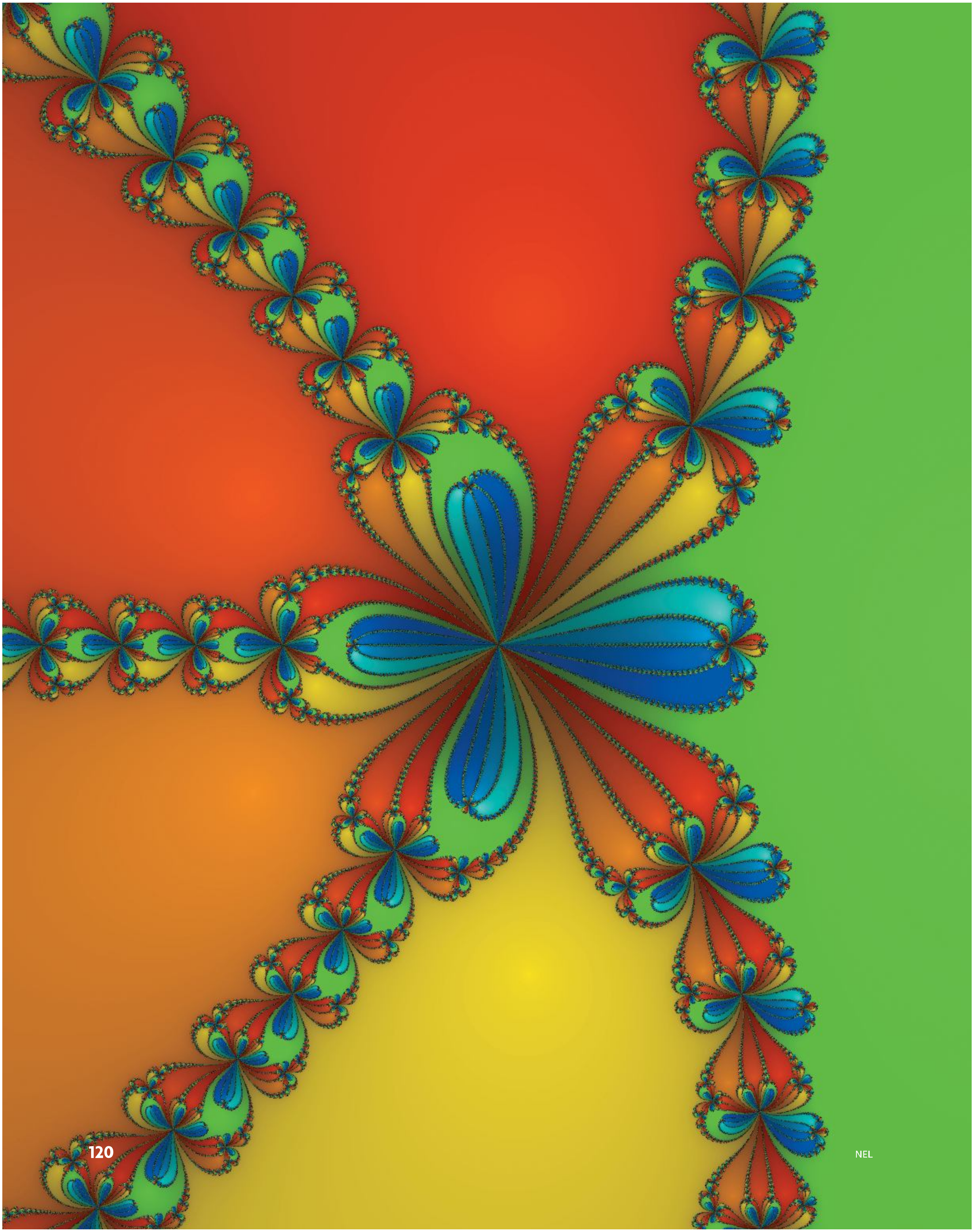
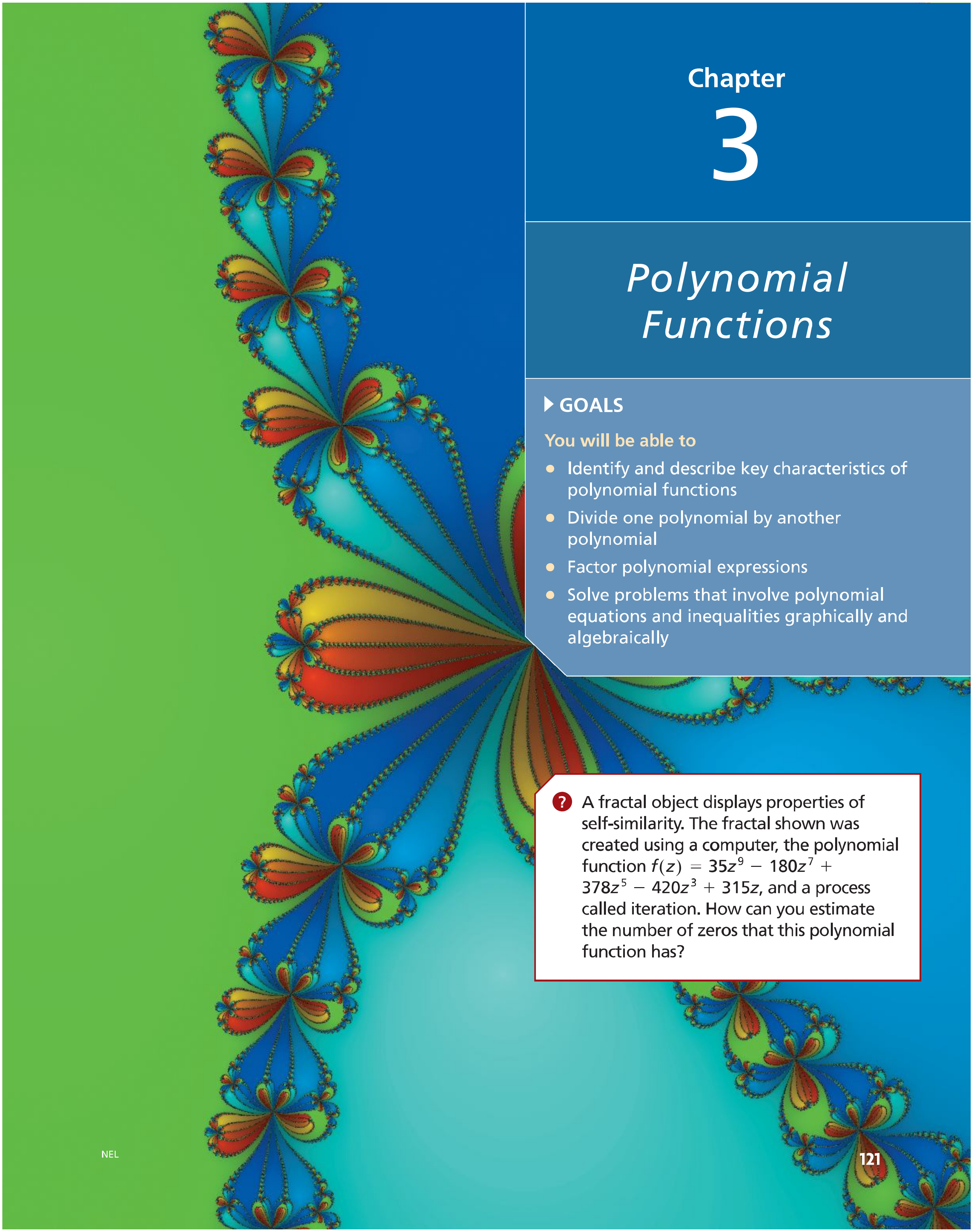




Chapter

3

Polynomial Functions

GOALS

You will be able to

- Identify and describe key characteristics of polynomial functions
- Divide one polynomial by another polynomial
- Factor polynomial expressions
- Solve problems that involve polynomial equations and inequalities graphically and algebraically

? A fractal object displays properties of self-similarity. The fractal shown was created using a computer, the polynomial function $f(z) = 35z^9 - 180z^7 + 378z^5 - 420z^3 + 315z$, and a process called iteration. How can you estimate the number of zeros that this polynomial function has?

3 Getting Started

Study *Aid*

- For help, see the Review of Essential Skills found at the Nelson Advanced Functions website.

Question	Appendix
1	R-2
2	R-3
3	R-6
4	R-8
5	R-9

SKILLS AND CONCEPTS You Need

1. Expand and simplify each of the following expressions.

a) $2x^2(3x - 11)$ **c)** $4x(2x - 5)(3x + 2)$

b) $(x - 4)(x + 6)$ **d)** $(5x - 4)(x^2 + 7x - 8)$

2. Factor each of the following expressions completely.

a) $x^2 + 3x - 28$ **b)** $2x^2 - 18x + 28$

3. Solve each of the following equations. Round your answer to two decimal places, if necessary.

a) $3x + 7 = x - 5$ **c)** $x^2 + 11x + 24 = 0$

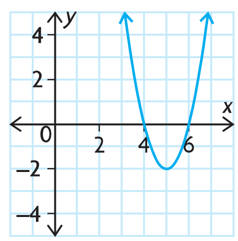
b) $(x + 3)(2x - 9) = 0$ **d)** $6x^2 + 22x = 8$

4. Describe the transformations that must be applied to $y = x^2$ to create the graph of each of the following functions.

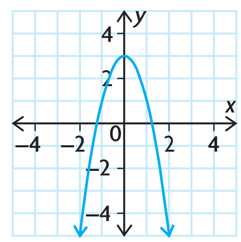
a) $y = \frac{1}{4}(x - 3)^2 + 9$ **b)** $y = \left(\frac{1}{2}x\right)^2 - 7$

5. Write the equation of each function shown below.

a)



b)



6. Graph each of the following functions.

a) $y = 3(x + 5)^2 - 4$ **b)** $y = 2x^2 - 12x + 5$

7. Use finite differences to classify each set of data as linear, quadratic, or other.

a)

x	y
−2	56.4
−1	50.6
0	45
1	39.6
2	34.4

b)

x	y
−2	11
−1	5
0	2
1	7
2	13

c)

x	y
−2	2
−1	6
0	18
1	54
2	162

d)

x	y
−2	7
−1	6.5
0	6
1	5.5
2	5

8. Create a concept web that shows the connections between each of the following for the function $f(x) = 3x^2 + 24x + 36$: the y-intercept, factored form, vertex form, axis of symmetry, direction of opening, zeroes, minimum value, value of the discriminant, and translations of the parent function.
On each arrow, write a brief description of the process you would use to obtain the information.

APPLYING *What You Know*

Examining Patterns

In the late 18th century, seven-year-old Carl Friedrich Gauss noticed a pattern that allowed him to determine the sum of the numbers from 1 to 100 very quickly. He realized that you could add 1 and 100, and then multiply by half of the largest number (50) to get 5050.

YOU WILL NEED

- graph paper

? Are there formulas for calculating the sum of the first n natural numbers and the sums of consecutive squares of natural numbers?

A. Copy and complete each table, then calculate the **finite differences** until they are constant.

B. Graph each relationship in part A on graph paper.

C. Use your graphs and finite differences to make a **conjecture** about the type of model that would fit the data in each table (linear, quadratic, or other).

D. Use a graphing calculator and the regression operation to verify your conjectures in part C.

E. Use the equations you found in part D to calculate the sum of the first five natural numbers and the sum of the squares of the first five natural numbers.

F. Verify that your calculations in part E are correct by comparing your sums with the values in both tables when $n = 5$.

G. Use the equation you found to verify that the sum of the natural numbers from 1 to 100 is 5050.

H. Use the equation you found to determine the sum of the squares of the natural numbers from 1 to 100.

Table 1

n	Sum up to n ($f(n)$)
1	1
2	$1 + 2 = 3$
3	$1 + 2 + 3 = 6$
4	
5	
6	
7	
8	
9	
10	

Table 2

n	Sum of the squares up to n^2 ($g(n)$)
1	1
2	$1^2 + 2^2 = 5$
3	$1^2 + 2^2 + 3^2 = 14$
4	
5	
6	
7	
8	
9	
10	

3.1 Exploring Polynomial Functions

YOU WILL NEED

- graphing calculator or graphing software

GOAL

Identify polynomial functions.

EXPLORE the Math

Beth knows that linear functions result in graphs of straight lines, while quadratic functions result in parabolas. She wonders what happens when the **degree** of a function is larger than 2. Beth searched for polynomials on the Internet and found the following table.

These are polynomial expressions.	These are not polynomial expressions.
$3x^2 - 5x + 3$	$\sqrt{x} + 5x^3$
$-4x + 5x^7 - 3x^4 + 2$	$\frac{1}{2x + 5}$
$\frac{2}{5}x^3 - 3x^5 + 4$	$6x^3 + 5x^2 - 3x + 2 + 4x^{-1}$
$\sqrt{4}x^3 - \frac{\sqrt{5}}{3}x^2 + 2x - \frac{1}{4}$	$\frac{3x^2 + 5x - 1}{2x^2 + x - 3}$
$3x - 5$	$4^x + 5$
-7	$\sin (x - 30)$
$-4x$	$x^2y + 3x - 4y^{-2}$
$(2x - 3)(x + 1)^2$	$3x^3 + 4x^{2.5}$

Communication *Tip*

A polynomial expression in one variable is usually written with the powers arranged from highest to lowest degrees, as in $5x^3 - 7x^2 + 4x + 3$. The phrase "polynomial expression" is often shortened to just "polynomial."

? What makes an expression a *polynomial* expression, and what do functions that involve polynomial expressions look like graphically and algebraically?

A. Look carefully at the expressions in the two columns of the table. What do all of the polynomial expressions have in common?

B. The expressions in the right column are not polynomials. How are they different from the polynomial expressions in the left column?

C. In your own words, define a polynomial expression.

D. The simplest polynomial functions are functions that contain a single term. Use a graphing calculator to graph each of the following polynomial functions. Then copy and complete the table.

Polynomial Function	Type	Sketch of Graph	Description of Graph	Domain and Range	Existence of Asymptotes?
$f(x) = x$	linear				
$f(x) = x^2$	quadratic				
$f(x) = x^3$	cubic				
$f(x) = x^4$	quartic				
$f(x) = x^5$	quintic				

polynomial functions

a function of the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers and n is a whole number; the equation of a polynomial function is defined by a polynomial expression, as in $f(x) = 5x^3 + 6x^2 - 3x + 7$

Communication Tip

Polynomial functions are named according to their degree. Polynomial functions of degree 1, 2, 3, 4, and 5 are commonly called linear, **quadratic**, **cubic**, **quartic**, and **quintic** functions, respectively.

E. Which polynomial functions in part D have similar graphical characteristics? How are the equations of these functions related?

F. For each function in part D, create a table of values for $-3 \leq x \leq 3$. Calculate the finite differences until they are constant. What do you notice?

G. Create equations for four different polynomial functions that are neither linear nor quadratic. Make sure that each function has a different degree and contains at least three terms. Graph each function on a graphing calculator, make a detailed sketch, and create a table of finite differences.

H. Create equations for four non-polynomial functions. Make detailed sketches of their graphs, and create a table of finite differences.

I. Compare and contrast the graphs, the equations, and the finite difference tables for the polynomial and non-polynomial functions you created. Explain how you can tell whether or not a function is a polynomial by looking at

i) its graph
ii) its equation
iii) its finite difference table

Reflecting

J. Explain how you can tell whether a polynomial equation is a function and not just a relation.

K. Why are the equations of the form $y = mx + b$ and $y = ax^2 + bx + c$ examples of polynomial functions?

L. As the degree of a polynomial function increases, describe what happens to

i) the graph of the function

ii) the finite differences

M. Would you change your definition in part C now, after having completed part G? Explain.

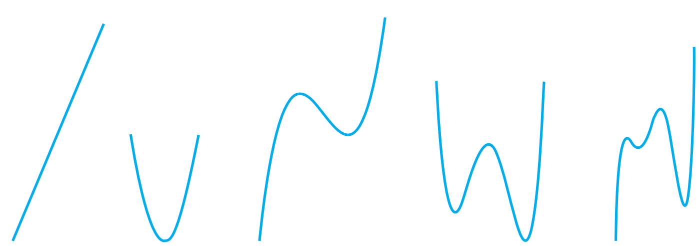
In Summary

Key Idea

- A polynomial in one variable is an expression of the form $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers and n is a whole number. The expression contains only one variable, with the powers arranged in descending order. For example, $2x + 5$, $3x^2 + 2x - 1$, and $5x^4 + 3x^3 - 6x^2 + 5x - 8$.

Need to Know

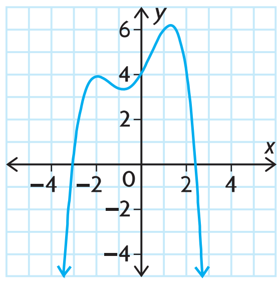
- In any polynomial expression, the exponents on the variable must be whole numbers.
- A polynomial function is any function that contains a polynomial expression in one variable. The degree of the function is the highest exponent in the expression. For example, $f(x) = 6x^3 - 3x^2 + 4x - 9$ has a degree of 3.
- The nth finite differences of a polynomial function of degree n are constant.
- The domain of a polynomial function is the set of real numbers, $\{x \in \mathbf{R}\}$.
- The range of a polynomial function may be all real numbers, or it may have a lower bound or an upper bound (but not both).
- The graphs of polynomial functions do not have horizontal or vertical asymptotes.
- The graphs of polynomial functions of degree zero are horizontal lines. The shape of other graphs depends on the degree of the function. Five typical shapes are shown for various degrees:

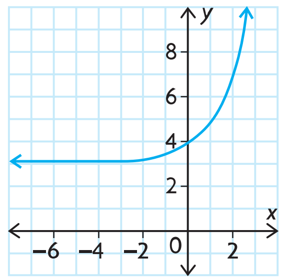


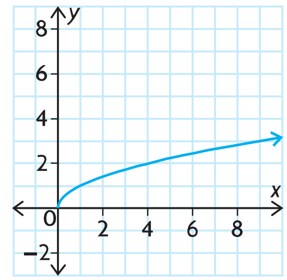
Linear ($n = 1$) Quadratic ($n = 2$) Cubic ($n = 3$) Quartic ($n = 4$) Quintic ($n = 5$)

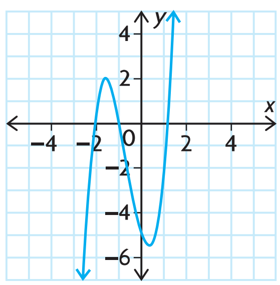
FURTHER Your Understanding

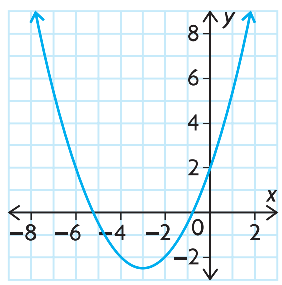
1. Determine which graphs represent polynomial functions. Explain how you know.

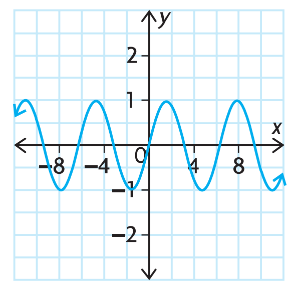
a) 

c) 

e) 

b) 

d) 

f) 

2. Determine whether each function is a polynomial function or another type of function. Justify your decision.

a) $f(x) = 2x^3 + x^2 - 5$

b) $f(x) = x^2 + 3x - 2$

c) $y = 2x - 7$

d) $y = \sqrt{x + 1}$

e) $y = \dfrac{x^2 - 4x + 1}{x + 2}$

f) $f(x) = x(x - 1)^2$

3. Use finite differences to determine the type of polynomial function that could model each relationship.

a) Michelle earns \$200 per week, plus 5% of sales.

Sales	0	500	1000	1500	2000
Earnings (\$)	200	225	250	275	300

b) A model rocket is launched from the roof of a school.

Time (s)	0	1	2	3	4
Height above Ground (m)	10	25	30	25	10

c) The volume of a box varies at different widths.

Width (cm)	1	2	3	4	5
Volume (cm^3)	200	225	250	275	300

d) The input for a function gives a certain output.

Input	0	1	2	3	4	5	6
Output	200	204	232	308	456	700	1064

4. Graph the function $y = 2^x$ on the domain $0 \leq x \leq 3$.

a) Explain why a person who sees only the graph you created (not the equation) might think that the graph represents a polynomial function.

b) Explain why this function is not a polynomial function.

5. Draw a graph of a polynomial function that satisfies all of the following characteristics:

- $f(-3) = 16, f(3) = 0$, and $f(-1) = 0$
- The y-intercept is 2.
- $f(x) \geq 0$ when $x < 3$.
- $f(x) \leq 0$ when $x > 3$.
- The domain is the set of real numbers.

6. Explain why there are many different graphs that fit different combinations of the characteristics in question 5. Draw two graphs that are different from each other, and explain how they satisfy some, but not all, of the characteristics in question 5.

7. Create equations for a linear, a quadratic, a cubic, and a quartic polynomial function that all share the same y-intercept of 5.

8. Complete the following chart to summarize your understanding of polynomials.

Definition	Characteristics
Polynomial	
Examples	Non-Examples

3.2 Characteristics of Polynomial Functions

GOAL

Investigate the turning points and end behaviours of polynomial functions.

YOU WILL NEED

- graphing calculator or graphing software

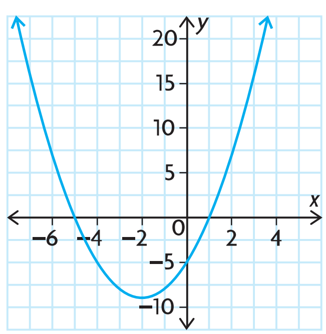
INVESTIGATE the Math

Karel knows that he can describe the graph of a linear function from its equation, using the slope and the y-intercept. He can also describe the graph of a quadratic function from its equation, using the vertex, y-intercept, and the direction of opening. Now he is wondering whether he can describe the graphs of polynomial functions of higher degree, using characteristics that can be predicted from their equations.

? How can you predict some of the characteristics of a polynomial function from its equation?

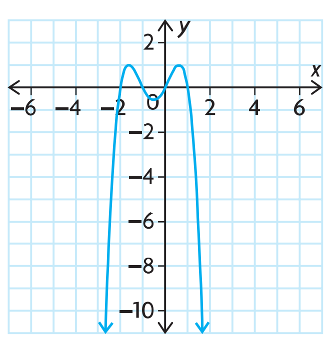
A. The graphs of some polynomial functions are shown below and on the following page.

a)



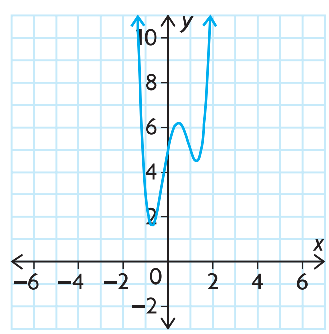
$f(x) = x^2 + 4x - 5$

b)



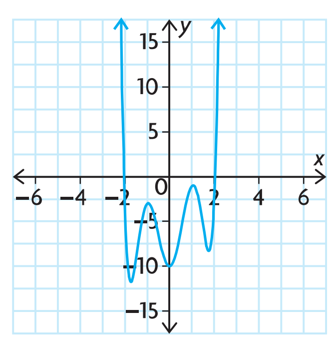
$f(x) = -x^4 - 2x^3 + x^2 + 2x$

c)



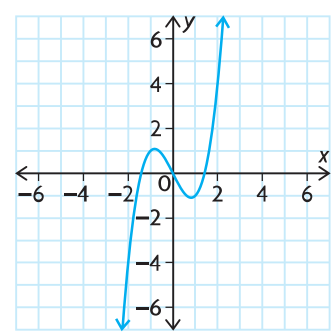
$f(x) = 3x^4 - 4x^3 - 4x^2 + 5x + 5$

d)



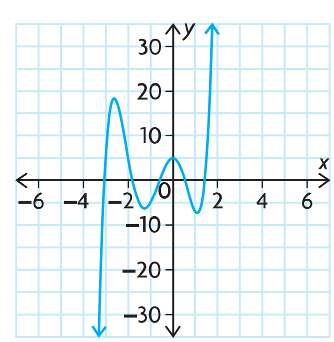
$P(x) = 2x^6 - 12x^4 + 18x^2 + x - 10$

e)



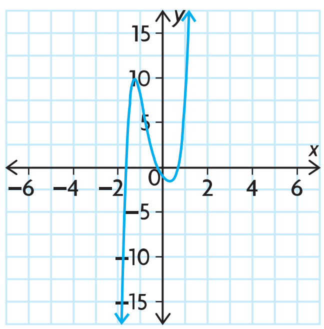
$f(x) = x^3 - 2x$

f)



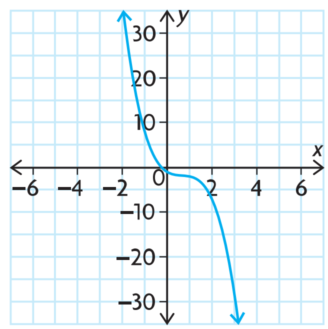
$f(x) = 2x^5 + 7x^4 - 3x^3 - 18x^2 + 5$

g)



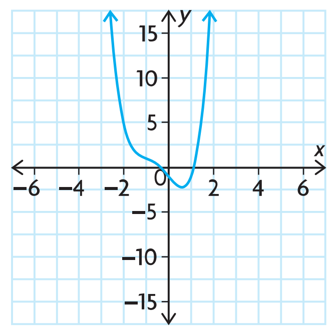
$f(x) = 5x^5 + 5x^4 - 2x^3 + 4x^2 - 3x$

h)



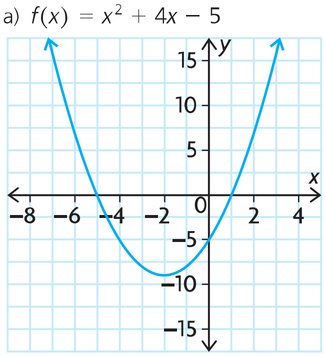
$f(x) = -2x^3 + 4x^2 - 3x - 1$

i)



$f(x) = x^4 + 2x^3 - 3x - 1$

Copy the following table, and complete it using the remaining equations and graphs given.

Equation and Graph	Degree	Even or Odd Degree?	Leading Coefficient	End Behaviours		Number of Turning Points
				$x \to -\infty$	$x \to +\infty$	
a) $f(x) = x^2 + 4x - 5$ 	2	even	$+1$	$y \to +\infty$	$y \to +\infty$	1

leading coefficient

the coefficient of the term with the highest degree in a polynomial

B. Describe any patterns that you see in your table.

C. Create two new polynomial functions of degree greater than 2, one of even degree and one of odd degree. Do these new polynomial functions support your observations in part B?

D. What do you think is the maximum number of turning points that a polynomial function of degree n can have?

E. Graph the following functions using a graphing calculator. Copy each graph and its equation into the appropriate column of a table like the one shown on the next page.

i) $f(x) = x^4 - 2x^2 + 1$

ii) $f(x) = x^3 + 3x^2 - 2x - 5$

iii) $f(x) = \frac{1}{2}x^{10} - \frac{1}{3}x^4 + x^2$

iv) $f(x) = x^3 + x$

v) $f(x) = -2x^6 + 3x^4$

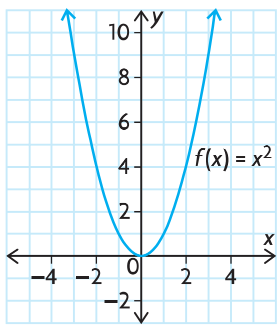
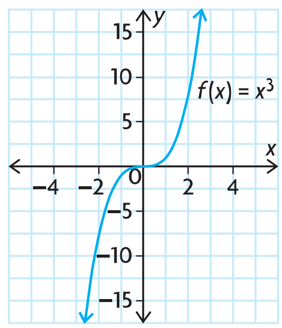
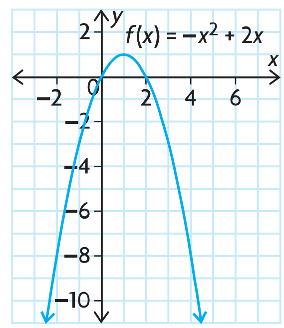
vi) $f(x) = x^5 - 3x$

vii) $f(x) = x^2 - 3x + 4$

viii) $f(x) = 2x^7 - 3x^3 + 2x$

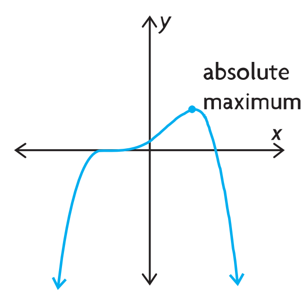
ix) $f(x) = -3x^4 + 2x^3 - 3x + 1$

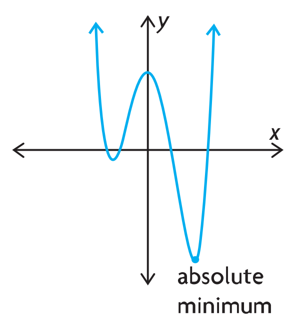
x) $f(x) = x^2 - x$

Even Functions	Odd Functions	Neither Even nor Odd Functions
(symmetry in the *y*-axis)	(rotational symmetry around the origin)	(neither of these symmetries)
		

absolute maximum/absolute minimum

the greatest/least value attained by a function for all values in its domain





F. Determine $f(-x)$ for each function in your table. Discuss any patterns that you see.

G. Is every function of even degree an even function? Why or why not?

H. Is every function of odd degree an odd function? Why or why not?

I. How can you use the equation of a polynomial function to describe its end behaviours, number of turning points, and symmetry?

Reflecting

J. Why must all polynomial functions of even degree have an absolute maximum or absolute minimum?

K. Why must all polynomial functions of odd degree have at least one zero?

L. Can the graph of a polynomial function have no zeros? Explain.

M. Examine all the graphs you have investigated and their equations. Is it possible to predict the maximum number of zeros that a graph will have if you are given its equation? Explain.

APPLY the Math

EXAMPLE 1 | Reasoning about characteristics of a given polynomial function

Describe the end behaviours of each function, the possible number of turning points, and the possible number of zeros. Use these characteristics to sketch possible graphs of the function.

a) $f(x) = -3x^5 + 4x^3 - 8x^2 + 7x - 5$ **b)** $g(x) = 2x^4 + x^2 + 2$

Solution

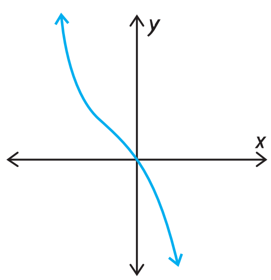
a) $f(x) = -3x^5 + 4x^3 - 8x^2 + 7x - 5$

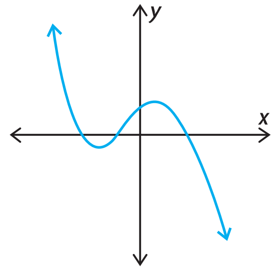
The degree is odd, so the function has opposite end behaviours. The leading coefficient is negative, so the graph must extend from the second quadrant to the fourth quadrant.

As $x \to -\infty, y \to +\infty$.

As $x \to +\infty, y \to -\infty$.

If x is a very large negative number, such as -1000, $-3x^5$ will have a large positive value and will have a greater effect on the value of the function than the other terms. Therefore, the graph will pass through the second quadrant. For very large positive values of x, $-3x^5$ will have a large negative value. Therefore, the graph will extend into the fourth quadrant.

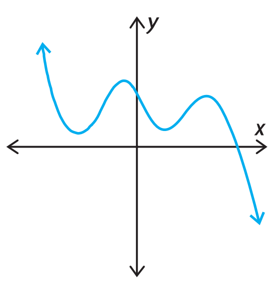


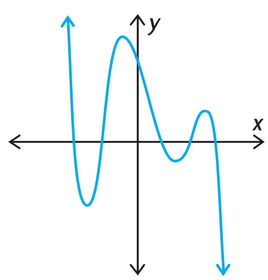


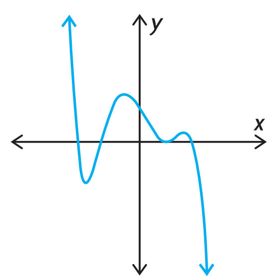
Using the end behaviours of the function, sketch possible graphs of a fifth-degree polynomial.

To pass through the second quadrant and extend into the fourth quadrant, the graph must have an even number of turning points.

$f(x)$ may have zero, two, or four turning points.







Since the function is a fifth-degree polynomial, it must have at least one zero and no more than five zeros.

$f(x)$ may have one, two, three, four, or five zeros.

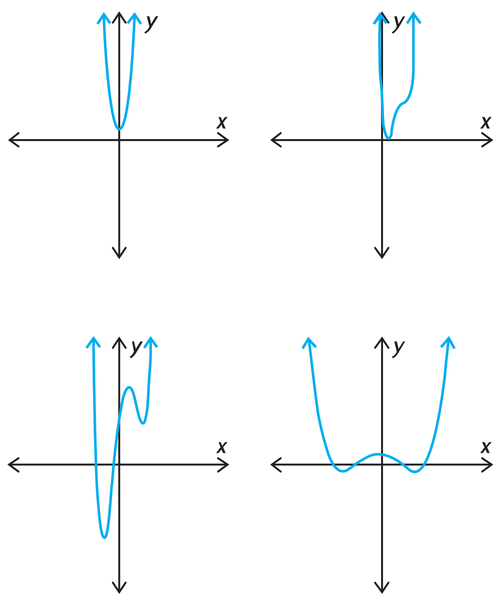
b) $g(x) = 2x^4 + x^2 + 2$

The degree is even, so the function has the same end behaviours. The leading coefficient is positive, so the graph must extend from the second quadrant to the first quadrant.

As $x \rightarrow -\infty, y \rightarrow +\infty$.

As $x \rightarrow +\propto, y \rightarrow +\infty$.

If x is a very large negative number, $2x^4$ will have a large positive value and will have a greater effect on the value of the function than the other terms. Therefore, the graph will pass through the second quadrant. For very large positive values of x, $2x^4$ will have a large positive value. Therefore, the graph will extend into the first quadrant.



Using the end behaviours of the function, sketch possible graphs of a fourth-degree polynomial.

To start in the second quadrant and end in the first quadrant, the graph must have an odd number of turning points.

Since the function is a fourth-degree polynomial, it may have anywhere from zero to four x-intercepts.

$f(x)$ may have one or three turning points and zero, one, two, three, or four zeros.

EXAMPLE 2 Reasoning about how given characteristics fit particular functions

What could the graph of a polynomial function that has range $\{y \in \mathbf{R} \mid y \leq 10\}$ and three turning points look like? What can you conclude about its equation?

Solution

End behaviours of the function:

As $x \to -\infty, y \to -\infty$.

As $x \to +\infty, y \to -\infty$.

Since the range has an upper limit, both ends of the function extend downward toward $-\infty$ in the third and fourth quadrants. For this to occur, the leading coefficient in the equation must be negative.

The function has at least two zeros.

Because the function has a maximum value that is positive and both ends extend downward, the function must cross the *x*-axis at least twice.

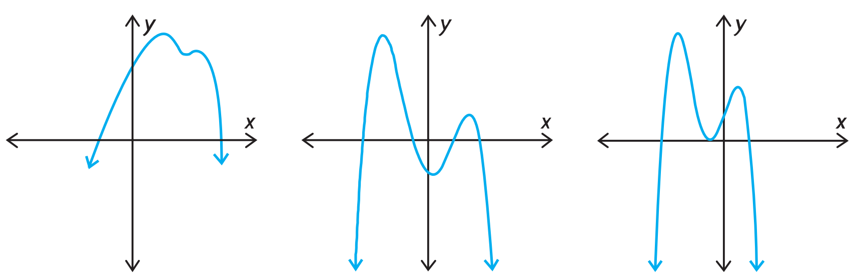
The function has an even degree.

Since the function has an absolute maximum, it must have an even degree. This is confirmed by the end behaviours, because they are the same.

The degree of the function is at least 4.

It is not possible to be sure about the degree of the function, but the degree must be at least one more than the number of turning points.

Here are some possible graphs of the function.



In Summary

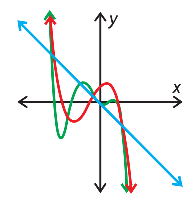
Key Ideas

- Polynomial functions of the same degree have similar characteristics.
- The degree and the leading coefficient in the equation of a polynomial function indicate the end behaviours of the graph.
- The degree of a polynomial function provides information about the shape, turning points, and zeros of the graph.

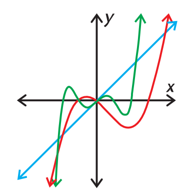
Need to Know

End Behaviours

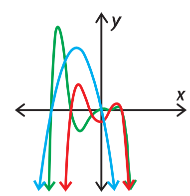
- An odd-degree polynomial function has opposite end behaviours.
  - If the leading coefficient is negative, then the function extends from the second quadrant to the fourth quadrant; that is, as $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$.



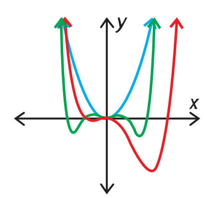
  - If the leading coefficient is positive, then the function extends from the third quadrant to the first quadrant; that is, as $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to \infty$.



- An even-degree polynomial function has the same end behaviours.
  - If the leading coefficient is negative, then the function extends from the third quadrant to the fourth quadrant; that is, as $x \to \pm\infty$, $y \to -\infty$.



  - If the leading coefficient is positive, then the function extends from the second quadrant to the first quadrant; that is, as $x \to \pm\infty$, $y \to \infty$.



Turning Points

- A polynomial function of degree n has at most $n - 1$ turning points.

Number of Zeros

- A polynomial function of degree n may have up to n distinct zeros.
- A polynomial function of odd degree must have at least one zero.
- A polynomial function of even degree may have no zeros.

Symmetry

- Some polynomial functions are symmetrical in the y-axis. These are even functions, where $f(-x) = f(x)$.
- Some polynomial functions have rotational symmetry about the origin. These are odd functions, where $f(-x) = -f(x)$.
- Most polynomial functions have no symmetrical properties. These are functions that are neither even nor odd, with no relationship between $f(-x)$ and $f(x)$.

CHECK Your Understanding

1. State the degree, leading coefficient, and end behaviours of each polynomial function.

a) $f(x) = -4x^4 + 3x^2 - 15x + 5$

b) $g(x) = 2x^5 - 4x^3 + 10x^2 - 13x + 8$

c) $p(x) = 4 - 5x + 4x^2 - 3x^3$

d) $h(x) = 2x(x - 5)(3x + 2)(4x - 3)$

2. a) Determine the minimum and maximum number of turning points for each function in question 1.

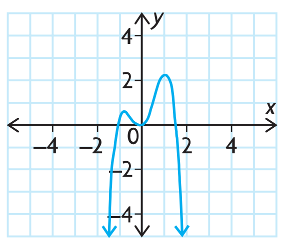
b) Determine the minimum and maximum number of zeros that each function in question 1 may have.

3. For each of the following graphs, decide if

a) the function has an even or odd degree

b) the leading coefficient is positive or negative

i)



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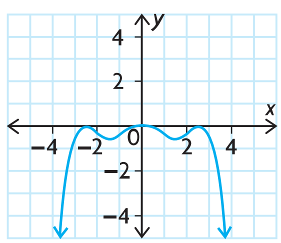

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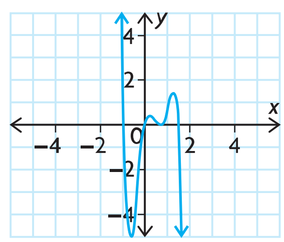

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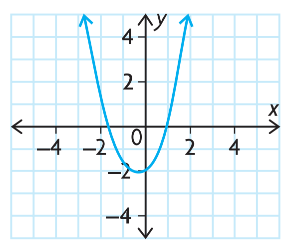

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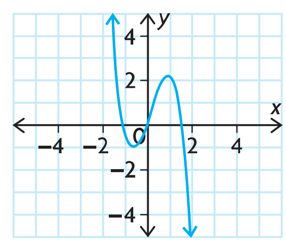

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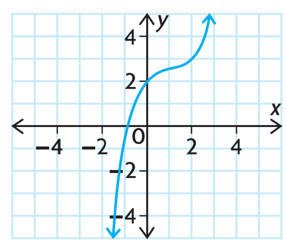


PRACTISING

4. K Describe the end behaviour of each polynomial function using the degree and the leading coefficient.

a) $f(x) = 2x^2 - 3x + 5$

b) $f(x) = -3x^3 + 2x^2 + 5x + 1$

c) $f(x) = 5x^3 - 2x^2 - 2x + 6$

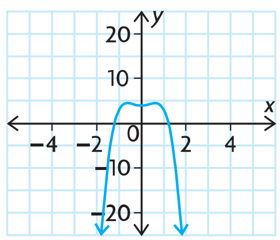
d) $f(x) = -2x^4 + 5x^3 - 2x^2 + 3x - 1$

e) $f(x) = 0.5x^4 + 2x^2 - 6$

f) $f(x) = -3x^5 + 2x^3 - 4x$

5. Use end behaviours, turning points, and zeros to match each polynomial equation with the most likely graph below. Explain.
   a) $y = 2x^3 - 4x^2 + 3x + 2$
   b) $y = -4x^4 + 3x^2 + 4$
   c) $y = x^2 + 3x - 5$
   d) $y = x^4 - x^3 - 4x^2 + 5x$
   e) $y = -2x^5 + 3x^4 + 6x^3 - 10x^2 + 2x + 5$
   f) $y = 3x^3 + 5x^2 - 3x + 1$

A



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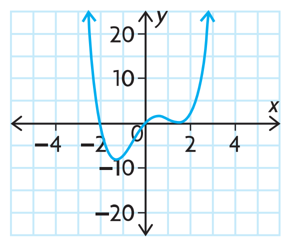

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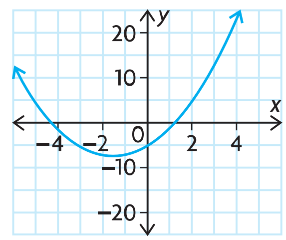

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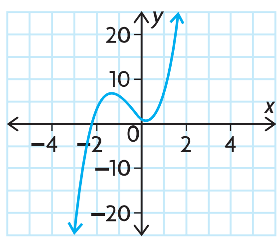

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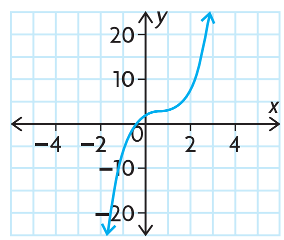

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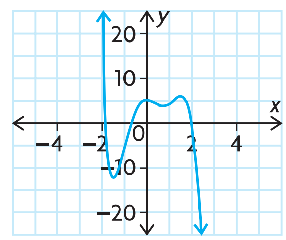


6. Give an example of a polynomial function that has each of the following end behaviours:
   a) As $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to \infty$.
   b) As $x \to \pm\infty$, $y \to \infty$.
   c) As $x \to \pm\infty$, $y \to -\infty$.
   d) As $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$.

7. Sketch the graph of a polynomial function that satisfies each set of conditions.
   a) degree 4, positive leading coefficient, 3 zeros, 3 turning points
   b) degree 4, negative leading coefficient, 2 zeros, 1 turning point
   c) degree 4, positive leading coefficient, 1 zero, 3 turning points
   d) degree 3, negative leading coefficient, 1 zero, no turning points
   e) degree 3, positive leading coefficient, 2 zeros, 2 turning points
   f) degree 4, two zeros, three turning points, Range $= \{y \in \mathbf{R} \mid y \leq 5\}$

8. C Explain why odd-degree polynomial functions can have only local maximums and minimums, but even-degree polynomial functions can have absolute maximums and minimums.

9. T Rei noticed that the graph of the function $f(x) = ax^b - cx$ is symmetrical with respect to the origin, and that it has some turning points. Does the graph have an odd or even number of turning points?

10. Sketch an example of a cubic function with a graph that intersects the x-axis at each number of points below.

a) only one point **b)** two different points **c)** three different points

11. Sketch an example of a quartic function with a graph that intersects the x-axis at each number of points below.

a) no points
b) only one point
c) two different points
d) three different points
e) four different points

12. The graph of a polynomial function has the following characteristics:

- Its domain and range are the set of all real numbers.
- There are turning points at $x = -2$, 0, and 3.

a) Draw the graphs of two different polynomial functions that have these three characteristics.
b) What additional characteristics would ensure that only one graph could be drawn?

13. A The mining town of Brighton was founded in 1900. Its population, y, in hundreds, is modelled by the equation $y = -0.1x^4 + 0.5x^3 + 0.4x^2 + 10x + 7$, where x is the number of years since 1900.

a) What was the population of the town in 1900?
b) Based on the equation, describe what happened to the population of Brighton over time. Justify your answer.

14. f is a polynomial function of degree n, where n is a positive even integer. Decide whether each of the following statements is true or false. If the statement is false, give an example that illustrates why it is false.

a) f is an even function.
b) f cannot be an odd function.
c) f will have at least one zero.
d) As $x \rightarrow \infty$, $y \rightarrow \infty$ and as $x \rightarrow -\infty$, $y \rightarrow \infty$.

15. If you needed to predict the graph or equation of a polynomial function and were only allowed to ask three questions about the function, what questions would you ask to help you the most? Why?

Extending

16. **a)** Suppose that $f(x) = ax^2 + bx + c$. What must be true about the coefficients if f is an even function?
b) Suppose that $g(x) = ax^3 + bx^2 + cx + d$. What must be true about the coefficients if g is an odd function?

3.3 Characteristics of Polynomial Functions in Factored Form

GOAL

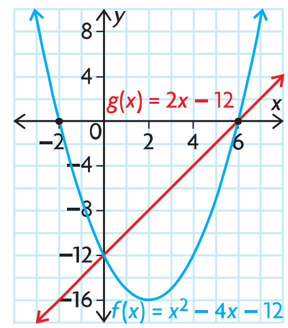
Determine the equation of a polynomial function that describes a particular graph or situation, and vice versa.

YOU WILL NEED

- graphing calculator or graphing software

INVESTIGATE the Math

The graphs of the functions $f(x) = x^2 - 4x - 12$ and $g(x) = 2x - 12$ are shown.



? What is the relationship between the real **roots** of a polynomial equation and the x-intercepts of the corresponding polynomial function?

A. Solve the equations $f(x) = 0$ and $g(x) = 0$ using the given functions. Compare your solutions with the graphs of the functions. What do you notice?

B. Create a cubic function from the family of polynomial functions of the form $h(x) = a(x - p)(x - q)(x - r)$.

C. Graph $y = h(x)$ on a graphing calculator. Describe the shape of the graph near each zero, and compare the shape to the order of each factor in the equation of the function.

D. Solve $h(x) = 0$, and compare your solutions with the zeros of the graph of the corresponding function. What do you notice?

E. Repeat parts B through D using a quartic function.

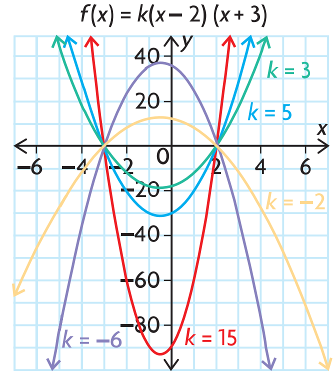
F. Repeat parts C and D using $m(x) = (x - 2)^2(x + 3)$. How would you describe the shape of the graph near the zero with the repeated factor?

G. Repeat parts C and D using $n(x) = (x - 2)^3(x + 3)$. How would you describe the shape of the graph near the zero with the repeated factor?

H. What relationship exists between the x-intercepts of the graph of a polynomial function and the roots of the corresponding equation?

family of polynomial functions

a set of polynomial functions whose equations have the same degree and whose graphs have common characteristics; for example, one type of quadratic family has the same zeros or x-intercepts



order

the exponent to which each factor in an algebraic expression is raised; for example, in $f(x) = (x - 3)^2(x - 1)$, the order of $(x - 3)$ is 2 and the order of $(x - 1)$ is 1

Reflecting

I. How does a squared factor in the equation of a polynomial function affect the shape of the graph near its corresponding zero?

J. How does a cubed factor in a polynomial function affect the shape of the graph near its corresponding zero?

K. Why does the relationship you described in part H make sense?

APPLY the Math

EXAMPLE 1 Using reasoning to draw a graph from the equation of a polynomial function

Sketch a possible graph of the function $f(x) = -(x + 2)(x - 1)(x - 3)^2$.

Solution

Let $x = 0$.

$$\begin{aligned} f(x) &= -(0 + 2)(0 - 1)(0 - 3)^2 \\ &= -(2)(-1)(-3)^2 \\ &= 18 \end{aligned}$$

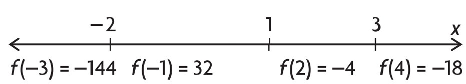
Calculate the y-intercept.

$$0 = -(x + 2)(x - 1)(x - 3)^2$$

$x = -2$, $x = 1$, or $x = 3$

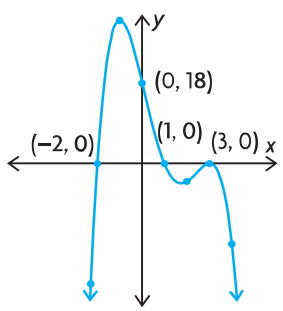
Determine the x-intercepts by letting $f(x) = 0$. Use the factors to solve the resulting equation for x.

Use values of x that fall between the x-intercepts as test values to determine the location of the function above or below the x-axis.



Since the function lies below the x-axis on both sides of $x = 3$, the graph must just touch the x-axis and not cross over at this point. The order of 2 on the factor $(x - 3)^2$ confirms the parabolic shape near $x = 3$.

Determine the end behaviours of the function.



$f(x) = -(x + 2)(x - 1)(x - 3)^2$

Because the degree is even and the leading coefficient is negative, the graph extends from third quadrant to the fourth quadrant; that is, as $x \rightarrow \pm\infty$, $y \rightarrow -\infty$.

This is a possible graph of $f(x)$ estimating the locations of the turning points.

EXAMPLE 2 Using reasoning to determine the equation of a function from given information

Write the equation of a cubic function that has zeros at -2, 3, and $\frac{2}{5}$. The function also has a y-intercept of 6.

Solution

$$f(x) = a(x + 2)(x - 3)(5x - 2)$$

Use the zeros of the function to create factors for the correct family of polynomials. Since this function has three zeros and it is cubic, the order of each factor must be 1.

$$6 = a(0 + 2)(0 - 3)(5(0) - 2)$$
$$6 = a(2)(-3)(-2)$$
$$6 = 12a$$
$$a = \frac{1}{2}$$

Use the y-intercept to calculate the value of a.

Substitute $x = 0$ and $y = 6$ into the equation, and solve for a.

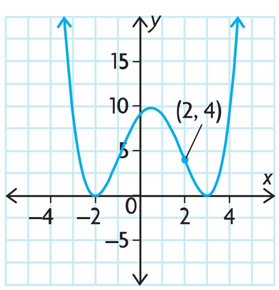
$$f(x) = \frac{1}{2}(x + 2)(x - 3)(5x - 2)$$

Write the equation in factored form.

EXAMPLE 3 Representing the graph of a polynomial function with its equation

a) Write the equation of the function shown below.

b) State the domain and range of the function.



Solution

a) $y = a(x + 2)^2(x - 3)^2$

Write the equation of the correct family of polynomials using factors created from the zeros.

Because the function must have positive values on both sides of the x-intercepts, the factors are squared. The parabolic shape of the graph near the zeros $x = -2$ and $x = 3$ confirms the order of 2 on the factors $(x + 2)^2$ and $(x - 3)^2$.

Let $x = 2$ and $y = 4$.

Substitute the coordinates of the point marked on the graph into the equation.

$$4 = a(2 + 2)^2(2 - 3)^2$$
$$4 = a(4)^2(-1)^2$$
$$4 = 16a$$
$$a = \frac{1}{4}$$

Solve to determine the value of a.

$$y = \frac{1}{4}(x + 2)^2(x - 3)^2$$

Write the equation in factored form.

b) Domain $= \{x \in \mathbf{R}\}$
Range $= \{y \in \mathbf{R} \mid y \geq 0\}$

All polynomial functions have their domain over the entire set of real numbers.

The graph has an absolute minimum value of 0 when $x = -2$ and $x = 3$. All other values of the function are greater than this.

EXAMPLE 4 Representing the equation of a polynomial function with its graph

Sketch the graph of $f(x) = x^4 + 2x^3$.

Solution

$$f(x) = x^4 + 2x^3$$
$$= x^3(x + 2)$$

Write the equation in factored form by dividing out the common factor of x^3.

The zeros are $x = 0$ and $x = -2$.

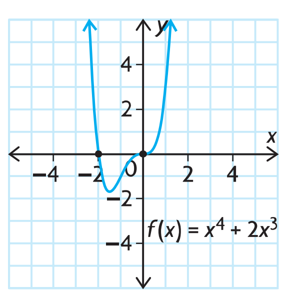
Determine the zeros, the order of the factors, and the shape of the graph near the zeros. The graph has a cubic shape at $x = 0$, since the factor x^3 has an order of 3. The graph has a linear shape near $x = -2$ since the factor $(x + 2)$ has an order of 1.

The y-intercept is $f(0) = 0^4 + 2(0)^3 = 0$.

Determine the y-intercept by letting $x = 0$.

End behaviours:
As $x \to \pm\infty$, $y \to \infty$.

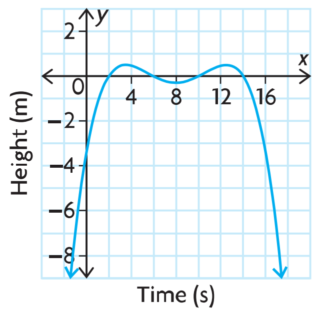
Determine the end behaviours. The function has an even degree, so the end behaviours are the same. The leading coefficient is positive, so the graph extends from the second quadrant to the first quadrant.



Use these characteristics to sketch a possible graph.

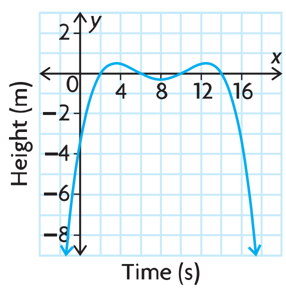
EXAMPLE 5 Representing a contextual situation with an equation of a polynomial function

While playing in the surf, a dolphin jumped twice into the air before diving deep below the surface of the water. The path of the dolphin is shown on the following graph.



Write the equation of the polynomial function that represents the height of the dolphin relative to the surface of the water.

Solution



$f(x) = a(x - 2)(x - 6)(x - 10)(x - 14)$

The zeros of the function are $x = 2, 6, 10$, and 14. These are the times when the dolphin breaks the surface of the water. Use the zeros to create the factors of a family of polynomial functions. Since the shape of the graph near each zero is linear, the order of each corresponding factor must be 1.

Let $f(3.5) = 0.5$.
The maximum height of the dolphin's leap was about 0.5 m when x was about 3.5 s.

Use the graph to estimate the maximum height of the dolphin's leap.

$$0.5 = a(3.5 - 2)(3.5 - 6)(3.5 - 10)(3.5 - 14)$$
$$0.5 = a(1.5)(-2.5)(-6.5)(-10.5)$$
$$0.5 = -255.9375a$$
$$a \doteq -0.002$$

Solve the equation to determine the value of a.

$$f(x) = -0.002(x - 2)(x - 6)(x - 10)(x - 14)$$

Write the equation in factored form.

In Summary

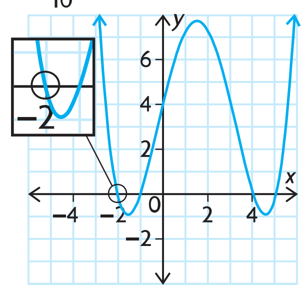
Key Idea

- The zeros of the polynomial function $y = f(x)$ are the same as the roots of the related polynomial equation, $f(x) = 0$.

Need to Know

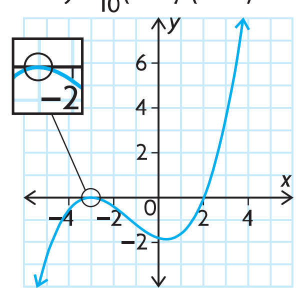
- To determine the equation of a polynomial function in factored form, follow these steps:
  - Substitute the zeros $(x_1, x_2, \ldots, x_n)$ into the general equation of the appropriate family of polynomial functions of the form $y = a(x - x_1)(x - x_2) \ldots (x - x_n)$.
  - Substitute the coordinates of an additional point for x and y, and solve for a to determine the equation.
- If any of the factors of a polynomial function are linear, then the corresponding x-intercept is a point where the curve passes through the x-axis. The graph has a linear shape near this x-intercept.

$y = \frac{1}{10}(x + 2)(x + 1)(x - 4)(x - 5)$



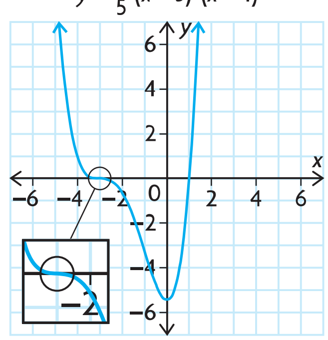
- If any of the factors of a polynomial function are squared, then the corresponding x-intercepts are turning points of the curve and the x-axis is tangent to the curve at these points. The graph has a parabolic shape near these x-intercepts.

$y = \frac{1}{10}(x + 3)^2(x - 2)$



- If any of the factors of a polynomial function are cubed, then the corresponding x-intercepts are points where the x-axis is tangent to the curve and also passes through the x-axis. The graph has a cubic shape near these x-intercepts.

$y = \frac{1}{5}(x + 3)^3(x - 1)$



CHECK Your Understanding

1. Match each equation with the most suitable graph. Explain your reasoning.

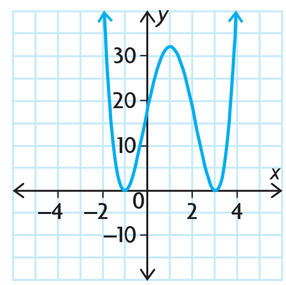
a) $f(x) = 2(x + 1)^2(x - 3)$

b) $f(x) = 2(x + 1)^2(x - 3)^2$

c) $f(x) = -2(x + 1)(x - 3)^2$

d) $f(x) = x(x + 1)(x - 3)(x - 5)$

A



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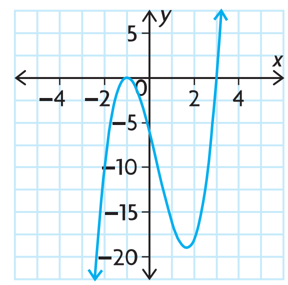

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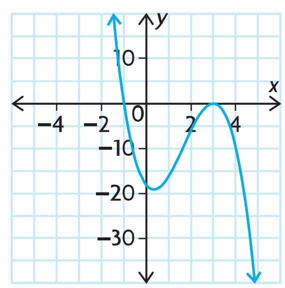

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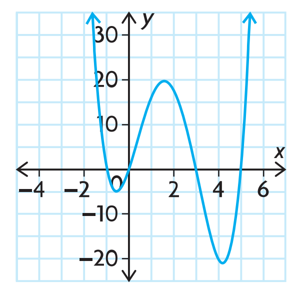


2. Sketch a possible graph of each function.

a) $f(x) = -(x - 4)(x - 1)(x + 5)$

b) $g(x) = x^2(x - 6)^3$

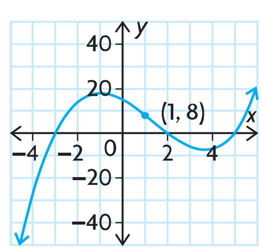
3. Each member of a family of quadratic functions has zeros at $x = -1$ and $x = 4$.

a) Write the equation of the family, and then state two functions that belong to the family.

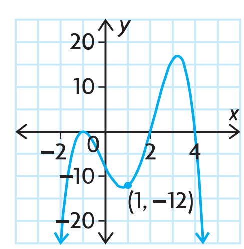
b) Determine the equation of the member of the family that passes through the point $(5, 9)$. Graph the function.

4. Write the equation of each function.

a)



b)



PRACTISING

5. Organize the following functions into families.

A	$y = 2(x - 3)(x + 5)$	**G**	$y = \frac{1}{2}(x - 3)(x + 5)$
B	$y = -1.8(x - 3)^2(x + 5)$	**H**	$y = -5(x + 8)(x)(x + 6)$
C	$y = -x(x + 6)(x + 8)$	**I**	$y = (x - 3)(x + 5)$
D	$y = 2(x + 5)(x + 3)^2$	**J**	$y = \frac{3}{5}(x + 5)(x + 3)^2$
E	$y = (x - 3)^2(x + 5)$	**K**	$y = \frac{x(x + 6)(x + 8)}{4}$
F	$y = x(x + 6)(x + 8)$	**L**	$y = 2(x + 5)(x^2 + 6x + 9)$

6. Sketch the graph of each function.

a) $y = x(x - 4)(x - 1)$
b) $y = -(x - 1)(x + 2)(x - 3)$
c) $y = x(x - 3)^2$
d) $y = (x + 1)^3$
e) $y = x(2x + 1)(x - 3)(x - 5)$
f) $y = x^2(3x - 2)^2$

7. a) Sketch an example of a cubic function with the given zeros. Then write the equation of the function.

i) $-3, 0, 2$ **iii)** $-1, 4$ (order 2)

ii) -2 (order 3) **iv)** $3, -\frac{1}{2}$ (order 2)

b) Are all the characteristics of the graphs unique? Explain.

8. Sketch an example of a quartic function with the given zeros, and write the equation of the function. Then write the equations of two other functions that belong to the same family.

a) $-5, -3, 2, 4$ **c)** $-2, \frac{3}{4}, 5$ (order 2)

b) -2 (order 2), 3 (order 2) **d)** 6 (order 4)

9. Sketch the graph of each function.

a) $y = 3x^3 - 48x$ **c)** $y = x^3 - 9x^2 + 27x - 27$

b) $y = x^4 + 4x^3 + 4x^2$ **d)** $y = -x^4 - 15x^3 - 75x^2 - 125x$

10. Sketch the graph of a polynomial function that satisfies each set of conditions.

a) degree 4, positive leading coefficient, 3 zeros, 3 turning points
b) degree 4, negative leading coefficient, 2 zeros, 1 turning point
c) degree 4, positive leading coefficient, 2 zeros, 3 turning points
d) degree 3, negative leading coefficient, 1 zero, no turning points

Year	Profit or Loss (in thousands of dollars)
1990	−216
1991	−88
1992	0
1993	54
1994	80
1995	84
1996	72
1997	50
1998	24
1999	0
2000	−16
2001	−18
2002	0
2003	44
2004	120

11. **A** A company's profits and losses during a 15-year period are shown in the table.

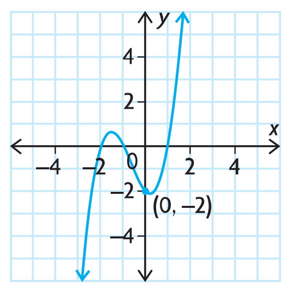
a) Sketch a graph of the data, using years since 1990 as the values of the independent variable.

b) If x represents the number of years since 1990 (with 1990 being year 0), write the polynomial equation that models the data.

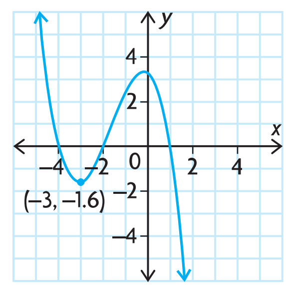
c) Is this trend likely to continue? What restrictions should be placed on the domain of the function so that it is realistic?

12. **K** Determine the equation of the polynomial function from each graph.

a)



b)



13. **a)** Determine the quadratic function that has zeros at -3 and -5, if $f(7) = -720$.

b) Determine the cubic function that has zeros at -2, 3, and 4, if $f(5) = 28$.

14. **T** The function $f(x) = kx^3 - 8x^2 - x + 3k + 1$ has a zero when $x = 2$. Determine the value of k. Graph $f(x)$, and determine all the zeros. Then rewrite $f(x)$ in factored form.

15. **C** Describe what you know about the graphs of each family of polynomials, in as much detail as possible.

a) $y = a(x - 2)^2(x - 4)^2$ **b)** $y = a(x + 4)(x - 3)^2$

Extending

16. Square corners cut from a 30 cm by 20 cm piece of cardboard create a box when the 4 remaining tabs are folded upwards. The volume of the box is $V(x) = x(30 - 2x)(20 - 2x)$, where x represents the height.

a) Calculate the volume of a box with a height of 2 cm.

b) Calculate the dimensions of a box with a volume of 1000 cm^3.

c) Solve $V(x) > 0$, and discuss the meaning of your solution in the context of the question.

d) State the restrictions in the context of the question.

3.4 Transformations of Cubic and Quartic Functions

GOAL

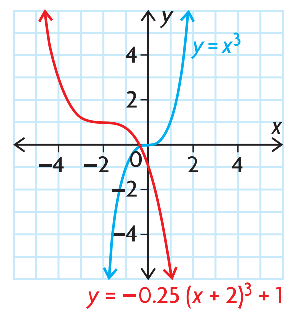
Describe and perform transformations on cubic and quartic functions.

YOU WILL NEED

- graphing calculator or graphing software

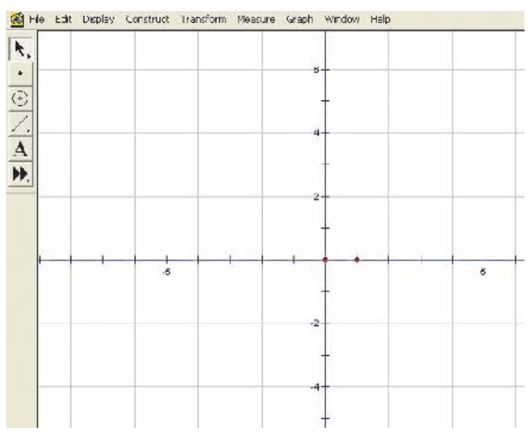
INVESTIGATE the Math

The graphs of the parent cubic function $y = x^3$ and a second function, which is a transformation of the parent function, are shown.

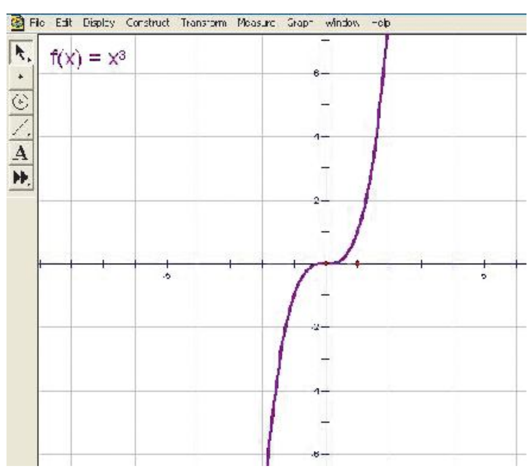


? How do the graphs of $y = a(k(x - d))^3 + c$ and $y = a(k(x - d))^4 + c$ relate to the graphs of $y = x^3$ and $y = x^4$?

A. Use dynamic geometry software to create a Cartesian grid with an x-axis and y-axis.



B. Plot $f(x) = x^3$.

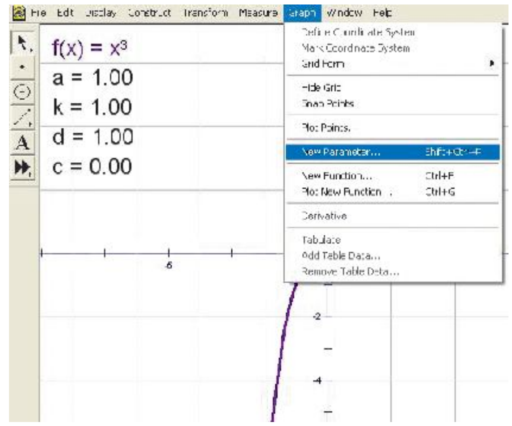


Tech *Support*

For information about how to use *The Geometer's Sketchpad* to plot functions, see Technical Appendix, T-19.

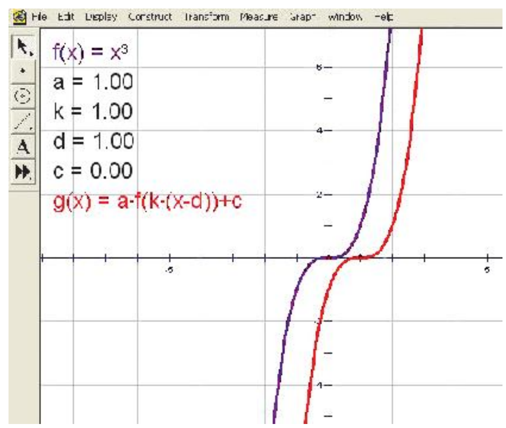
C. Define four new parameters: $a = 1$, $k = 1$, $d = 1$, and $c = 0$.

D. Create and plot $g(x) = a(k(x - d))^3 + c$. Describe how the new graph, $g(x)$, is related to the graph of the parent function, $f(x)$.



E. Make a conjecture about how changing the parameter a in the function $g(x)$ will affect the graph of the parent function, $f(x)$.

F. Change the value of a at least four times using integers and rational numbers. Record the effect of each change on the graph. Make sure that you use both positive and negative values.



G. Repeat parts E and F for each of the other parameters (k, d, and c).

H. Repeat parts A to G for the quartic function $f(x) = x^4$.

Reflecting

I. Describe the transformations that must be applied to the graph of the function $f(x) = x^3$ to create the graph of $y = a(k(x - d))^3 + c$.

J. Describe the transformations that must be applied to the graph of the function $f(x) = x^4$ to create the graph of $y = a(k(x - d))^4 + c$.

K. Do you think your descriptions in parts I and J can be applied to transformations of the function $f(x) = x^n$ for all possible values of n? Explain.

APPLY the Math

EXAMPLE 1 Using reasoning to determine transformations

Describe the transformations that must be applied to $y = x^3$ to graph $y = -8\left(\frac{1}{2}x + 1\right)^3 - 3$, and then graph this function.

Solution A: Using the equation as given

$$y = -8\left(\frac{1}{2}x + 1\right)^3 - 3$$

$$y = -8\left(\frac{1}{2}(x + 2)\right)^3 - 3$$

Factor the coefficient of x so that the function is in the form $y = a(k(x - d))^3 + c$.

$y = x^3$ is

- vertically stretched by a factor of 8 and reflected in the x-axis
- horizontally stretched by a factor of 2
- translated 2 units left
- translated 3 units down

$a = -8$

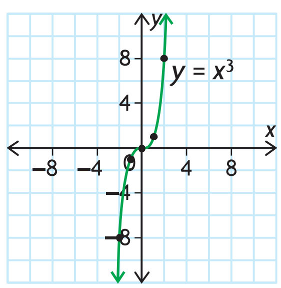
$k = \frac{1}{2}$

$d = -2$

$c = -3$

$y = x^3$

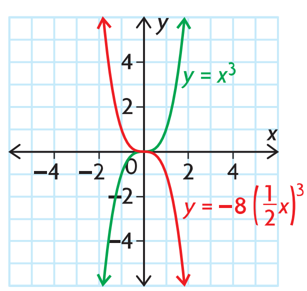
x	y
−2	−8
−1	−1
0	0
1	1
2	8



Begin with the parent function to be transformed and its key points.

$(x, y) \rightarrow (2x, -8y)$

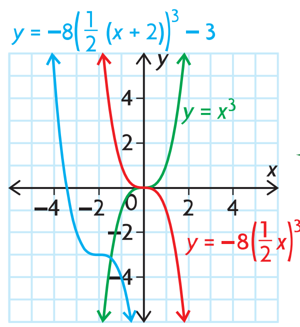
$y = x^3$	$y = -8\left(\frac{1}{2}x\right)^3$
(−2, −8)	(2(−2), −8(−8)) = (−4, 64)
(−1, −1)	(2(−1), −8(−1)) = (−2, 8)
(0, 0)	(2(0), −8(0)) = (0, 0)
(1, 1)	(2(1), −8(1)) = (2, −8)
(2, 8)	(2(2), −8(8)) = (4, −64)



Perform the stretches, reflections, and compressions first. Multiply the x-coordinates of the key points by 2. Multiply the y-coordinates of the key points by −8.

$(2x, -8y) \rightarrow (2x - 2, -8y - 3)$

$y = -8\left(\frac{1}{2}x\right)^3$	$y = -8\left(\frac{1}{2}(x + 2)\right)^3 - 3$
$(-4, 64)$	$(-4 - 2, 64 - 3) = (-6, 61)$
$(-2, 8)$	$(-2 - 2, 8 - 3) = (-4, 5)$
$(0, 0)$	$(0 - 2, 0 - 3) = (-2, -3)$
$(2, -8)$	$(2 - 2, -8 - 3) = (0, -11)$
$(4, -64)$	$(4 - 2, -64 - 3) = (2, -67)$



Perform the translations last. Subtract 2 from the x-coordinate and 3 from the y-coordinate of each point on the red graph to obtain the corresponding point on the blue graph.

Solution B: Simplifying the equation first

$$y = -8\left(\frac{1}{2}x + 1\right)^3 - 3$$

$$y = -8\left(\frac{1}{2}(x + 2)\right)^3 - 3$$

$$y = -8\left(\frac{1}{2}\right)^3(x + 2)^3 - 3$$

$$y = -(x + 2)^3 - 3$$

$$y = -(x + 2)^3 - 3$$

Factor out the coefficient of x, and apply the exponent to both parts of the product.

Simplify.

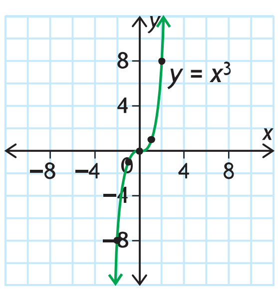
$y = x^3$ is

- vertically reflected in the x-axis
- translated 2 units to the left
- translated 3 units down

$a = -1$
$d = -2$
$c = -3$

$y = x^3$

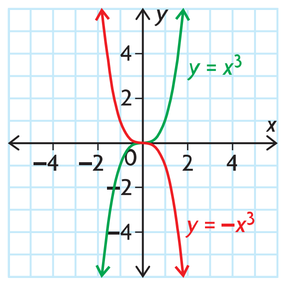
x	y
-2	-8
-1	-1
0	0
1	1
2	8



Begin with the graph of the parent function to be transformed.

$(x, y) \rightarrow (x, -y)$

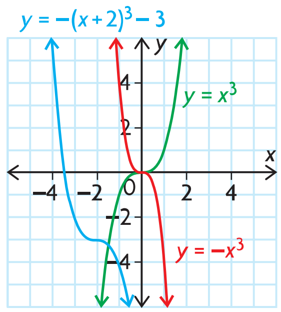
$y = x^3$	$y = -x^3$
$(-2, -8)$	$(-2, -(-8)) = (-2, 8)$
$(-1, -1)$	$(-1, -(-1)) = (-1, 1)$
$(0, 0)$	$(0, -(0)) = (0, 0)$
$(1, 1)$	$(1, -(1)) = (1, -1)$
$(2, 8)$	$(2, -(8)) = (2, -8)$



Apply the reflection in the *x*-axis first. Multiply the *y*-coordinate of each key point on the green graph to obtain the corresponding point on the red graph.

$(x, -y) \rightarrow (x - 2, -y + 3)$

$y = -x^3$	$y = -(x + 2)^3 - 3$
$(-2, 8)$	$(-2 - 2, 8 - 3) = (-4, 5)$
$(-1, 1)$	$(-1 - 2, 1 - 3) = (-3, -2)$
$(0, 0)$	$(0 - 2, 0 - 3) = (-2, -3)$
$(1, -1)$	$(1 - 2, -1 - 3) = (-1, -4)$
$(2, -8)$	$(2 - 2, -8 - 3) = (0, -11)$



Apply the translations last. Subtract 2 from the *x*-coordinate and 3 from the *y*-coordinate of each point on the red graph to obtain the corresponding point on the blue graph.

Note that the final graph, shown in blue, is the same from the two different solutions shown above. Two different sets of transformations have resulted in the same final graph.

EXAMPLE 2 Selecting a strategy to determine the roots of a quartic function

Determine the *x*-intercept(s) of the function $y = 3(x + 6)^4 - 48$.

Solution A: Using algebra

$y = 3(x + 6)^4 - 48$

$0 = 3(x + 6)^4 - 48$

Let $y = 0$, and solve for x.

$48 = 3(x + 6)^4$

$\frac{48}{3} = \frac{3(x + 6)^4}{3}$

Use inverse operations to isolate $(x + 6)^4$. Add 48 to both sides, and then divide both sides by 3.

$16 = (x + 6)^4$

$\pm\sqrt[4]{16} = \sqrt[4]{(x + 6)^4}$

Take the fourth root of both sides.

$\pm 2 = x + 6$

$2 = x + 6$ and $-2 = x + 6$

$2 - 6 = x \qquad -2 - 6 = x$

$-4 = x \qquad -8 = x$

Any even root of a number has both a positive value and a negative value.

The *x*-intercepts are -4 and -8.

Solution B: Using transformations and a graphing calculator

Enter the equation into a graphing calculator.

To graph $y = 3(x + 6)^4 - 48$, $y = x^4$ must be
- vertically stretched by a factor of 3 since $a = 3$
- translated 6 units to the left and 48 units down, since $d = -6$ and $c = -48$

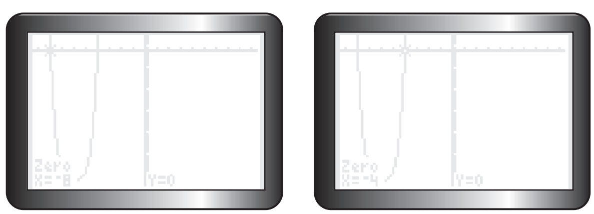
Use transformations to help determine suitable window settings.

$(x, y) \rightarrow (x - 6, 3y - 48)$
$(0, 0) \rightarrow (0 - 6, 3(0) - 48)$
$= (-6, -48)$

Determine the new location of the turning point, (0, 0), of the parent function. Subtract 6 from the x-coordinate. Multiply the y-coordinate by 3, and subtract 48.

Since $a > 0$, the graph opens up.

Adjust the window settings so that Xmin is to the left of $x = -6$ and Ymin is below $y = -48$.



Graph the function. Use the zero operation to determine the locations of the zeros.

Tech Support

For help using the zero operation on a graphing calculator to determine the zeros of a function, see Technical Appendix, T-8.

The x-intercepts are -8 and -4.

In Summary

Key Ideas

- The polynomial function $y = a(k(x - d))^n + c$ can be graphed by applying transformations to the graph of the parent function $y = x^n$, where $n \in \mathbf{N}$. Each point (x, y) on the graph of the parent function changes to $\left(\frac{x}{k} + d, ay + c\right)$.
- When using transformations to graph a function in the fewest steps, you can apply a and k together, and then c and d together.

Need to Know

- In $y = a(k(x - d))^n + c$,
  - the value of a represents a vertical stretch/compression and possibly a vertical reflection
  - the value of k represents a horizontal stretch/compression and possibly a horizontal reflection
  - the value of d represents a horizontal translation
  - the value of c represents a vertical translation

CHECK Your Understanding

1. Match each function with the most likely graph. Explain your reasoning.

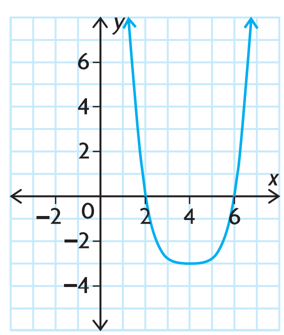
a) $y = 2(x - 3)^3 + 1$

b) $y = -\frac{1}{3}(x + 1)^3 - 1$

c) $y = 0.2(x - 4)^4 - 3$

d) $y = -1.5(x + 3)^4 + 4$

A



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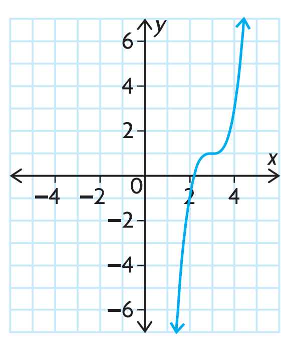

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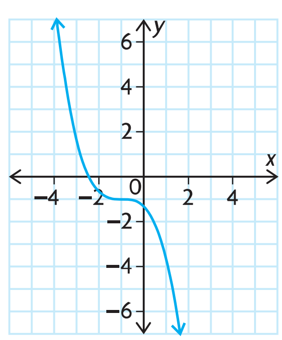
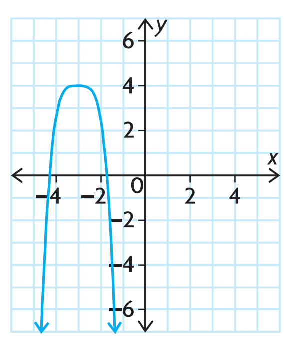


D



2. State the parent function that must be transformed to create the graph of each of the following functions. Then describe the transformations that must be applied to the parent function.

a) $y = \frac{5}{4}x^4 + 3$

b) $y = 3x - 4$

c) $y = (3x + 4)^3 - 7$

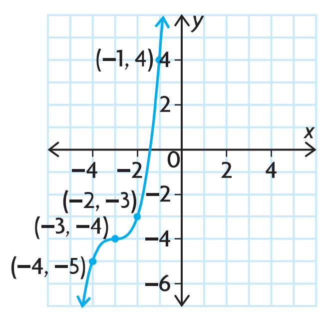
d) $y = -(x + 8)^4$

e) $y = -4.8(x - 3)(x - 3)$

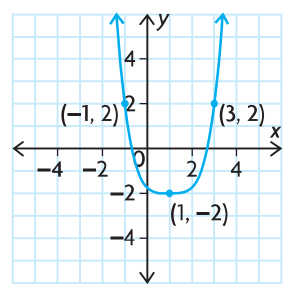
f) $y = 2\left(\frac{1}{5}x + 7\right)^3 - 4$

3. Describe the transformations that were applied to the parent function to create each of the following graphs. Then write the equation of the transformed function.

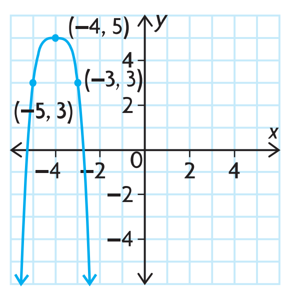
a) parent function: $y = x^3$



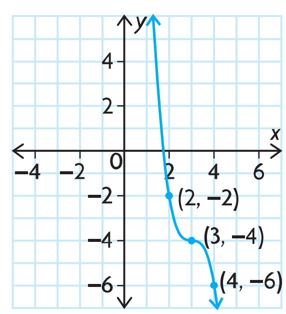
c) parent function: $y = x^4$



b) parent function: $y = x^4$



d) parent function: $y = x^3$



PRACTISING

4. Describe the transformations that were applied to $y = x^3$ to create each of the following functions.
K

a) $y = 12(x - 9)^3 - 7$

b) $y = \left(\frac{7}{8}(x + 1)\right)^3 + 3$

c) $y = -2(x - 6)^3 - 8$

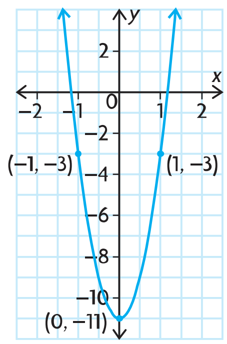
d) $y = (x + 9)(x + 9)(x + 9)$

e) $y = -2(-3(x - 4))^3 - 5$

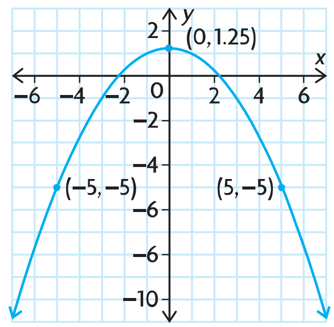
f) $y = \left(\frac{3}{4}(x - 10)\right)^3$

5. For each graph, determine the equation of the function in the form $y = a(x - h)^2 + k$. Then describe the transformations that were applied to $y = x^2$ to obtain each graph.

a)



b)



6. The function $y = x^3$ has undergone the following sets of transformations. If $y = x^3$ passes through the points $(-1, -1)$, $(0, 0)$, and $(2, 8)$, list the coordinates of these transformed points on each new curve.

a) vertically compressed by a factor of $\frac{1}{2}$, horizontally compressed by a factor of $\frac{1}{5}$, and horizontally translated 6 units to the left

b) reflected in the y-axis, horizontally stretched by a factor of 2, and vertically translated 3 units up

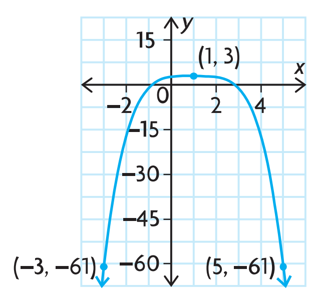
c) reflected in the x-axis, vertically stretched by a factor of 3, horizontally translated 4 units to the right, and vertically translated $\frac{1}{2}$ of a unit down

d) vertically compressed by a factor of $\frac{1}{10}$, horizontally stretched by a factor of 7, and vertically translated 2 units down

e) reflected in the y-axis, reflected in the x-axis, and vertically translated $\frac{9}{10}$ of a unit up

f) horizontally stretched by a factor of 7, horizontally translated 4 units to the left, and vertically translated 2 units down

7. T The graph shown is a result of transformations applied to $y = x^4$. Determine the equation of this transformed function.



8. A Dikembe has reflected the function $g(x) = x^3$ in the x-axis, vertically compressed it by a factor of $\frac{2}{3}$, horizontally translated it 13 units to the right, and vertically translated it 13 units down. Three points on the resulting curve are $\left(11, -\frac{23}{3}\right)$, $(13, -13)$, and $\left(15, -\frac{55}{3}\right)$. Determine the original coordinates of these three points on $g(x)$.

9. Determine the x-intercepts of each of the following polynomial functions. Round to two decimal places, if necessary.
   a) $y = 2(x + 3)^4 - 2$
   b) $y = (x - 2)^3 - 8$
   c) $y = -3(x + 1)^4 + 48$
   d) $y = -5(x + 6)^4 - 10$
   e) $y = 4(x - 8)^4 - 12$
   f) $y = -(2x + 5)^3 - 20$

10. Consider the function $y = 2(x - 4)^n + 1$, $n \in \mathbf{N}$.
    a) How many zeros will the function have if $n = 3$? Explain how you know.
    b) How many zeros will the function have if $n = 4$? Explain how you know.
    c) Make a general statement about the number of zeros that the function will have, for any value of n. Explain your reasoning.

11. a) For what values of n will the reflection of the function $y = x^n$ in the x-axis be the same as its reflection in the y-axis. Explain your reasoning.
    b) For what values of n will the reflections be different? Explain your reasoning.

12. Consider the function $y = x^3$.
    C a) Use algebraic and graphical examples to describe all the transformations that could be applied to this function.
    b) Explain why just creating a single table of values is not always the best way to sketch the graph of a function.

Extending

13. Can you create the graph of the function $y = 2(x - 1)(x + 4)(x - 5)$ by transforming the function $y = (x - 4)(x + 1)(x - 8)$? Explain.

14. Transform the graph of the function $y = (x - 1)^2(x + 1)^2$ to determine the roots of the function $y = 2(x - 1)^2(x + 1)^2 + 1$.

15. The function $f(x) = x^2$ was transformed by vertically stretching it, horizontally compressing it, horizontally translating it, and vertically translating it. The resulting function was then transformed again by reflecting it in the x-axis, vertically compressing it by a factor of $\frac{4}{5}$, horizontally compressing it by a factor of $\frac{1}{2}$, and vertically translating it 6 units down. The equation of the final function is $f(x) = -4(4(x + 3))^2 - 5$. What was the equation of the function after it was transformed the first time?

3 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: How can you tell whether an expression is a polynomial?

A: A polynomial is an expression in which the coefficients are real numbers and the exponents on the variables are whole numbers.

For example, consider the following expressions:

$3x^2 - 5x^3 + \frac{1}{2}x, \quad \sqrt{x} + 4x^2 - 3, \quad \frac{x+3}{2x-5}, \quad \sqrt{5}x^2 + 8x - 10$

Only two of these expressions, $3x^2 - 5x^3 + \frac{1}{2}x$ and $\sqrt{5}x^2 + 8x - 10$, are polynomials.

Study *Aid*

- See Lesson 3.1.
- Try Mid-Chapter Review Questions 1 and 2.

Q: How can you describe the characteristics of the graph of a polynomial function by looking at its equation?

A: The degree of the function and the sign of the leading coefficient can be used to determine the end behaviours of the graph.

- If the degree of the function is odd and the leading coefficient is
  - negative, then the function extends from above the x-axis to below the x-axis
  - positive, then the function extends from below the x-axis to above the x-axis
- If the degree of the function is even and the leading coefficient is
  - negative, then both ends of the function are below the x-axis
  - positive, then both ends of the function are above the x-axis
- For any polynomial function, the maximum number of turning points is one less than the degree of the function.
- If the degree of the function is
  - odd, then there must be an even number of turning points
  - even, then there must be an odd number of turning points

Study *Aid*

- See Lesson 3.2, Example 1.
- Try Mid-Chapter Review Questions 3 and 4.

Q: How can you sketch the graph of a polynomial function that is in factored form?

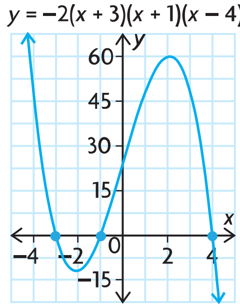
A: The factors of the function can be used to determine the real roots of the corresponding polynomial equation. These roots are the x-intercepts of the graph. Use other characteristics of the function, such as end behaviours, turning points, and the order of each factor, to approximate the shape of the graph.

Study *Aid*

- See Lesson 3.3, Example 2.
- Try Mid-Chapter Review Questions 5, 6, and 7.

For example, to sketch the graph of $y = -2(x + 3)(x + 1)(x - 4)$, first determine the x-intercepts. They are -3, -1, and 4. Because the order of each factor is 1, the graph has a linear shape near each zero. Because the leading coefficient is -2 and the degree is 3, the graph extends from the second quadrant to the fourth quadrant. There are, at most, two turning points.

$y = -2(x + 3)(x + 1)(x - 4)$

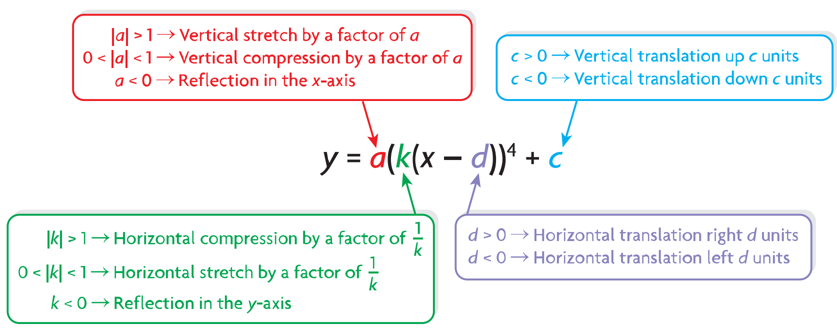


Study Aid

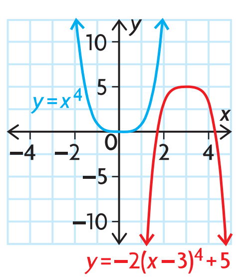
- See Lesson 3.4, Examples 1 and 2.
- Try Mid-Chapter Review Questions 8 and 9.

Q: How can you sketch the graph of a polynomial function using transformations?

A: If the equation is in the form $y = a(k(x - d))^n + c$, then transform the graph of $y = x^n$ as follows:



For example, to sketch the graph of $y = -2(x - 3)^4 + 5$, vertically stretch the graph of $y = x^4$ by a factor of 2, reflect it through the x-axis, and then translate it 3 units to the right and 5 units up. As a result of these transformations, every point (x, y) on the graph of $y = x^4$ changes to $(x + 3, -2y + 5)$.



PRACTICE Questions

Lesson 3.1

1. Determine whether or not each function is a polynomial function. If it is not a polynomial function, explain why.

a) $f(x) = \frac{2}{3}x^4 + x^2 - 1$
b) $f(x) = x^{\frac{5}{2}} - 7x^2 + 3$
c) $f(x) = \sqrt{10x^3} - 16x^2 + 15$
d) $f(x) = \frac{x^2 + 4x + 2}{x - 2}$

2. For each of the following, give an example of a polynomial function that has the characteristics described.

a) a function of degree 3 that has four terms
b) a function of degree 4 that has three terms
c) a function of degree 6 that has two terms
d) a function of degree 5 that has five terms

Lesson 3.2

3. State the end behaviours of each of the following functions.

a) $f(x) = -11x^3 + x^2 - 2$
b) $f(x) = 70x^2 - 67$
c) $f(x) = x^3 - 1000$
d) $f(x) = -13x^4 - 4x^3 - 2x^2 + x + 5$

4. State whether each function has an even number of turning points or an odd number of turning points.

a) $f(x) = 6x^3 + 2x$
b) $f(x) = -20x^6 - 5x^3 + x^2 - 17$
c) $f(x) = 22x^4 - 4x^3 + 3x^2 - 2x + 2$
d) $f(x) = -x^5 + x^4 - x^3 + x^2 - x + 1$

Lesson 3.3

5. Sketch a possible graph of each of the following functions.

a) $f(x) = -(x - 8)(x + 1)$
b) $f(x) = 3(x + 3)(x + 3)(x - 1)$
c) $f(x) = (x + 2)(x - 4)(x + 2)(x - 4)$
d) $f(x) = -4(2x + 5)(x - 2)(x + 4)$

6. If the value of k is unknown, which of the following characteristics of the graph of $f(x) = k(x + 14)(x - 13)(x + 15)(x - 16)$ cannot be determined: the x-intercepts, the shape of the graph near each zero, the end behaviours, or the maximum number of turning points?

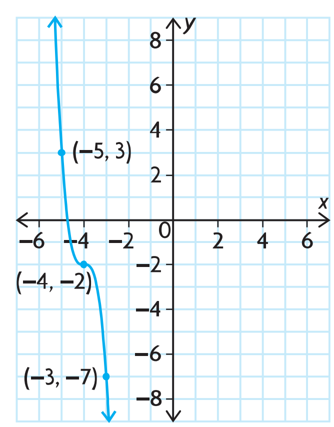
7. Determine the equation of the polynomial function that has the following zeros and passes through the point $(7, 5000)$: $x = 2$ (order 1), $x = -3$ (order 2), and $x = 5$ (order 1).

Lesson 3.4

8. Describe the transformations that were applied to $y = x^4$ to get each of the following functions.

a) $y = -25(3(x + 4))^4 - 60$
b) $y = 8\left(\frac{3}{4}x\right)^4 + 43$
c) $y = (-13x + 26)^4 + 13$
d) $y = \frac{8}{11}(-x)^4 - 1$

9. Describe the transformations that were applied to $y = x^3$ to produce the following graph.



3.5 Dividing Polynomials

GOAL

Use a variety of strategies to determine the quotient when one polynomial is divided by another polynomial.

LEARN ABOUT the Math

Recall that long division can be used to determine the quotient of two numbers. For example, $107 \div 4$ can be evaluated as follows:

$$\begin{array}{rl} & 26 \leftarrow \text{quotient} \\ \text{divisor} \rightarrow 4 & \overline{)107} \leftarrow \text{dividend} \\ & \underline{8\downarrow} \\ & 27 \\ & \underline{24} \\ & 3 \leftarrow \text{remainder} \end{array}$$

Every division statement that involves numbers can be rewritten using multiplication and addition. The multiplication is the quotient, and the addition is the remainder. For example, since $107 = (4)(26) + 3$, then $\frac{107}{4} = 26 + \frac{3}{4}$. The quotient is 26, and the remainder is 3.

? How can you use a similar strategy to determine the quotient of $(3x^3 - 5x^2 - 7x - 1) \div (x - 3)$?

EXAMPLE 1 Selecting a strategy to divide a polynomial by a binomial

Determine the quotient of $(3x^3 - 5x^2 - 7x - 1) \div (x - 3)$.

Solution A: Using polynomial division

$$x - 3\overline{)3x^3 - 5x^2 - 7x - 1}$$

Focus on the *first terms* of the dividend and the divisor, and then determine the quotient when these terms are divided. Here, the first term of the dividend is $3x^3$ and the first term of the divisor is x.

$$\begin{array}{r} 3x^2 \\ x - 3\overline{)3x^3 - 5x^2 - 7x - 1} \end{array}$$

Since $3x^3 \div x = 3x^2$, this becomes the first term of the quotient. Place $3x^2$ above the term of the dividend with the same degree.

$$\begin{array}{r} 3x^2 \\ x-3\overline{)\,3x^3 - 5x^2 - 7x - 1} \\ \underline{3x^3 - 9x^2} \\ 4x^2 \end{array}$$

Multiply $3x^2$ by the divisor, and write the answer below the dividend. Make sure that you line up "like terms." $3x^2(x - 3) = 3x^3 - 9x^2$. Subtract this product from the dividend.

$$\begin{array}{r} 3x^2 + 4x \\ x-3\overline{)\,3x^3 - 5x^2 - 7x - 1} \\ \underline{-3x^3 + 9x^2} \downarrow \\ 4x^2 - 7x \\ \underline{4x^2 - 12x} \\ 5x \end{array}$$

Now focus on x in the divisor $x - 3$ and $4x^2$ in the expression $4x^2 - 7x$. Determine the quotient when these terms are divided. Since $4x^2 \div x = 4x$, place $4x$ above the x in the dividend. Multiply $4x$ by the divisor. Write the answer below the last line (making sure that you line up like terms), and then subtract.

$$\begin{array}{rr} & 3x^2 + 4x + 5 \\ & x-3\overline{)\,3x^3 - 5x^2 - 7x - 1} \\ 3x^2(x-3) \rightarrow & \underline{3x^3 - 9x^2} \downarrow \quad \downarrow \\ & 4x^2 - 7x \\ 4x(x-3) \rightarrow & \underline{4x^2 - 12x} \\ & 5x - 1 \\ 5(x-3) \rightarrow & \underline{5x - 15} \\ & 14 \end{array}$$

Repeat this process until the degree of the remainder is less than the degree of the divisor.

Since the divisor has degree 1, the remainder should be a constant.

$$3x^3 - 5x^2 - 7x - 1 = (x - 3)(3x^2 + 4x + 5) + 14$$

Write the multiplication statement that shows how the divisor, dividend, quotient, and remainder are all related.

$$\begin{aligned} &(x - 3)(3x^2 + 4x + 5) + 14 \\ &= 3x^3 + 4x^2 + 5x - 9x^2 - 12x - 15 + 14 \\ &= 3x^3 - 5x^2 - 7x - 1 \end{aligned}$$

To check, expand and simplify the right side of the division statement.

The result is the dividend, which confirms the division was done correctly.

Solution B: Using synthetic division

$(3x^3 - 5x^2 - 7x - 1) \div (x - 3) \rightarrow k = 3$

$$\begin{array}{r|rrrr} 3 & 3 & -5 & -7 & -1 \\ & & & & \\ \hline & & & & \end{array}$$

Synthetic division is an efficient way to divide a polynomial by a binomial of the form $(x - k)$.

Create a chart that contains the coefficients of the dividend, as shown. The dividend and binomial must be written with its terms in descending order, by degree.

$$\begin{array}{r|rrrr} 3 & 3 & -5 & -7 & -1 \\ & \downarrow & & & \\ & {}_{\times 3 \nearrow} & 9 & & \\ \hline & 3 & & & \end{array}$$

Bring the first term down. This is now the coefficient of the first term of the quotient.

Multiply it by k, and write the answer below the second term of the dividend.

$$\begin{array}{r|rrrr} 3 & 3 & -5 & -7 & -1 \\ & & \text{add}\downarrow & & \\ & & 9 & & \\ \hline & 3 & 4 & & \end{array}$$

Now add the terms together.

$$\begin{array}{r|rrrr} 3 & 3 & -5 & -7 & -1 \\ & & & \text{add}\downarrow & \\ & & 9 & {}_{\times 3 \nearrow} 12 & \\ \hline & 3 & 4 & 5 & \end{array}$$

Repeat this process for the answer you just obtained.

$$\begin{array}{r|rrrr} 3 & 3 & -5 & -7 & -1 \\ & & & & \text{add}\downarrow \\ & & 9 & 12 & {}_{\times 3 \nearrow} 15 \\ \hline & 3 & 4 & 5 & 14 \end{array}$$

Repeat this process one last time.

$$\begin{array}{r|rrrr} 3 & 3 & -5 & -7 & -1 \\ & & 9 & 12 & 15 \\ \hline & 3 & 4 & 5 & 14 \end{array}$$

coefficients of quotient: 3, 4, 5; remainder: 14

$3x^2 + 4x + 5$

The last number below the chart is the remainder. The first numbers are the coefficients of the quotient, starting with the degree that is one less than the original dividend.

$3x^3 - 5x^2 - 7x - 1$
$= (x - 3)(3x^2 + 4x + 5) + 14$

Write the corresponding multiplication statement.

Reflecting

A. When dividing an *n*th degree polynomial by a *k*th degree polynomial, what degree is the quotient? What degree is the remainder?

B. If you divide a number by another number and the remainder is zero, what can you conclude? Do you think you can make the same conclusion for polynomials? Explain.

C. If you had a divisor of $x + 5$, what value of k would you use in synthetic division?

APPLY the Math

EXAMPLE 2 Selecting a strategy to determine the remainder in polynomial division

Determine the remainder of $\dfrac{5x - 2x^3 + 3 + x^4}{1 + 2x + x^2}$.

Solution

$$x^2 + 2x + 1 \overline{)\, x^4 - 2x^3 + 0x^2 + 5x + 3}$$

Write the terms of the dividend and the quotient in descending order, by degree.

Since there is no x^2-term in the dividend, use 0 as the coefficient of x^2 to make the like terms line up properly.

$$\begin{array}{rl} & x^2 \\ x^2 + 2x + 1 \big) & x^4 - 2x^3 + 0x^2 + 5x + 3 \\ & \underline{x^4 + 2x^3 + 1x^2} \downarrow \\ & \quad -4x^3 - 1x^2 + 5x \end{array}$$

Follow the same steps that you use for long division with numbers.

$$\begin{array}{rl} & x^2 - 4x + 7 \\ x^2 + 2x + 1 \big) & x^4 - 2x^3 + 0x^2 + 5x + 3 \\ x^2(x^2 + 2x + 1) \rightarrow & \underline{x^4 + 2x^3 + 1x^2} \\ & \quad -4x^3 - 1x^2 + 5x \\ -4x(x^2 + 2x + 1) \rightarrow & \quad \underline{-4x^3 - 8x^2 - 4x} \\ & \qquad 7x^2 + 9x + 3 \\ 7(x^2 + 2x + 1) \rightarrow & \qquad \underline{7x^2 + 14x + 7} \\ & \qquad\quad -5x - 4 \end{array}$$

Repeat this process until the degree of the remainder is less than the degree of the divisor.

Since the divisor was degree 2, the remainder should be degree 1.

The remainder is $-5x - 4$.

$$\begin{aligned} &x^4 - 2x^3 + 5x + 3 \\ &\quad = (x^2 + 2x + 1)(x^2 - 4x + 7) - 5x - 4 \end{aligned}$$

Write the corresponding division statement.

EXAMPLE 3 Selecting a strategy to determine whether one polynomial is a factor of another polynomial

Determine whether $x + 2$ is a factor of $13x - 2x^3 + x^4 - 6$.

Solution

Use synthetic division to divide $13x - 2x^3 + x^4 - 6$ by $x + 2$.

Rearrange the terms of the dividend in descending order. An x^2-term, with a coefficient of 0, needs to be added to the dividend.

$$(x^4 - 2x^3 + 0x^2 + 13x - 6) \div (x + 2)$$

-2	1	-2	0	13	-6
	$\downarrow$	-2	8	-16	6
	1	-4	8	-3	0

In this example, $k = -2$. Multiply and add to complete the chart.

$x + 2$ is a factor, and so is $x^3 - 4x^2 + 8x - 3$.

$$(x + 2)(x^3 - 4x^2 + 8x - 3)$$
$$= x^4 - 2x^3 + 0x^2 + 13x - 6$$

The quotient is $x^3 - 4x^2 + 8x - 3$, and the remainder is 0. Since the remainder is 0, $x + 2$ must be a factor of the dividend.

EXAMPLE 4 Selecting a strategy to determine the factors of a polynomial

$2x + 3$ is one factor of the function $f(x) = 6x^3 + 5x^2 - 16x - 15$. Determine the other factors. Then determine the zeros, and sketch a graph of the polynomial.

Solution

$$(2x + 3) = 2\left(x + \frac{3}{2}\right)$$
$$= 2\left(x - \left(-\frac{3}{2}\right)\right) \rightarrow k = -\frac{3}{2}$$

To use synthetic division, the divisor must be of the form $(x - k)$. Rewrite the divisor by dividing out the common factor 2 (the coefficient of x).

The division can now be done in two steps.

$$\begin{array}{r|rrrr} -\frac{3}{2} & 6 & 5 & -16 & -15 \\ & \downarrow & -9 & 6 & 15 \\ \hline & 6 & -4 & -10 & 0 \end{array}$$

First, divide $6x^3 + 5x^2 - 16x - 15$ by $\left(x + \frac{3}{2}\right)$. This means that

$$\frac{6x^3 + 5x^2 - 16x - 15}{\left(x + \frac{3}{2}\right)} = 6x^2 - 4x - 10 + \frac{0}{\left(x + \frac{3}{2}\right)}.$$

$$\frac{1}{2} \times \left[\frac{6x^3 + 5x^2 - 16x - 15}{\left(x + \frac{3}{2}\right)}\right]$$

$$= \frac{1}{2} \times \left[6x^2 - 4x - 10 + \frac{0}{\left(x + \frac{3}{2}\right)}\right]$$

$$\frac{6x^3 + 5x^2 - 16x - 15}{2\left(x + \frac{3}{2}\right)} = \frac{6x^2}{2} - \frac{4x}{2} - \frac{10}{2} + \frac{0}{2\left(x + \frac{3}{2}\right)}$$

$$\frac{6x^3 + 5x^2 - 16x - 15}{(2x + 3)} = 3x^2 - 2x - 5 + 0$$

Second, since the original divisor was $(2x + 3)$ or $2\left(x + \frac{3}{2}\right)$, multiply both sides by $\frac{1}{2}$ to get the correct multiplication statement.

Notice what this means—we only needed to divide our solution by 2 in the synthetic division.

$$\begin{array}{r|rrrr} -\frac{3}{2} & 6 & 5 & -16 & -15 \\ & \downarrow & -9 & 6 & 15 \\ \hline & 6 & -4 & -10 & 0 \\ & \div 2 & \div 2 & \div 2 & \div 2 \\ & 3 & -2 & -5 & 0 \end{array}$$

There is no remainder, which verifies that $2x + 3$ is a factor of the dividend.

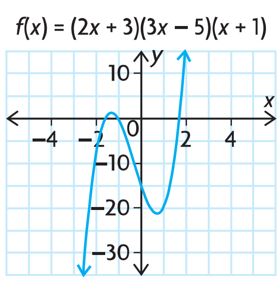
$$\begin{aligned} f(x) &= 6x^3 + 5x^2 - 16x - 15 \\ &= (2x + 3)(3x^2 - 2x - 5) \\ &= (2x + 3)(3x - 5)(x + 1) \end{aligned}$$

Factor the quotient.

Since $f(x) = (2x + 3)(3x - 5)(x + 1)$, the zeros are $-\frac{3}{2}$, $\frac{5}{3}$, and -1.

Determine the zeros by setting each factor equal to zero and solving for x.

An approximate graph of $y = f(x)$ is shown below.



Use the zeros to locate and plot the x-intercepts. Determine the y-intercept, and plot this point. Examine the standard and factored forms of the equation to determine the end behaviours of the function and the shape of the graph near the zeros. Sketch the graph.

In Summary

Key Idea

- Polynomials can be divided in much the same way that numbers are divided.

Need to Know

- A polynomial can be divided by a polynomial of the same degree or less.
- Synthetic division is a shorter form of polynomial division. It can only be used when the divisor is linear (that is, $(x - k)$ or $(ax - k)$).
- When using polynomial or synthetic division,
  - terms should be arranged in descending order of degree, in both the divisor and the dividend, to make the division easier to perform
  - zero must be used as the coefficient of any missing powers of the variable in both the divisor and the dividend
- If the remainder of polynomial or synthetic division is zero, both the divisor and the quotient are factors of the dividend.

CHECK *Your Understanding*

1. **a)** Divide $x^4 - 16x^3 + 4x^2 + 10x - 11$ by each of the following binomials.

i) $x - 2$ **ii)** $x + 4$ **iii)** $x - 1$

b) Are any of the binomials in part a) factors of $x^4 - 16x^3 + 4x^2 + 10x - 11$? Explain.

2. State the degree of the quotient for each of the following division statements, if possible.

a) $(x^4 - 15x^3 + 2x^2 + 12x - 10) \div (x^2 - 4)$

b) $(5x^3 - 4x^2 + 3x - 4) \div (x + 3)$

c) $(x^4 - 7x^3 + 2x^2 + 9x) \div (x^3 - x^2 + 2x + 1)$

d) $(2x^2 + 5x - 4) \div (x^4 + 3x^3 - 5x^2 + 4x - 2)$

3. Complete the divisions in question 2, if possible.

4. Complete the following table.

Dividend	Divisor	Quotient	Remainder
$2x^3 - 5x^2 + 8x + 4$	$x + 3$	$2x^2 - 11x + 41$	
	$2x + 4$	$3x^3 - 5x + 8$	-3
$6x^4 + 2x^3 + 3x^2 - 11x - 9$		$2x^3 + x - 4$	-5
$3x^3 + x^2 - 6x + 16$	$x + 2$		8

PRACTISING

5. Calculate each of the following using long division.

K

a) $(x^3 - 2x + 1) \div (x - 4)$
b) $(x^3 + 2x^2 - 6x + 1) \div (x + 2)$
c) $(2x^3 + 5x^2 - 4x - 5) \div (2x + 1)$
d) $(x^4 + 3x^3 - 2x^2 + 5x - 1) \div (x^2 + 7)$
e) $(x^4 + 6x^2 - 8x + 12) \div (x^3 - x^2 - x + 1)$
f) $(x^5 + 4x^4 + 9x + 8) \div (x^4 + x^3 + x^2 + x - 2)$

6. Calculate each of the following using synthetic division.
a) $(x^3 - 7x - 6) \div (x - 3)$
b) $(2x^3 - 7x^2 - 7x + 19) \div (x - 1)$
c) $(6x^4 + 13x^3 - 34x^2 - 47x + 28) \div (x + 3)$
d) $(2x^3 + x^2 - 22x + 20) \div (2x - 3)$
e) $(12x^4 - 56x^3 + 59x^2 + 9x - 18) \div (2x + 1)$
f) $(6x^3 - 2x - 15x^2 + 5) \div (2x - 5)$

7. Each divisor was divided into another polynomial, resulting in the given quotient and remainder. Find the other polynomial (the dividend).
a) divisor: $x + 10$, quotient: $x^2 - 6x + 9$, remainder: -1
b) divisor: $3x - 2$, quotient: $x^3 + x - 12$, remainder: 15
c) divisor: $5x + 2$, quotient: $x^3 + 4x^2 - 5x + 6$, remainder: $x - 2$
d) divisor: $x^2 + 7x - 2$, quotient: $x^4 + x^3 - 11x + 4$, remainder: $x^2 - x + 5$

8. Determine the remainder, r, to make each multiplication statement true.
a) $(2x - 3)(3x + 5) + r = 6x^2 + x + 5$
b) $(x + 3)(x + 5) + r = x^2 + 9x - 7$
c) $(x + 3)(x^2 - 1) + r = x^3 + 3x^2 - x - 3$
d) $(x^2 + 1)(2x^3 - 1) + r = 2x^5 + 2x^3 + x^2 + 1$

9. Each dividend was divided by another polynomial, resulting in the given quotient and remainder. Find the other polynomial (the divisor).
a) dividend: $5x^3 + x^2 + 3$, quotient: $5x^2 - 14x + 42$, remainder: -123
b) dividend: $10x^4 - x^2 + 20x - 2$, quotient: $10x^3 - 100x^2 + 999x - 9970$, remainder: 99 698
c) dividend: $x^4 + x^3 - 10x^2 - 1$, quotient: $x^3 - 3x^2 + 2x - 8$, remainder: 31
d) dividend: $x^3 + x^2 + 7x - 7$, quotient: $x^2 + 3x + 13$, remainder: 19

10. Determine whether each binomial is a factor of the given polynomial.

a) $x + 5, x^3 + 6x^2 - x - 30$

b) $x + 2, x^4 - 5x^2 + 4$

c) $x - 2, x^4 - 5x^2 + 6$

d) $2x - 1, 2x^4 - x^3 - 4x^2 + 2x + 1$

e) $3x + 5, 3x^6 + 5x^5 + 9x^2 + 17x - 1$

f) $5x - 1, 5x^4 - x^3 + 10x - 10$

11. **A** The volume of a rectangular box is $(x^3 + 6x^2 + 11x + 6)$ cm^3. The box is $(x + 3)$ cm long and $(x + 2)$ cm wide. How high is the box?

12. a) $8x^3 + 10x^2 - px - 5$ is divisible by $2x + 1$. There is no remainder. Find the value of p.

b) When $x^6 + x^4 - 2x^2 + k$ is divided by $1 + x^2$, the remainder is 5. Find the value of k.

13. The polynomial $x^3 + px^2 - x - 2$, $p \in \mathbf{R}$, has $x - 1$ as a factor. What is the value of p?

14. **T** Let $f(x) = x^n - 1$, where n is an integer and $n \geq 1$. Is $f(x)$ always divisible by $x - 1$? Justify your decision.

15. If the divisor of a polynomial, $f(x)$, is $x - 4$, then the quotient is $x^2 + x - 6$ and the remainder is 7.

a) Write the division statement.

b) Rewrite the division statement by factoring the quotient.

c) Graph $f(x)$ using your results in part b).

16. **C** Use an example to show how synthetic division is essentially the same as regular polynomial division.

Extending

17. The volume of a cylindrical can is $(4\pi x^3 + 28\pi x^2 + 65\pi x + 50\pi)$ cm^3. The can is $(x + 2)$ cm high. What is the radius?

18. Divide.

a) $(x^4 + x^3y - xy^3 - y^4) \div (x^2 - y^2)$

b) $(x^4 - 2x^3y + 2x^2y^2 - 2xy^3 + y^4) \div (x^2 + y^2)$

19. Is $x - y$ a factor of $x^3 - y^3$? Justify your answer.

20. If $f(x) = (x + 5)q(x) + (x + 3)$, what is the first multiple of $(x + 5)$ that is greater than $f(x)$?

3.6 Factoring Polynomials

GOAL

Make connections between a polynomial function and its remainder when divided by a binomial.

YOU WILL NEED

- graphing calculator

INVESTIGATE the Math

Consider the polynomial function $f(x) = x^3 + 4x^2 + x - 6$.

? How can you determine the factors of a polynomial function of degree 3 or greater?

A. For $f(x) = x^3 + 4x^2 + x - 6$, determine $f(2)$.

B. Determine the quotient of $\frac{f(x)}{x - 2}$, and state the remainder of the division. What do you notice?

C. Predict what the remainder of the division $\frac{f(x)}{x + 2}$ will be. What does this tell you about the relationship between $f(x)$ and $x + 2$?

D. Copy and complete the following table by choosing eight additional values of x. Use both positive and negative values. Leave space to add more columns in part E.

a	$f(a)$
2	20
−2	

E. Add the following two columns to your table, and complete your table for the other values of x.

a	$f(a)$	$\frac{f(x)}{x-a}$	Remainder
2	20	$\frac{f(x)}{x-2} = x^2 + 6x + 13 + \frac{20}{x-2}$	20

F. For which values of a in your table is $x - a$ a factor of $f(x)$? Can you see a pattern? Explain how you know there is a pattern.

G. How do the values of a that you identified in part F relate to the graph of $f(x)$?

H. Use your table and/or the graph to determine all the factors of $f(x)$.

I. Create a new factorable function, $g(x)$, and check whether the pattern you saw in part F exists for your new function.

Reflecting

J. What is the relationship between $f(a)$ and the quotient $\frac{f(x)}{x-a}$?

K. What is the value of $f(a)$ when $x - a$ is a factor?

L. How can you use your answer in part K to determine the factors of a polynomial?

EXAMPLE 1 Using reasoning to determine a remainder

Determine the remainder when $x^3 + 7x^2 + 2x - 5$ is divided by $x + 7$.

Solution

Let $f(x) = x^3 + 7x^2 + 2x - 5$. ← Assign a function name to the expression given.

$f(x) = (x + 7)(\text{quotient}) + \text{remainder}$ ← $f(x)$ can be written as a division statement, with the divisor $x + 7$ multiplied by some quotient plus some remainder.

$f(-7) = (0)(\text{quotient}) + \text{remainder}$ ← If $x = -7$, the divisor will be equal to 0 and the value of the function will be equal to the remainder.

$= 0 + \text{remainder}$

$= \text{remainder}$

$f(-7) = (-7)^3 + 7(-7)^2 + 2(-7) - 5$

$= -19$

The remainder is -19.

remainder theorem
when a polynomial, $f(x)$, is divided by $x - a$, the remainder is equal to $f(a)$. If the remainder is zero, then $x - a$ is a factor of the polynomial. This can be used to help factor polynomials.

From Example 1, when $f(x)$ is divided by $x - 7$, the remainder is $f(7)$. This can be generalized into a theorem, known as the remainder theorem.

EXAMPLE 2 Selecting tools and strategies to factor a polynomial

Factor $x^3 - 5x^2 - 2x + 24$ completely.

Solution

Let $f(x) = x^3 - 5x^2 - 2x + 24$.
Possible values of a: $\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$
$x - a$ is a factor if $f(a) = 0$.

Factors of $f(x)$ will be of the form $x - a$, since the leading coefficient of $f(x)$ is 1. Since a must divide into the constant term, the possible values of a are the factors of 24.

Using a graphing calculator makes this process much faster. Enter the equation into Y1.

In the home screen, enter Y1(1). This will give you the remainder when $f(x)$ is divided by $x - 1$.

$f(1) = 18$, so $x - 1$ is not a factor.

$f(-1) = 20$
$f(2) = 8$
$f(-2) = 0$

Therefore, $x + 2$ is a factor.

Repeat this process until you find a value of a that results in a remainder of zero. The factor will be of the form $x - a$.

$$\begin{array}{r|rrrr} -2 & 1 & -5 & -2 & 24 \\ & \downarrow & -2 & 14 & -24 \\ \hline & 1 & -7 & 12 & 0 \end{array}$$

Use synthetic or regular polynomial division to divide $f(x)$ by $x + 2$.

$$\begin{aligned} f(x) &= (x + 2)(x^2 - 7x + 12) \\ &= (x + 2)(x - 4)(x - 3) \end{aligned}$$

Factor the quotient.

factor theorem

$x - a$ is a factor of $f(x)$, if and only if $f(a) = 0$

Communication Tip

"A if and only if B" means that if A is true, then B is also true, and if B is true, then A is also true.

So "$x - a$ is a factor of $f(x)$, if and only if $f(a) = 0$" means that if $x - a$ is a factor of $f(x)$, then $f(a) = 0$, and if $f(a) = 0$, then $x - a$ is a factor of $f(x)$.

The **factor theorem** is a special case of the remainder theorem.

EXAMPLE 3 Connecting the factor theorem to characteristics of the graph of a polynomial function

Sketch a graph of the function $y = 4x^4 + 6x^3 - 6x^2 - 4x$.

Solution

$$y = 4x^4 + 6x^3 - 6x^2 - 4x$$
$$= 2x(2x^3 + 3x^2 - 3x - 2)$$

First, divide out any common factors of the polynomial.

Let $f(x) = 2x^3 + 3x^2 - 3x - 2$.

$$f(1) = 2(1)^3 + 3(1)^2 - 3(1) - 2$$
$$= 0$$

Use the factor theorem to factor the remaining cubic.

$x - 1$ is a factor.

1	2	3	−3	−2
	↓	2	5	2
	2	5	2	0

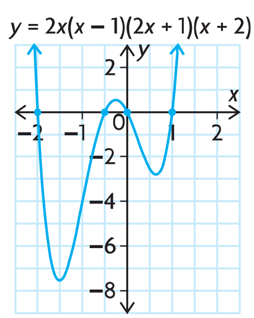
Divide to determine the other factors.

$$y = 2x(x - 1)(2x^2 + 5x + 2)$$
$$= 2x(x - 1)(2x + 1)(x + 2)$$

Factor the quotient.

The function has zeros at $x = 0, 1, -\frac{1}{2}$, and -2.

State the zeros.



Sketch a graph using the zeros and other characteristics from the standard and factored forms of the polynomial equations.

Since the degree is even and the leading coefficient is positive, the graph extends from the second quadrant to the first quadrant.

The function $y = 4x^4 + 6x^3 - 6x^2 - 4x + 0$ has a y-intercept of 0.

Each factor of $y = 2x(x - 1)(2x + 1)(x + 2)$ is order 1, so the graph has a linear shape near each zero.

EXAMPLE 4 Using a grouping strategy to factor polynomials

Factor $x^4 - 6x^3 + 2x^2 - 12x$.

Solution

$x^4 - 6x^3 + 2x^2 - 12x$

$= (x^4 - 6x^3) + (2x^2 - 12x)$ ← Group the first two terms and last two terms together.

$= x^3(x - 6) + 2x(x - 6)$ ← Divide out the common factors from each binomial.

$= (x - 6)(x^3 + 2x)$ ← Divide out the common factor of $x - 6$.

$= x(x - 6)(x^2 + 2)$ ← Divide out the common factor of x.

EXAMPLE 5 Connecting to prior knowledge to solve a problem

When $2x^3 - mx^2 + nx - 2$ is divided by $x + 1$, the remainder is -12 and $x - 2$ is a factor. Determine the values of m and n.

Solution

Let $f(x) = 2x^3 - mx^2 + nx - 2$.

$(x + 1) \rightarrow$ remainder -12	$(x - 2) \rightarrow$ remainder 0
$f(-1) = -12$	$f(2) = 0$
$2(-1)^3 - m(-1)^2 + n(-1) - 2 = -12$	$2(2)^3 - m(2)^2 + n(2) - 2 = 0$
$-2 - m - n - 2 = -12$	$16 - 4m + 2n - 2 = 0$
① $8 - n = m$	② $-4m + 2n = -14$

Set up two equations using the information given.

Simplify both equations.

Now you have a linear system of two equations in two unknowns.

Substitute equation $8 - n$ from equation ① into m in equation ②.

$-4(8 - n) + 2n = -14$ ← Solve this system of equations. Use substitution.

$-32 + 4n + 2n = -14$

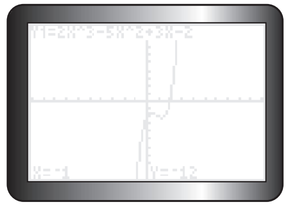
$6n = 18$

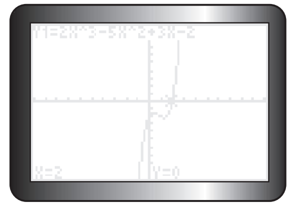
$n = 3$

$8 - (3) = m$
$5 = m$

Substitute $n = 3$ into ①.

To check, verify that $f(-1) = -12$ and $f(2) = 0$.





$n = 3$ and $m = 5$

The original polynomial is $f(x) = 2x^3 - 5x^2 + 3x - 2$.

In Summary

Key Ideas

- The remainder theorem: When a polynomial, $f(x)$, is divided by $x - a$, the remainder is equal to $f(a)$.
- The factor theorem: $x - a$ is a factor of $f(x)$, if and only if $f(a) = 0$.

Need to Know

- To factor a polynomial, $f(x)$, of degree 3 or greater,
  - use the Factor Theorem to determine a factor of $f(x)$
  - divide $f(x)$ by $x - a$
  - factor the quotient, if possible
- If a polynomial, $f(x)$, has a degree greater than 3, it may be necessary to use the factor theorem more than once.
- Not all polynomial functions are factorable.

CHECK Your Understanding

1. a) Given $f(x) = x^4 + 5x^3 + 3x^2 - 7x + 10$, determine the remainder when $f(x)$ is divided by each of the following binomials, without dividing.

i) $x - 2$

ii) $x + 4$

iii) $x - 1$

b) Are any of the binomials in part a) factors of $f(x)$? Explain.

2. Which of the following functions are divisible by $x - 1$?

a) $f(x) = x^4 - 15x^3 + 2x^2 + 12x - 10$

b) $g(x) = 5x^3 - 4x^2 + 3x - 4$

c) $h(x) = x^4 - 7x^3 + 2x^2 + 9x$

d) $j(x) = x^3 - 1$

3. Determine all the factors of the function $f(x) = x^3 + 2x^2 - 5x - 6$.

PRACTISING

4. State the remainder when $x + 2$ is divided into each polynomial.

K

a) $x^2 + 7x + 9$

b) $6x^3 + 19x^2 + 11x - 11$

c) $x^4 - 5x^2 + 4$

d) $x^4 - 2x^3 - 11x^2 + 10x - 2$

e) $x^3 + 3x^2 - 10x + 6$

f) $4x^4 + 12x^3 - 13x^2 - 33x + 18$

5. Determine whether $2x - 5$ is a factor of each polynomial.

a) $2x^3 - 5x^2 - 2x + 5$

b) $3x^3 + 2x^2 - 3x - 2$

c) $2x^4 - 7x^3 - 13x^2 + 63x - 45$

d) $6x^4 + x^3 - 7x^2 - x + 1$

6. Factor each polynomial using the factor theorem.

a) $x^3 - 3x^2 - 10x + 24$

b) $4x^3 + 12x^2 - x - 15$

c) $x^4 + 8x^3 + 4x^2 - 48x$

d) $4x^4 + 7x^3 - 80x^2 - 21x + 270$

e) $x^5 - 5x^4 - 7x^3 + 29x^2 + 30x$

f) $x^4 + 2x^3 - 23x^2 - 24x + 144$

7. Factor fully.

a) $f(x) = x^3 + 9x^2 + 8x - 60$

b) $f(x) = x^3 - 7x - 6$

c) $f(x) = x^4 - 5x^2 + 4$

d) $f(x) = x^4 + 3x^3 - 38x^2 + 24x + 64$

e) $f(x) = x^3 - x^2 + x - 1$

f) $f(x) = x^5 - x^4 + 2x^3 - 2x^2 + x - 1$

8. Use the factored form of $f(x)$ to sketch the graph of each function in question 7.

9. The polynomial $12x^3 + kx^2 - x - 6$ has $2x - 1$ as one of its factors. Determine the value of k.

10. When $ax^3 - x^2 + 2x + b$ is divided by $x - 1$, the remainder is 10. When it is divided by $x - 2$, the remainder is 51. Find a and b.

A

11. Determine a general rule to help decide whether $x - a$ and $x + a$ are factors of $x^n - a^n$ and $x^n + a^n$.

T

12. The function $f(x) = ax^3 - x^2 + bx - 24$ has three factors. Two of these factors are $x - 2$ and $x + 4$. Determine the values of a and b, and then determine the other factor.

13. Consider the function $f(x) = x^3 + 4x^2 + kx - 4$. The remainder from $f(x) \div (x + 2)$ is twice the remainder from $f(x) \div (x - 2)$. Determine the value of k.

14. Show that $x - a$ is a factor of $x^4 - a^4$.

15. Explain why the factor theorem works.

C

Extending

16. Use the factor theorem to prove that $x^2 - x - 2$ is a factor of $x^3 - 6x^2 + 3x + 10$.

17. Prove that $x + a$ is a factor of $(x + a)^5 + (x + c)^5 + (a - c)^5$.

3.7 Factoring a Sum or Difference of Cubes

GOAL

Factor the sum and difference of cubes.

LEARN ABOUT the Math

Megan has been completing her factoring homework, but she is stuck on the fourth question. She would prefer not to use the factor theorem, so she is hoping that there is a shortcut for factoring this type of polynomial.

$4x^2 - 9$	$16y^2 - 25$
$= (2x + 3)(2x - 3)$	$= (4y + 5)(4y - 5)$
$a^2 - 100$	$x^3 - 27$
$= (a + 10)(a - 10)$	$= ?$

? How can you factor a sum of cubes or a difference of cubes in one step?

EXAMPLE 1 Selecting a strategy to factor a sum or difference of cubes

Factor the expressions $(ax)^3 - b^3$ and $(ax)^3 + b^3$ for your choice of values of a and b.

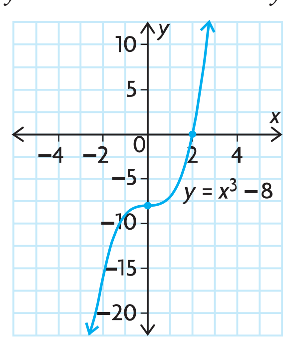
Solution A: Using a graph to factor a difference of cubes

Let $a = 1$ and $b = 2$.

Then, $(ax)^3 - b^3 = x^3 - 8$.

Substitute values of a and b.

$y = x^3 - 8$ is the same as $y = x^3$, translated 8 units down.



Use transformations to graph the function.

The graph of $y = x^3 - 8$ shows an x-intercept, which can be used to create one factor of the polynomial.

The only x-intercept is at 2, so $(x - 2)$ is one factor.

2	1	0	0	−8
	↓	2	4	8
	1	2	4	0

Divide $x^3 - 8$ by $x - 2$ to determine the other factor.

$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$

If $x^2 + 2x + 4 = 0$, then the roots are $\frac{-2 \pm \sqrt{4 - 16}}{2}$.

$x^2 + 2x + 4$ cannot be factored further because the corresponding equation does not have real roots. The discriminant $b^2 - 4ac$ is equal to -12.

This is verified by the graph.

$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$

Solution B: Using the factor theorem to factor a sum of cubes

Let $a = 2$ and $b = 3$.

Substitute values of a and b.

This gives the expression $f(x) = 8x^3 + 27$.

$f\left(-\frac{3}{2}\right) = 0$, so $2x + 3$ is a factor.

Use the factor theorem to determine one factor of $f(x)$.

$-\frac{3}{2}$	8	0	0	27
	↓	−12	18	−27
	8	−12	18	0

Divide to determine the other factors.

$f(x) = \left(x + \frac{3}{2}\right)(8x^2 - 12x + 18)$

$= \left(x + \frac{3}{2}\right)(2)(4x^2 - 6x + 9)$

$f(x) = (2x + 3)(4x^2 - 6x + 9)$

Multiplying $\left(x + \frac{3}{2}\right)$ by 2 results in the equivalent factor, $2x + 3$.

Solution C: Using a general solution

Let $a = 1$.

Substitute a value of 1 for a.

Then, $(ax)^3 - b^3 = x^3 - b^3$.

Let $f(x) = x^3 - b^3$.

$f(b) = 0$, so $(x - b)$ is a factor.

Use the factor theorem to determine one factor of $f(x)$.

b	1	0	0	$-b^3$
	$\downarrow$	b	b^2	b^3
	1	b	b^2	0

Divide to determine the other factors.

$f(x) = (x - b)(x^2 + bx + b^2)$

$x^3 - b^3 = (x - b)(x^2 + bx + b^2)$

If $b = 3$, $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$.
If $b = 5$, $x^3 - 125 = (x - 5)(x^2 + 5x + 25)$.

Substitute different values of b.

Reflecting

A. Why would an expression such as $x^3 - 8$ be called a *difference of cubes?*

B. Why would an expression such as $8x^3 + 27$ be called a *sum of cubes?*

C. Why was the quadratic formula useful for determining that the second factor could not be factored further?

D. State a general factorization for the difference of cubes, $A^3 - B^3$, and for the sum of cubes, $A^3 + B^3$.

APPLY the Math

EXAMPLE 2 Selecting a strategy to factor a polynomial

Factor the expression $27x^3 + 125$.

Solution

$27x^3 + 125$
$= (3x)^3 + (5)^3$
This is a sum of cubes.

Any sum of cubes can be factored as follows: $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$

$= (3x + 5)(9x^2 - 15x + 25)$

Use this factorization to write the two factors, if $A = 3x$ and $B = 5$.

EXAMPLE 3 Connecting prior knowledge to factor a polynomial

Factor $7x^4 - 448x$.

Solution

$7x^4 - 448x$

$= 7x(x^3 - 64)$ ← Divide out the common factor. This leaves a difference of cubes. Any difference of cubes can be factored as follows: $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$

$= 7x(x - 4)(x^2 + 4x + 16)$ ← Use this factorization to write the factors, if $A = x$ and $B = 4$.

EXAMPLE 4 Selecting a strategy to factor a polynomial that is not obviously a cubic

Factor the expression $x^9 - 512$ completely.

Solution

$x^9 - 512$

$= (x^3)^3 - (8)^3$ ← Write the expression as the difference of two cubes.

$= (x^3 - 8)(x^6 + 8x^3 + 64)$ ← Use the factorization $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$ to factor the expression, if $A = x^3$ and $B = 8$.

$= (x - 2)(x^2 + 2x + 4)(x^6 + 8x^3 + 64)$ ← $(x^3 - 8)$ is also a difference of cubes, so factor it further using the pattern where $A = x$ and $B = 2$.

In Summary

Key Ideas

- An expression that contains two perfect cubes that are added together is called a sum of cubes and can be factored as follows:

  $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$

- An expression that contains perfect cubes where one is subtracted from the other is called a difference of cubes and can be factored as follows:

  $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$

CHECK Your Understanding

1. Using Solution C for Example 1 as a model, determine the factors of $x^3 + b^3$.

2. Factor each of the following expressions.

a) $x^3 - 64$
b) $x^3 - 125$
c) $x^3 + 8$
d) $8x^3 - 27$
e) $64x^3 - 125$
f) $x^3 + 1$
g) $27x^3 + 8$
h) $1000x^3 + 729$
i) $216x^3 - 8$

3. Factor each expression.

a) $64x^3 + 27y^3$
b) $-3x^4 + 24x$
c) $(x + 5)^3 - (2x + 1)^3$
d) $x^6 + 64$

PRACTISING

4. Factor.

K **a)** $x^3 - 343$
b) $216x^3 - 1$
c) $x^3 + 1000$
d) $125x^3 - 512$
e) $64x^3 - 1331$
f) $343x^3 + 27$
g) $512x^3 + 1$
h) $1331x^3 + 1728$
i) $512 - 1331x^3$

5. Factor each expression.

a) $\frac{1}{27}x^3 - \frac{8}{125}$
b) $-432x^5 - 128x^2$
c) $(x - 3)^3 + (3x - 2)^3$
d) $\frac{1}{512}x^9 - 512$

6. Jarred claims that the expression
A $\frac{(a + b)(a^2 - ab + b^2) + (a - b)(a^2 + ab + b^2)}{2a^3}$ is equivalent to 1.
Do you agree or disagree with Jarred? Justify your decision.

7. 1729 is a very interesting number. It is the smallest whole number that can be expressed as a sum of two cubes in two ways: $1^3 + 12^3$ and $9^3 + 10^3$. Use the factorization for the sum of cubes to verify that these sums are equal.

8. Prove that $(x^2 + y^2)(x^4 - x^2y^2 + y^4)(x^{12} - x^6y^6 + y^{12}) + 2x^9y^9$
T equals $(x^9 + y^9)^2$ using the factorization for the sum of cubes.

9. Some students might argue that if you know how to factor a sum
C of cubes, then you do not need to know how to factor a difference of cubes. Explain why you agree or disagree.

Extending

10. The number 1729, in question 7, is called a taxicab number.

a) Use the Internet to find out why 1729 is called a taxicab number.
b) Are there other taxicab numbers? If so, what are they?

3 Chapter Review

FREQUENTLY ASKED Questions

Q: How can you divide polynomials?

A: You can divide polynomials using an algorithm similar to long division with numbers. If the divisor is a binomial, then you can use synthetic division.

For example, you can divide $3x^3 + 2x - 17$ by $x - 2$ as follows:

Study Aid

- See Lesson 3.5, Examples 1, 2, and 3.
- Try Chapter Review Questions 10, 11, and 12.

Using Synthetic Division

$(x - 2) \rightarrow k = 2$

$3x^3 + 2x - 17$
$= 3x^3 + 0x^2 + 2x - 17$

$$\begin{array}{r|rrrr} 2 & 3 & 0 & 2 & -17 \\ & \downarrow & 6 & 12 & 28 \\ \hline & 3 & 6 & 14 & 11 \end{array}$$

$3x^3 + 2x - 17$
$= (x - 2)(3x^2 + 6x + 14) + 11$

Using Regular Polynomial Division

$$\begin{array}{rl} & 3x^2 + 6x + 14 \\ & x - 2\overline{)3x^3 + 0x^2 + 2x - 17} \\ 3x^2(x-2) \rightarrow & \underline{3x^3 - 6x^2} \\ & 6x^2 + 2x \\ 6x(x-2) \rightarrow & \underline{6x^2 - 12x} \\ & 14x - 17 \\ 14(x-2) \rightarrow & \underline{14x - 28} \\ & 11 \end{array}$$

$3x^3 + 2x - 17 = (x - 2)(3x^2 + 6x + 14) + 11$

Q: How do you factor a polynomial of degree 3 or greater?

A1: Use the factor theorem to determine one factor of the polynomial, and then divide to determine the other factors.

For example, to factor $x^3 - 6x^2 - 13x + 42$, let $f(x) = x^3 - 6x^2 - 13x + 42$ and determine the first possible factor by finding a number the makes $f(x) = 0$.

Study Aid

- See Lesson 3.6, Example 2.
- Try Chapter Review Questions 14 and 15.

Possibilities: $\pm1, \pm2, \pm3, \pm6, \pm7, \pm14, \pm21, \pm42$

$f(2) = (2)^3 - 6(2)^2 - 13(2) + 42 = 0$, so $x - 2$ is a factor.

Use synthetic division to find the other factor.

$$\begin{array}{r|rrrr} 2 & 1 & -6 & -13 & 42 \\ & \downarrow & 2 & 8 & -42 \\ \hline & 1 & -4 & -21 & 0 \end{array}$$

$f(x) = (x - 2)(x^2 - 4x - 21)$ Factor the quotient.
$= (x - 2)(x - 7)(x + 3)$

A2: Factor using the sum or difference of cubes pattern when appropriate:

$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$
$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$

For example, you can factor $27x^3 - 64$ as follows:

$27x^3 - 64$
$= (3x)^3 - (4)^3$
$= (3x - 4)(9x^2 + 12x + 16)$

Study Aid

- See Lesson 3.7, Examples 2 and 3.
- Try Chapter Review Questions 16, 17, and 18.

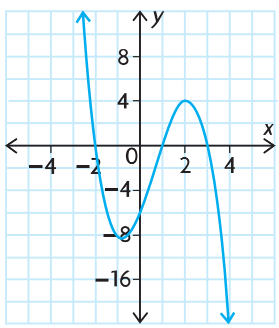
PRACTICE Questions

Lesson 3.1

1. Draw the graph of a polynomial function that has all of the following characteristics:
- $f(2) = 10, f(-3) = 0$, and $f(4) = 0$
- The y-intercept is 0.
- $f(x) > 0$ when $x < -3$ and $0 < x < 4$.
- $f(x) < 0$ when $-3 < x < 0$ and $x > 4$.
- The range is the set of real numbers.

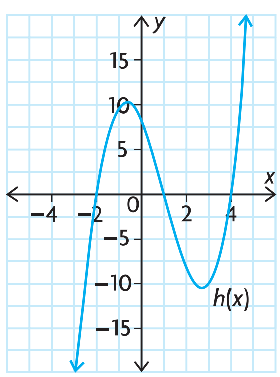
Lesson 3.2

2. Describe the end behaviours of this function.

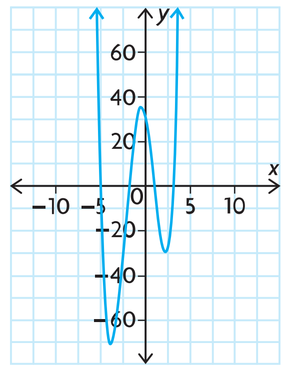


3. State the possible degree of each function, the sign of the leading coefficient, and the number of turning points.

a)



b)



Lesson 3.3

4. For each of the following, write the equations of three cubic functions that have the given zeros and belong to the same family of functions.

a) $-3, 6, 4$ **b)** $5, -1, -2$ **c)** $-7, 2, 3$ **d)** $9, -5, -4$

5. For each of the following, write the equations of three quartic functions that have the given zeros and belong to the same family of functions.

a) $-6, 2, 5, 8$ **b)** $4, -8, 1, 2$ **c)** $0, -1, 9, 10$ **d)** $-3, 3, -6, 6$

6. Sketch the graph of $f(x) = (x - 3)(x + 2)(x + 5)$ using the zeros and end behaviours.

7. Determine the equation of the function with zeros at ± 1 and -2, and a y-intercept of -6. Then sketch the function.

Lesson 3.4

8. Describe the transformations that were applied to $y = x^2$ to obtain each of the following functions.

a) $y = -2(x - 1)^2 + 23$

b) $y = \left(\frac{12}{13}(x + 9)\right)^2 - 14$

c) $y = x^2 - 8x + 16$

d) $y = \left(x + \frac{3}{7}\right)\left(x + \frac{3}{7}\right)$

e) $y = 40(-7(x - 10))^2 + 9$

9. The function $y = x^3$ has undergone each of the following sets of transformations. List three points on the resulting function.

a) vertically stretched by a factor of 25, horizontally compressed by a factor of $\frac{5}{6}$, horizontally translated 3 units to the right

b) reflected in the y-axis, horizontally stretched by a factor of 7, vertically translated 19 units down

c) reflected in the x-axis, vertically compressed by a factor of $\frac{6}{11}$, horizontally translated 5 units to the left, vertically translated 16 units up

d) vertically stretched by a factor of 100, horizontally stretched by a factor of 2, vertically translated 14 units up

e) reflected in the y-axis, vertically translated 45 units down

f) reflected in the x-axis, horizontally compressed by a factor of $\frac{7}{10}$, horizontally translated 12 units to the right, vertically translated 6 units up

Lesson 3.5

10. Calculate each of the following using long division.

a) $(2x^3 + 5x^2 + 3x - 4) \div (x + 5)$

b) $(x^4 + 4x^3 - 3x^2 - 6x - 7) \div (x^2 - 8)$

c) $(2x^4 - 2x^2 + 3x - 16) \div (x^3 + 3x^2 + 3x - 3)$

d) $(x^5 - 8x^3 - 7x - 6) \div (x^4 + 4x^3 + 4x^2 - x - 3)$

11. Divide each polynomial by $x + 2$ using synthetic division.

a) $2x^3 + 5x^2 - x - 5$

b) $3x^3 + 13x^2 + 17x + 3$

c) $2x^4 + 5x^3 - 16x^2 - 45x - 18$

d) $2x^3 + 4x^2 - 5x - 4$

12. Each divisor was divided into another polynomial, resulting in the given quotient and remainder. Determine the dividend.

a) divisor: $x - 9$, quotient: $2x^2 + 11x - 8$, remainder: 3

b) divisor: $4x + 3$, quotient: $x^3 - 2x + 7$, remainder: -4

c) divisor: $3x - 4$, quotient: $x^3 + 6x^2 - 6x - 7$, remainder: 5

d) divisor: $3x^2 + x - 5$, quotient: $x^4 - 4x^3 + 9x - 3$, remainder: $2x - 1$

Lesson 3.6

13. Without dividing, determine the remainder when $x^3 + 2x^2 - 6x + 1$ is divided by $x + 2$.

14. Factor each polynomial using the factor theorem.

a) $x^3 - 5x^2 - 22x - 16$

b) $2x^3 + x^2 - 27x - 36$

c) $3x^4 - 19x^3 + 38x^2 - 24x$

d) $x^4 + 11x^3 + 36x^2 + 16x - 64$

15. Factor fully.

a) $8x^3 - 10x^2 - 17x + 10$

b) $2x^3 + 7x^2 - 7x - 30$

c) $x^4 - 7x^3 + 9x^2 + 27x - 54$

d) $4x^4 + 4x^3 - 35x^2 - 36x - 9$

Lesson 3.7

16. Factor each difference of cubes.

a) $64x^3 - 27$

b) $512x^3 - 125$

c) $343x^3 - 1728$

d) $1331x^3 - 1$

17. Factor each sum of cubes.

a) $1000x^3 + 343$

b) $1728x^3 + 125$

c) $27x^3 + 1331$

d) $216x^3 + 2197$

18. **a)** Factor the expression $x^6 - y^6$ completely by treating it as a difference of squares.

b) Factor the same expression by treating it as a difference of cubes.

c) Explain any similarities or differences in your final results.