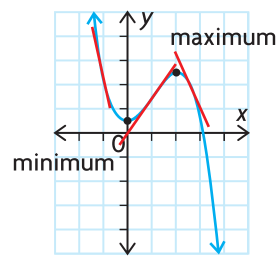
Key Idea

- The instantaneous rate of change is zero at both a maximum point and a minimum point. As a result, the tangent lines drawn at these points will be horizontal lines.



Need to Know

- If the instantaneous rate of change is negative before the value where the rate of change is zero and positive after this value, then a minimum occurs. Graphically, the tangent lines must have a negative slope before the minimum point and a positive slope after.
- If the instantaneous rate of change is positive before the value where the rate of change is zero and negative after this value, then a maximum occurs. Graphically, the tangent lines must have a positive slope before the maximum point and a negative slope after.



CHECK Your Understanding

1. The cost of running an assembly line can be modelled by the function $C(x) = 0.3x^2 - 0.9x + 1.675$, where $C(x)$ is the cost per hour in thousands of dollars and x is the number of items produced per hour in thousands. The most economical production level occurs when 1500 items are produced. Verify this using the appropriate calculations for rate of change in cost.

2. For a person at rest, the function $P(t) = -20\cos(300°t) + 100$ models blood pressure, in millimetres of mercury (mm Hg), at time t seconds. What is the rate of change in blood pressure at 3 s?

3. If a function has a maximum value at $(a, f(a))$, what do you know about the slopes of the tangent lines at the following points?
 a) points to the left of, and very close to, $(a, f(a))$
 b) points to the right of, and very close to, $(a, f(a))$

4. If a function has a minimum value at $(a, f(a))$, what do you know about the slopes of the tangent lines at the following points?
 a) points to the left of, and very close to, $(a, f(a))$
 b) points to the right of, and very close to, $(a, f(a))$

PRACTISING

5. **K** For each function, the point given is the maximum or minimum. Use the difference quotient to verify that the slope of the tangent at this point is zero.

a) $f(x) = 0.5x^2 + 6x + 7.5$; $(-6, -10.5)$

b) $f(x) = -6x^2 + 6x + 9$; $(0.5, 10.5)$

c) $f(x) = 5 \sin(x)$; $(90°, 5)$

d) $f(x) = -4.5 \cos(2x)$; $(0°, -4.5)$

6. Use an algebraic strategy to verify that the point given for each function is either a maximum or a minimum.

a) $f(x) = x^2 - 4x + 5$; $(2, 1)$

b) $f(x) = -x^2 - 12x + 5.75$; $(-6, 41.75)$

c) $f(x) = x^2 - 9x$; $(4.5, -20.25)$

d) $f(x) = 3 \cos(x)$; $(0°, 3)$

e) $f(x) = x^3 - 3x$; $(-1, 2)$

f) $f(x) = -x^3 + 12x - 1$; $(2, 15)$

7. **A** A pilot who is flying at an altitude of 10 000 feet is forced to eject from his airplane. The path that his ejection seat takes is modelled by the equation $h(t) = -16t^2 + 90t + 10\,000$, where $h(t)$ is his altitude in feet and t is the time since his ejection in seconds. Estimate at what time, t, the pilot is at a maximum altitude. Explain how the maximum altitude is related to the slope of the tangent line at certain points.

8. **a)** Graph each function using a graphing calculator. Then find the minimum or maximum point for the function.

i) $f(x) = x^2 + 10x - 15$

ii) $f(x) = -3x^2 + 45x + 16$

iii) $f(x) = 4x^2 - 26x - 3$

iv) $f(x) = -0.5x^2 + 6x + 0.45$

b) Draw tangent lines on either side of the points you found in part a).

c) Explain how the tangent lines you drew confirm the existence of the minimum or maximum points you found in part a).

9. **T** **a)** Find the maximum *and* minimum values for each exponential growth or decay equation on the given interval.

i) $y = 100(0.85)^t$, for $0 \leq t \leq 5$

ii) $y = 35(1.15)^x$, for $0 \leq x \leq 10$

b) Examine your answers for part a). Use your answers to hypothesize about where the maximum value will occur in a given range of values, $a \leq x \leq b$. Explain and support your hypothesis thoroughly.

10. The height of a diver above the water is modelled by the function $h(t) = -5t^2 + 5t + 10$, where t represents the time in seconds and $h(t)$ represents the height in metres. Use the appropriate calculations for the rate of change in height to show that the diver reaches her maximum height at $t = 0.5$ s.

11. The top of a flagpole sways back and forth in high winds. The function $f(t) = 8 \sin (180°t)$ represents the displacement, in centimetres, that the flagpole sways from vertical, where t is the time in seconds. The flagpole is vertical when $f(t) = 0$. It is 8 cm to the right of its resting place when $f(t) = 8$, and 8 cm to the left of its resting place when $f(t) = -8$. If the flagpole is observed for 2 s, it appears to be farthest to the left when $t = 1.5$ s. Is this observation correct? Justify your answer using the appropriate calculations for the rate of change in displacement.

12. C The weekly revenue for battery sales at Discount H hardware store can be modelled by the function $R(x) = -x^2 + 10x + 30\,000$, where revenue, R, and the cost of a package of batteries, x, are in dollars. The maximum revenue occurs when a package of batteries costs \$5. Write detailed instructions, using the appropriate calculations for the rate of change in revenue, to verify that the maximum revenue occurs when a package of batteries costs \$5. Exchange instructions with a partner. Follow your partner's instructions to verify when the maximum revenue occurs.

Extending

13. Explain how to determine the value of x that gives a maximum for a transformed sine function in the form $y = a \sin (k(x - d)) + c$, if the maximum for $y = \sin x$ occurs at $(90°, 1)$.

14. The speedometer in a car shows the vehicle's instantaneous velocity, or rate of change in position, at any moment. Every 5 s, Myra records the speedometer reading in a vehicle driven by a friend. She then plots these values. When Myra begins considering rates of change shown on her graph, what quantity is she looking at? Explain what different scenarios on Myra's graph mean, such as, her graph is increasing, but the rate of change between points on her graph is decreasing.

15. Estimate the instantaneous rate of change for $f(x) = x^2$ at $x = -2, -1, 2$, and 3. Does there appear to be a rule for determining the instantaneous rate of change for the function at given values of x? If so, state the rule. Repeat for $f(x) = x^3$.

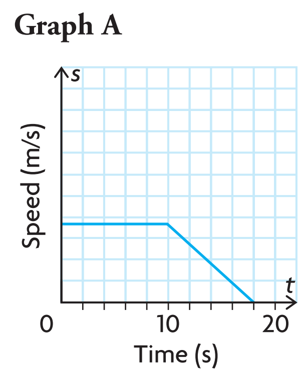
2 Chapter Review

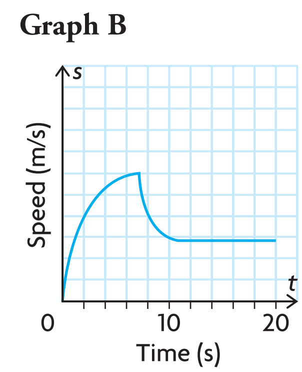
FREQUENTLY ASKED Questions

Study *Aid*

- See Lesson 2.4, Examples 1, 2, and 3.
- Try Chapter Review Questions 8, 9, and 10.

Q: What descriptions could be given to produce the following speed versus time graphs? Explain.





A: Graph A: A person walks at the same rate for 10 s and then slows down and comes to a stop at 18 s. This is shown in the graph because the horizontal line means that the person is walking at the same rate, and the straight line with a negative slope means that the person is slowing down at a constant rate.

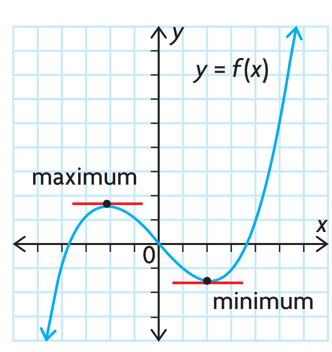
Graph B: A person walks, increasing speed at a variable rate for 8 s and then decreasing speed at a variable rate. From 11 s to 20 s, the person walks at the same rate.

Study *Aid*

- See Lesson 2.5, Examples 1, 2, and 3.
- Try Chapter Review Questions 11 and 13.

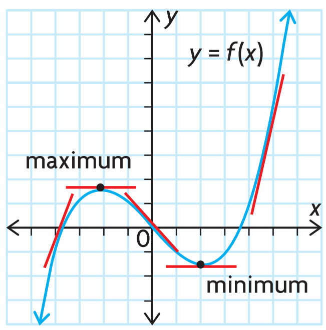
Q: How can you verify, for a given value of the independent variable, where a maximum or minimum occurs using rate of change calculations?

A: Check to see if the instantaneous rate of change is equal to zero at any point where a maximum or minimum might occur. If it does, then a maximum or minimum could occur there. Graphically, the tangent line must be horizontal at this point.



If the instantaneous rate of change is positive before the point where the rate of change is zero, and negative after, then a maximum occurs. Graphically, the tangent lines must have a positive slope before the maximum point and a negative slope after.

If the instantaneous rate of change is negative before the point where the rate of change is zero, and positive after, then a minimum occurs. Graphically, the tangent lines must have a negative slope before the minimum point and a positive slope after.



Q: When solving problems that require you to estimate the value for the instantaneous rate of change in a relationship at a specific point, what does the sign of this estimated value indicate?

A: The sign of the estimated value of the instantaneous rate of change gives you information about what is happening to the values of the dependent variable in the relationship at that exact point in time. If the instantaneous rate of change is positive (indicated graphically by a tangent line that rises from left to right), then the values for the dependent variable are increasing. If the instantaneous rate of change is negative (indicated graphically by a tangent line that falls from left to right), then the values for the dependent variable are decreasing.

Q: Can the difference quotient $\frac{f(a + h) - f(a)}{h}$ be used to determine both average and instantaneous rates of change?

A: Yes. For any function $y = f(x)$, the difference quotient provides a formula for calculating the average rate of change between two points $(a, f(a))$ and $(a + h, f(a + h))$. In both the case of average and instantaneous rate of change, h is the difference between the values for the independent variable that define the interval on which the rate of change is being calculated. In the case of instantaneous rate of change, h is made arbitrarily small so that this interval is close to 0. The calculation approximates the instantaneous rate of change when two points on $y = f(x)$ are chosen that are very, very close to each other.

Study Aid

- See Lesson 2.5, Example 2.
- Try Chapter Review Question 12.

PRACTICE Questions

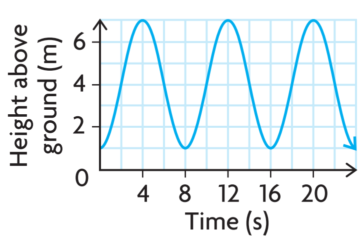
Lesson 2.1

1. The following table shows the daily number of watches sold at a shop and the amount of money made from the sales.

Number of Watches (w)	Revenue (r) ($)
25	437.50
17	297.50
20	350.00
12	210.00
24	420.00

a) Does the data in the table appear to follow a linear relation? Explain.
b) Graph the data. How does the graph compare with your hypothesis?
c) What is the average rate of change in revenue from $w = 20$ to $w = 25$?
d) What is the cost of one watch, and how does this cost relate to the graph?

2. The graph shows the height above the ground of a person riding a Ferris wheel.



a) Calculate the average rate of change in height on the interval [0, 4].
b) Calculate the average rate of change in height on the interval [4, 8].
c) Discuss the similarities and differences in your answers to parts a) and b).

3. A company is opening a new office. The initial expense to set up the office is $10 000, and the company will spend another $2500 each month in utilities until the new office opens.
a) Write the equation that represents the company's total expenses in terms of months until the office opens.
b) What is the average rate of change in the company's expenses from $3 \leq m \leq 6$?
c) Do you expect this rate of change to vary? Why or why not?

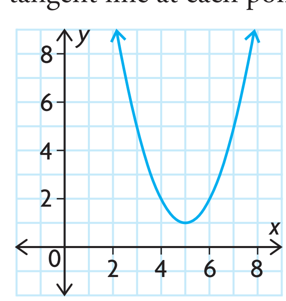
Lesson 2.2

4. An investment's value, $V(t)$, is modelled by the function $V(t) = 2500(1.15)^t$, where t is the number of years after funds are invested.
a) To find the instantaneous rate of change in the value of the investment at $t = 4$, what intervals on either side of 4 would you choose? Why?
b) Use your intervals from part a) to find the instantaneous rate of change in the value of the investment at $t = 4$.

5. The height, in centimetres, of a piston attached to a turning wheel at time t, in seconds, is modelled by the equation $y = 2 \sin(120°t)$.
a) Examine the equation, and select a strategy for finding the instantaneous rate of change in the piston's height at $t = 12$ s.
b) Use your strategy from part a) to find the instantaneous rate of change at $t = 12$ s.

Lesson 2.3

6. For the graph shown, estimate the slope of the tangent line at each point.



a) (4, 2)
b) (5, 1)
c) (7, 5)

7. Use a graphing calculator to graph the equation $y = 5x^2 + 3x + 7$. Then use your calculator to estimate the instantaneous rate of change for each value of x.

a) $x = -4$
b) $x = -2$
c) $x = -0.3$
d) $x = 2$

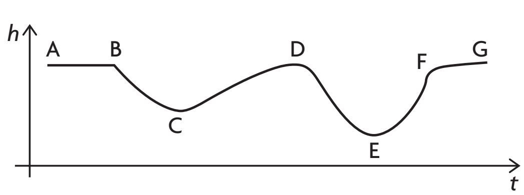
Lesson 2.4

8. A sculptor makes a vase for flowers. The radius and circumference of the vase increase as the height of the vase increases. The vase is filled with water. Draw a possible graph of the height of the water as time increases.

9. A newspaper carrier delivers papers on her bicycle. She bikes to the first neighbourhood at a rate of 10 km/h. She slows down at a constant rate over a period of 7 s, to a speed of 5 km/h, so that she can deliver her papers. After travelling at this rate for 3 s, she sees one of her customers and decides to stop. She slows at a constant rate until she stops. It takes her 6 s to stop.

a) Draw a graph of the newspaper carrier's rate over time for the time period after she arrives at the first neighbourhood.
b) What is the average rate of change in speed over the first 7 s?
c) What is the average rate of change in speed from second 7 to 12 seconds.
d) What is the instantaneous rate of change in speed at 12 s?

10. The graph shows the height of a roller coaster versus time. Describe how the vertical speed of the roller coaster will vary as it travels along the track from A to G. Sketch a graph to show the vertical speed of the roller coaster.



Lesson 2.5

11. A maximum or minimum is given for each of the following functions. Select a strategy, and verify whether the point given is a maximum or a minimum.

a) $f(x) = x^2 - 10x + 7$; $(5, -18)$
b) $g(x) = -x^2 - 6x - 4$; $(-3, 5)$
c) $h(x) = -2x^2 + 68x + 75$; $(17, 653)$
d) $j(x) = \sin(-2x)$; $(45°, -1)$
e) $k(x) = -4\cos(x + 25)$; $(-25°, -4)$
f) $m(x) = \frac{1}{20}(x^3 + 2x^2 - 15x)$; $\left(-3, \frac{9}{5}\right)$

12. a) For each function, find the equation for the slope of the secant line between any general point on the function $(a + h, f(a + h))$ and the given x-coordinate of another point.

i) $f(x) = x^2 - 30x$; $a = 2$
ii) $g(x) = -4x^2 - 56x + 16$; $a = -1$

b) Use each slope equation you found in part a) to estimate the slope of the tangent line at the point with the given x-coordinate.

13. a) Explain how the instantaneous rates of change differ on either side of a maximum point of a function.

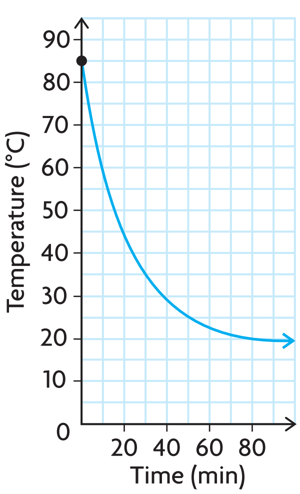
b) Explain how the instantaneous rates of change differ on either side of a minimum point of a function.

14. a) Use graphing technology to graph $f(x) = x^4 - 2x^2$.

b) Use the graph to estimate the locations of the maximum and minimum values of this function.
c) Explain how tangent lines can be used to verify the locations you identified in part b).
d) Confirm your estimates by using the maximum and minimum operations on the graphing calculator.

2 Chapter Self-Test

1. A speedboat driver is testing a new boat. He begins the test by steadily increasing the boat's speed until he reaches 3 kn (knots) over a period of 1 min. Because he is in a no-wake zone, he stays at this speed for 5 min. After leaving the no-wake zone, he steadily increases the speed of the boat to 25 kn over a period of 2 min. He stays at this speed for 5 min and then increases the speed of the boat to 45 kn over a period of 1 min. After staying at this speed for 5 min, he decelerates the boat at a steady rate over a period of 4 min until he comes to a stop.
   a) Draw a graph of the boat's speed versus time. Remember to label your data points.
   b) What is the average rate of change in speed from $t = 6$ to $t = 8$ and from $t = 8$ to $t = 13$? How are the two rates different? What does this tell you about the speed of the boat during these two intervals of time?
   c) What is the instantaneous rate of change in speed at $t = 7$?



2. A cup of hot cocoa left on a desk in a classroom had its temperature measured once every minute. The graph shows the relationship between the temperature of the cocoa, in degrees Celsius, and time, in minutes.
   a) Determine the slope of the secant line that passes through the points (5, 70) and (50, 25).
   b) What does the answer to part a) mean in this context?
   c) Estimate the slope of the tangent line at the point (30, 35).
   d) What does the answer to part b) mean in this context?
   e) Discuss what happens to the rate at which the cup of cocoa cools over the 90 min period.

3. The profit $P(x)$ of a cosmetics company, in thousands of dollars, is given by $P(x) = -5x^2 + 400x - 2550$, where x is the amount spent on advertising in thousands of dollars.
   a) Calculate the average rate of change in profit on the interval $8 \leq x \leq 10$.
   b) Estimate the instantaneous rate of change in profit when $x = 50$.
   c) Discuss the significance of the signs in your answers to parts a) and b).

4. Estimate the instantaneous rate of change for each function at each point given. Identify any point that is a maximum/minimum value.
   a) $h(p) = 2p^2 + 3p$; $p = -1, -0.75$, and 1
   b) $k(x) = -0.75x^2 + 1.5x + 13$; $x = -2$, 4, and 1