# 1.1 Functions

**YOU WILL NEED**
- graph paper
- graphing calculator (optional)

**GOAL**

Represent and describe functions and their characteristics.

## *LEARN ABOUT* the Math

Jonathan and Tina are building an outdoor skating rink. They have enough materials to make a rectangular rink with an area of about 1800 $m^2$, and they do not want to purchase any additional materials. They know, from past experience, that a good rink must be approximately 30 m longer than it is wide.

? What dimensions should they use to make their rink?

### EXAMPLE 1 Representing a situation using a mathematical model

Determine the dimensions that Jonathan and Tina should use to make their rink.

#### Solution A: Using an algebraic model

Let $x$ represent the length. Let $y$ represent the width.

$$A = xy$$
$$1800 = xy$$
$$\frac{1800}{x} = y$$

We know the area must be 1800 $m^2$, so if we let the width be the **independent variable**, we can write an expression for the length.

The width, in terms of $x$, is $\frac{1800}{x}$.

Let $f(x)$ represent the difference between the length and the width.

$$f(x) = x - \frac{1800}{x},$$

where $f(x) = 30$.

$$x - \frac{1800}{x} = 30$$

Using **function notation**, write an equation for the difference in length and width. The relation is a **function** because each input produces a unique output. In this case the difference or value of the function must be 30.

$$x(x) - x\left(\frac{1800}{x}\right) = x(30)$$

To solve the equation, multiply all the terms in the equation by the lowest common denominator, *x*, to eliminate any rational expressions.

$$x^2 - 1800 = 30x$$
$$x^2 - 30x - 1800 = 0$$
$$(x - 60)(x + 30) = 0$$

This results in a quadratic equation. Rearrange the equation so that it is in the form $ax^2 + bx + c = 0$. Factor the left side.

$$x - 60 = 0 \text{ or } x + 30 = 0$$
$$x = 60 \text{ or } x = -30$$

Solve for each factor. $x = -30$ is outside the domain of the function, since length cannot be negative. This is an inadmissible solution.

The length is 60 m.

$$y = \frac{1800}{60} = 30$$

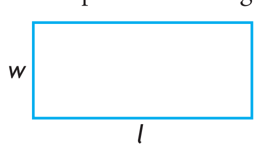
Calculate the width.

The width is 30 m.

The dimensions that are 30 m apart and will produce an area of 1800 $m^2$ are 60 m × 30 m.

## Solution B: Using a numerical model

Let $l$ represent the length. Let $w$ represent the width.




Length is the independent variable.
Its domain is $0 < l < 1800$.
Width is the dependent variable.

$$A = lw$$
$$1800 = lw$$
$$\frac{1800}{l} = w$$

Write an equation for the width in terms of length for a fixed area of 1800 $m^2$.

Guess 1: $l = 200$

$$w = \frac{1800}{200} = 9$$

Check: $l - w = 200 - 9 \neq 30$

Use different values for the length to calculate possible widths. Check to see if the difference between the length and width is 30.

Guess 2: $l = 100$

$$w = \frac{1800}{100} = 18$$

Check: $l - w = 100 - 18 \neq 30$

| Area (m²) | Length (m) | Width (m) | Length – Width |
|---|---|---|---|
| 1800 | 100 | 18 | 82 |
| 1800 | 90 | 20 | 70 |
| 1800 | 80 | 22.5 | 57.5 |
| 1800 | 70 | 25.71 | 44.29 |
| 1800 | 60 | 30 | 30 |
| 1800 | 50 | 36 | 14 |
| 1800 | 40 | 45 | −5 |
| 1800 | 30 | 60 | −30 |
| 1800 | 20 | 90 | −70 |

Create a table of values to investigate the difference between the length and the width for a variety of lengths.

The dimensions that are 30 m apart and produce an area of 1800 $m^2$ are 60 m × 30 m.

A function can also be represented with a graph. A graph provides a visual display of how the variables in the function are related.

## Solution C: Using a graphical model

Let $x$ represent the length. Let $y$ represent the width.

$$A = xy$$
$$1800 = xy$$
$$\frac{1800}{x} = y$$

Using length ($x$) as the independent variable, write an expression for width ($y$).

The width, in terms of $x$, is $\frac{1800}{x}$.

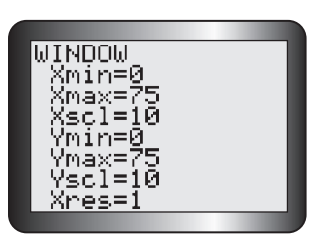
Let $f(x)$ represent the difference between the dimensions.

$$f(x) = x - \frac{1800}{x}$$

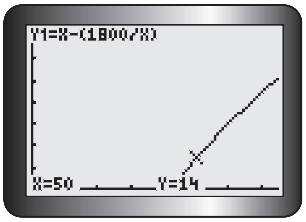
Determine the appropriate window settings to graph $f(x)$ on a graphing calculator.

The value for $x$ (length of rink) will be positive but surely less than 75 m, so we use Xmin = 0 and Xmax = 75.

We use the same settings for the range of f($x$), for simplicity.



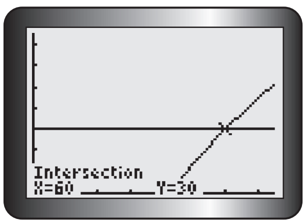
Graph the difference function.

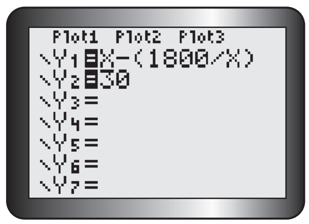



Use the TRACE feature on the graph to investigate points with the ordered pairs (length, length − width) on $f(x)$.

A length of 50 m gives a 14 m difference between the length and the width.

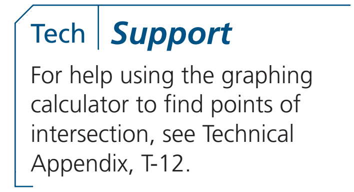
Determine the length that exceeds the width by 30 m.




To determine the length that is 30 m longer than the width, graph $g(x) = 30$ in $Y_2$ and locate the point of intersection for $g(x)$ and $f(x)$.

The dimensions that are 30 m apart and produce an area of 1800 $m^2$ are 60 m × 30 m.




## Reflecting

**A.** Would the function change if width was used as the independent variable instead of length? Explain.

**B.** Is it necessary to restrict the domain and range in this problem? Explain.

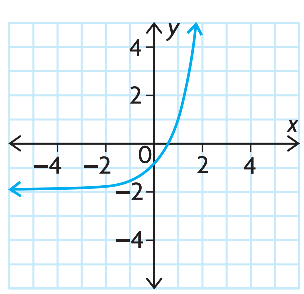
**C.** Why was it useful to think of the relationship between the length and the width as a function to solve this problem?

# *APPLY* the Math

## EXAMPLE 2 | Using reasoning to decide whether a relation is a function

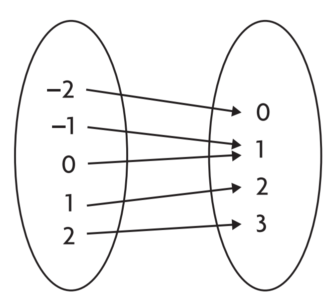
Decide whether each of the following relations is a function. State the domain and range.

**a)**



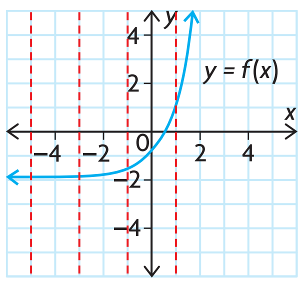

**b)** $y = \frac{1}{x^2}$

**c)**



## Solution

**a)**




Apply the **vertical line test**. Any vertical line drawn on the graph of a function passes through, at most, a single point. This indicates that each number in the domain corresponds to only one number in the range, which is the condition for the relation to be a function.

The graph represents an **exponential function**.

$D = \{x \in \mathbf{R}\}$
$R = \{y \in \mathbf{R} \mid y > -2\}$

Since the graph of this function has no breaks, or vertical **asymptotes**, and continues indefinitely in both the positive and negative direction, its domain consists of all the **real numbers**.

The function has a **horizontal asymptote** defined by the equation $y = -2$. All its values lie above this horizontal line.

**b)**

| ***x*** | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| ***f*(*x*)** | $\frac{1}{9}$ | $\frac{1}{4}$ | 1 | undefined | 1 | $\frac{1}{4}$ | $\frac{1}{9}$ |

Create a table of values.

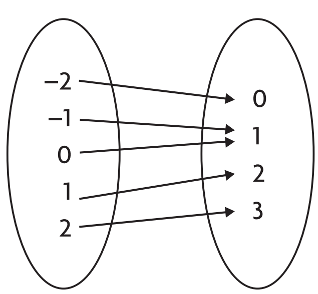
The table indicates that each number in the domain corresponds to only one number in the range.

$f(x) = \frac{1}{x^2}$ is a function.

$D = \{x \in \mathbf{R} \mid x \neq 0\}$
$R = \{y \in \mathbf{R} \mid y > 0\}$

$f(x) = \frac{1}{x^2}$ has a **vertical asymptote** defined by $x = 0$. Its domain consists of all the real numbers, except 0. It has a horizontal asymptote defined by the equation $y = 0$. All its values are positive, since $x$ is squared, so they lie above this horizontal line.

**c)**




The mapping diagram indicates that each number in the domain corresponds to only one number in the range.

A function can have converging arrows but cannot have diverging arrows in a mapping diagram.

This is a function.

$D = \{-2, -1, 0, 1, 2\}$
$R = \{0, 1, 2, 3\}$

The first oval represents the elements found in the domain. The second oval represents the elements found in the range.

## EXAMPLE 3 Using reasoning to determine the domain and range of a function

Naill rides a Ferris wheel that has a diameter of 6 m. The axle of the Ferris wheel is 4 m above the ground. The Ferris wheel takes 90 s to make one complete rotation, and Naill rides for 10 rotations. What are the domain and range of the function that models Naill's height above the ground, in terms of time, while he rides the Ferris wheel?

### Solution

$h(t) = a \sin [k(t - d)] + c$

or

$h(t) = a \cos [k(t - d)] + c$

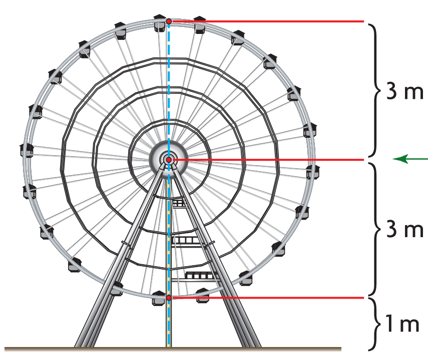
This situation involves circular motion, which can be modelled by a sine or cosine function.

$D = \{t \in \mathbf{R} \mid 0 \leq t \leq 900\}$

Examine the conditions on the independent variable time to determine the domain. Time cannot be negative, so the lower boundary is 0. The wheel rotates once every 90 s, and Naill rides for 10 complete rotations.

$90 \times 10 = 900$

The upper boundary is 900 s.




$R = \{h(t) \in \mathbf{R} \mid 1 \leq h(t) \leq 7\}$

Examine the conditions on the dependent variable height to determine the range. The radius of the wheel is 3 m. Since the axle is located 4 m above the ground, the lowest height that Naill can be above the ground is the difference between the height of the axle and the radius of the wheel: $4 - 3 = 1$ m. This is the lower boundary of the range.

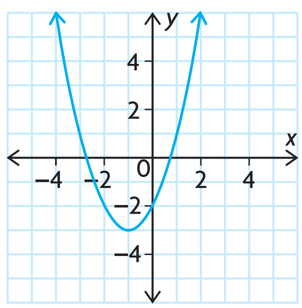
The greatest height he reaches is the sum of the height of the axle and the radius of the wheel: $4 + 3 = 7$ m. This is the upper boundary of the range.

## In Summary

### Key Ideas

- A function is a relation in which there is a unique output for each input. This means that each value of the independent variable (the domain) must correspond to one, and only one, value of the dependent variable (the range).
- Functions can be represented graphically, numerically, or algebraically.

**Graphical Example**



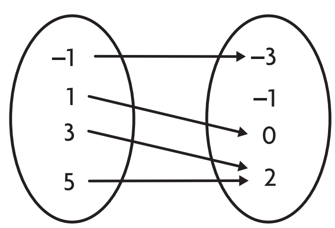

**Numerical Examples**

Set of ordered pairs:
$\{(1, 3), (3, 5), (-2, 9), (5, 11)\}$

Table of values:

| $x$ | $y$ |
|---|---|
| −2 | 4 |
| −1 | 1 |
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |

Mapping diagram:

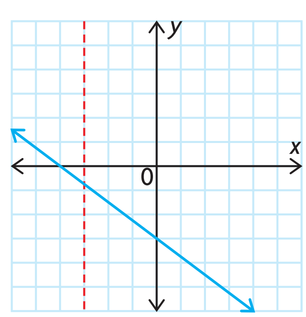



**Algebraic Examples**

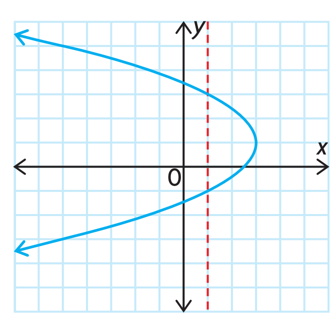
$y = 2 \sin (3x) + 4$
or
$f(x) = 2 \sin (3x) + 4$

### Need to Know

- Function notation, $f(x)$, is used to represent the values of the dependent variable in a function, so $y = f(x)$.
- You can use the vertical line test to check whether a graph represents a function. A graph represents a function if every vertical line intersects the graph in, at most, one point. This shows that there is only one element in the range for each element in the domain.
- The domain and range of a function depend on the type of function.
- The domain and range of a function that models a particular situation may need to be restricted, based on the situation. For example, negative values may not have meaning when dealing with variables such as time.



function

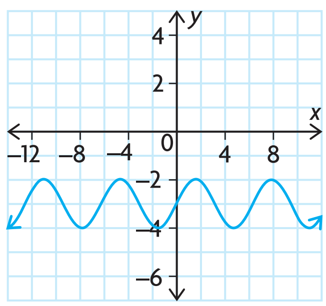


not a function

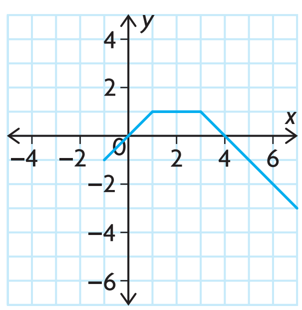
## *CHECK Your Understanding*

**1.** State the domain and range of each relation. Then determine whether the relation is a function, and justify your answer.

**a)**



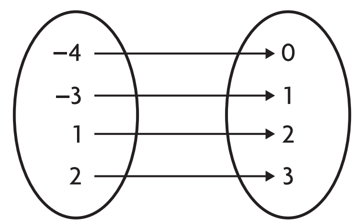

**b)**




**c)** $\{(1, 4), (1, 9), (2, 7), (3, -5), (4, 11)\}$

**d)** $y = 3x - 5$

**e)**




**f)** $y = -5x^2$

**2.** State the domain and range of each relation. Then determine whether the relation is a function, and justify your answer.

**a)** $y = -2(x + 1)^2 - 3$

**b)** $y = \dfrac{1}{x + 3}$

**c)** $y = 2^{-x}$

**d)** $y = \cos x + 1$

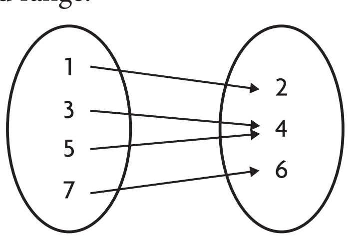
**e)** $x^2 + y^2 = 9$

**f)** $y = 2 \sin x$

## *PRACTISING*

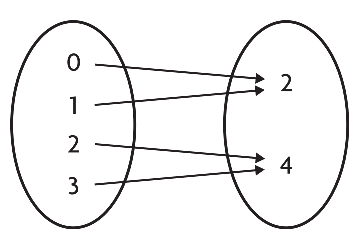
**3.** Determine whether each relation is a function, and state its domain and range.

**a)**



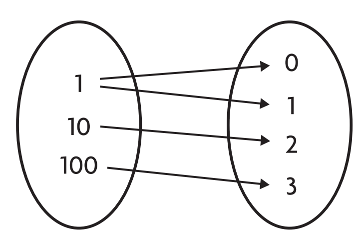

**b)** $\{(2, 3), (1, 3), (5, 6), (0, -1)\}$

**c)**




**d)** $\{(2, 5), (6, 1), (2, 7), (8, 3)\}$

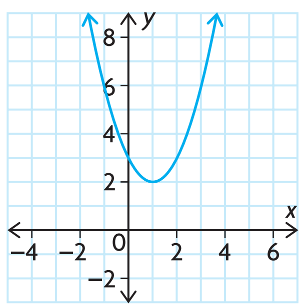
**e)**



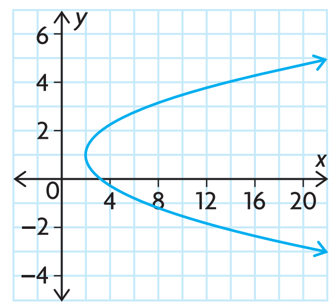

**f)** $\{(1, 2), (2, 1), (3, 4), (4, 3)\}$

**4.** Determine whether each relation is a function, and state its domain
**K** and range.

**a)**




**b)**




**c)** $x^2 = 2y + 1$

**d)** $x = y^2$

**e)** $y = \frac{3}{x}$

**f)** $f(x) = 3x + 1$

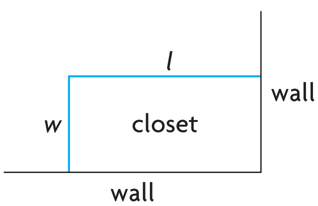
**5.** Determine the equations that describe the following function rules:

**a)** The input is 3 less than the output.

**b)** The output is 5 less than the input multiplied by 2.

**c)** Subtract 2 from the input and then multiply by 3 to find the output.

**d)** The sum of the input and output is 5.




**6.** Martin wants to build an additional closet in a corner of his bedroom. Because the closet will be in a corner, only two new walls need to be built. The total length of the two new walls must be 12 m. Martin wants the length of the closet to be twice as long as the width, as shown in the diagram.

**a)** Explain why $l = 2w$.

**b)** Let the function $f(l)$ be the sum of the length and the width. Find the equation for $f(l)$.

**c)** Graph $y = f(l)$.

**d)** Find the desired length and width.

**7.** The following table gives Tina's height above the ground while riding a
**A** Ferris wheel, in relation to the time she was riding it.

| Time (s) | 0 | 20 | 40 | 60 | 80 | 100 | 120 | 140 | 160 | 180 | 200 | 220 | 240 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Height (m) | 5 | 10 | 5 | 0 | 5 | 10 | 5 | 0 | 5 | 10 | 5 | 0 | 5 |

**a)** Draw a graph of the relation, using time as the independent variable and height as the dependent variable.

**b)** What is the domain?

**c)** What is the range?

**d)** Is this relation a function? Justify your answer.

**e)** Another student sketched a graph, but used height as the independent variable. What does this graph look like?

**f)** Is the relation in part e) a function? Justify your answer.

**8.** Consider what happens to a relation when the coordinates of all its ordered pairs are switched.

**a)** Give an example of a function that is still a function when its coordinates are switched.

**b)** Give an example of a function that is no longer a function when its coordinates are switched.

**c)** Give an example of a relation that is not a function, but becomes a function when its coordinates are switched.

**9.** Explain why a relation that fails the vertical line test is not a function.

**10.** Consider the relation between $x$ and $y$ that consists of all points $(x, y)$ such that the distance from $(x, y)$ to the origin is 5.

**a)** Is (4, 3) in the relation? Explain.

**b)** Is (1, 5) in the relation? Explain.

**c)** Is the relation a function? Explain.

**11.** The table below lists all the ordered pairs that belong to the function $g(x)$.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $g(x)$ | 3 | 4 | 7 | 12 | 19 | 28 |

**a)** Determine an equation for $g(x)$.

**b)** Does $g(3) - g(2) = g(3 - 2)$? Explain.

**12.** T The factors of 4 are 1, 2, and 4. The sum of the factors is $1 + 2 + 4 = 7$. The sum of the factors is called the sigma function. Therefore, $f(4) = 7$.

**a)** Find $f(6)$, $f(7)$, and $f(8)$.

**b)** Is $f(15) = f(3) \times f(5)$?

**c)** Is $f(12) = f(3) \times f(4)$?

**d)** Are there others that will work?

**13.** C Make a concept map to show what you have learned about functions. Put "FUNCTION" in the centre of your concept map, and include the following words:

| | | |
|---|---|---|
| algebraic model | graphical model | numerical model |
| dependent variable | independent variable | range |
| domain | mapping model | vertical line test |
| function notation | | |

## Extending

**14.** Consider the relations $x^2 + y^2 = 25$ and $y = \sqrt{25 - x^2}$. Draw the graphs of these relations, and determine whether each relation is a function. State the domain and range of each relation.

**15.** You already know that $y$ is a function of $x$ if and only if the graph passes the vertical line test. When is $x$ a function of $y$? Explain.

### Communication *Tip*

A concept map is a type of web diagram used for exploring knowledge and gathering and sharing information. A concept map consists of cells that contain a concept, item, or question and links. The labelled links are labelled and denote direction with an arrow symbol. The labelled links explain the relationship between the cells. The arrow describes the direction of the relationship and reads like a sentence.