1.3 Properties of Graphs of Functions

GOAL

Compare and contrast the properties of various types of functions.

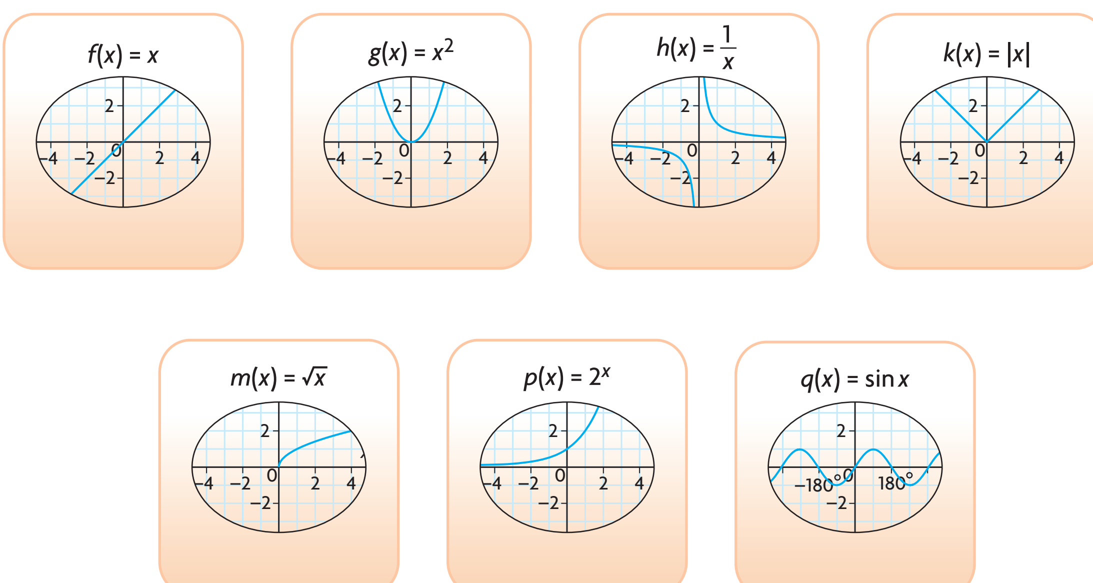
YOU WILL NEED

- graphing calculator

INVESTIGATE the Math

Two students created a game that they called "Which function am I?" In this game, players turn over cards that are placed face down and match the characteristics and properties with the correct functions. The winner is the player who has the most pairs at the end of the game.

The students have studied the following parent functions:



? Which criteria could the students use to differentiate between these different types of functions?

A. Graph each of these parent functions on a graphing calculator, and sketch its graph. State the domain and range of each function, and determine its zeros and y-intercepts.

B. Determine the intervals of increase and the intervals of decrease for each of the parent functions.

interval of increase

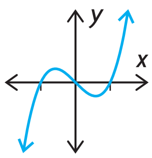
the interval(s) within a function's domain, where the y-values of the function get larger, moving from left to right

interval of decrease

the interval(s) within a function's domain, where the y-values of the function get smaller, moving from left to right

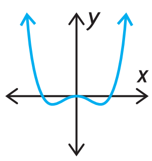
odd function

any function that has rotational symmetry about the origin; algebraically, all odd functions have the property $f(-x) = -f(x)$



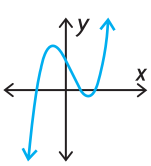
even function

any function that is symmetric about the y-axis; algebraically, all even functions have the property $f(-x) = f(x)$



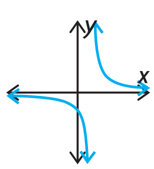
continuous function

any function that does not contain any holes or breaks over its entire domain



discontinuity

a break in the graph of a function is called a point of discontinuity



C. State whether each parent function is an odd function, an even function, or neither.

D. Do any of the functions have vertical or horizontal asymptotes? If so, what are the equations of these asymptotes?

E. Which graphs are continuous? Which have discontinuities?

F. Complete the following statements to describe the end behaviour of each parent function.

a) As x increases to large positive values, y . . .

b) As x decreases to large negative values, y . . .

Communication | *Tip*

It is often convenient to use the symbol for infinity, ∞, and the following notation to write the end behaviour of a function:

- For "As x increases to large positive values, y . . . ," write "As $x \to \infty$, $y \to$ . . . "
- For "As x decreases to large negative values, y . . . ," write "As $x \to -\infty$, $y \to$ . . . "

G. Summarize your findings.

Reflecting

H. Which of the parent functions can be distinguished by their domain? Which can be distinguished by their range? Which can be distinguished by their zeros?

I. An increasing function is one in which the function's values increase from left to right over its entire domain. A decreasing function is one in which the function's values decrease from left to right over its entire domain. Which of the parent functions are increasing functions? Which are decreasing functions?

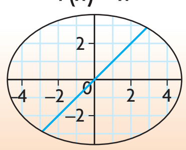
J. Which properties of each function would make the function easy to identify from a description of it?

APPLY the Math

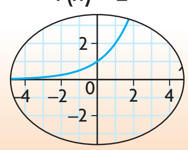
EXAMPLE 1 Connecting the graph of a function with its characteristics

Match each parent function card with a characteristic of its graph. Each card may only be used for one parent function.

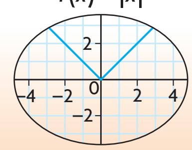
$f(x) = x$



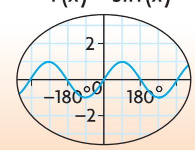
$f(x) = 2^x$



$f(x) = |x|$



$f(x) = \sin(x)$



Range: $\{y \in \mathrm{R} \mid y \geq 0\}$

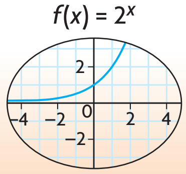
Domain: $\{x \in \mathrm{R}\}$

Infinite Number of Zeros

As $x \to -\infty$, $y \to 0$.

Solution

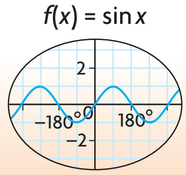
$f(x) = 2^x$



As $x \to -\infty$, $y \to 0$.

This property describes the end behaviour: as x becomes negatively large, y approaches zero. The function must have a horizontal asymptote defined by $y = 0$. The function must be $y = 2^x$.

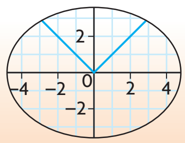
$f(x) = \sin x$



Infinite Number of Zeros

The sine function is periodic and continues infinitely, intersecting the x-axis an infinite number of times.

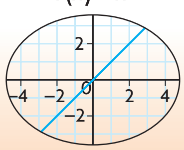
$f(x) = |x|$



Range:
$\{y \in \mathrm{R} \mid y \geq 0\}$

The range $\{y \in \mathbf{R} \mid y \geq 0\}$ indicates that all y-values of the function must be non-negative. This is true for both $f(x) = 2^x$ and $f(x) = |x|$. However, $f(x) = 2^x$ has the x-axis as its horizontal asymptote, so $y \neq 0$. Choose $f(x) = |x|$.

$f(x) = x$



Domain:
$\{x \in \mathrm{R}\}$

The remaining function, $f(x) = x$, matches the property $\{x \in \mathbf{R}\}$.

If you are given some characteristics of a function, you may be able to determine the equation of the function.

EXAMPLE 2 Using reasoning to determine the equation of a parent function

State which of the parent functions in this lesson have the following characteristics:

a) Domain $= \{x \in \mathbf{R}\}$

b) Range $= \{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$

Solution

a) Domain $= \{x \in \mathbf{R}\}$

$f(x) = x$

$g(x) = x^2$

~~$h(x) = \frac{1}{x}$~~ (Domain $= \{x \in \mathbf{R} \mid x \neq 0\}$)

$k(x) = |x|$

~~$m(x) = \sqrt{x}$~~ (Domain $= \{x \in \mathbf{R} \mid x \geq 0\}$)

$p(x) = 2^x$

$q(x) = \sin x$

There are five parent functions that match this characteristic and two that do not.

b) Range $= \{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$

~~$f(x) = x$~~ (Range $= \{y \in \mathbf{R}\}$)

~~$g(x) = x^2$~~ (Range $= \{y \in \mathbf{R} \mid y \geq 0\}$)

~~$k(x) = |x|$~~ (Range $= \{y \in \mathbf{R} \mid y \geq 0\}$)

~~$p(x) = 2^x$~~ (Range $= \{y \in \mathbf{R} \mid y \geq 0\}$)

$q(x) = \sin x$

Of these five functions, only the sine function has the range $\{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$.

Visualizing what the graph of a function looks like can help you remember some of the characteristics of the function.

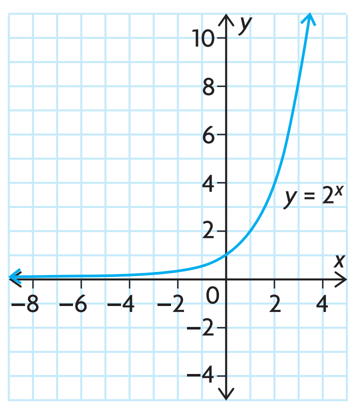
EXAMPLE 3 Connecting the characteristics of a function with its equation

Which of the following are characteristics of the parent function $p(x) = 2^x$? Justify your reasoning.

a) The graph is decreasing for all values in the domain of $p(x)$.
b) The graph is continuous for all values in the domain of $p(x)$.
c) The function $p(x)$ is an even function.
d) The function $p(x)$ has no zeros.

Solution

$p(x) = 2^x$



The function $p(x)$ is an exponential function with a base that is greater than 1.

This type of function is increasing for all values in its domain.

a) This function is increasing for all values in the domain of $p(x)$.

b) The graph is continuous for all values in the domain of $p(x)$.

This function has no breaks.

c) The function $p(x)$ is not an even function.

This type of function is not symmetric about the y-axis. $f(-x) = 2^{-x}$. This substitution does not result in $f(x)$.

d) The function $p(x)$ has no zeros.

As x approaches negative infinity, the graph gets arbitrarily close to the x-axis but does not intersect it.

Only b) and d) are characteristics of $p(x)$.

EXAMPLE 4 Connecting the characteristics of a function with its equation and its graph

Determine a possible transformed parent function that has the following characteristics, and sketch the function:

- $D = \{x \in \mathbf{R}\}$
- $R = \{y \in \mathbf{R} \mid y \geq -2\}$
- decreasing on the interval $(-\infty, 0)$
- increasing on the interval $(0, \infty)$

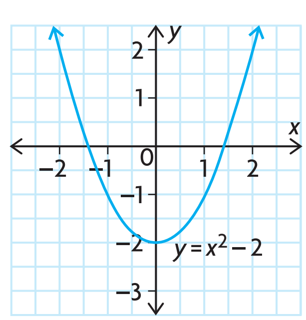
Solution

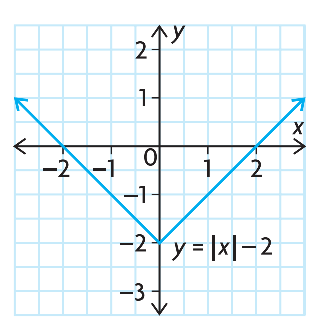
~~$f(x) = x$~~
$g(x) = x^2$
$k(x) = |x|$
~~$p(x) = 2^x$~~
~~$q(x) = \sin x$~~

List the functions that have domain $\{x \in \mathbf{R}\}$. Eliminate the functions that cannot have the range $\{y \in \mathbf{R} \mid y \geq -2\}$. Each of the remaining functions can be translated down two units to have this range.

Function	Intervals of Increase	Intervals of Decrease
$g(x) = x^2$	$(0, \infty)$	$(-\infty, 0)$
$k(x) = \|x\|$	$(0, \infty)$	$(-\infty, 0)$

State the intervals of increase and decrease for the two remaining functions. Check to see if these intervals match the given conditions. There are two possible parent functions that have the given characteristics.





Sketch the graph of each parent function shifted 2 units down.

Communication *Tip*

The interval $(-\infty, 0)$ is described using interval notation and is equivalent to $x < 0$ in set notation. The use of round brackets in interval notation indicates that the endpoint is not included in the interval. The use of square brackets in interval notation indicates that the endpoint is included in the interval. For example, $[-3, 5)$ is equivalent to $-3 \leq x < 5$.

In Summary

Key Idea

Functions can be categorized based on their graphical characteristics:

- domain and range
- *x*-intercepts and *y*-intercepts
- continuity and discontinuity
- intervals of increase and decrease
- symmetry (even/odd)
- end behaviour

Need to Know

- Given a set of graphical characteristics, the type of function that has these characteristics can be determined by eliminating those that do not have these characteristics.
- Some characteristics are more helpful than others when determining the type of function.

CHECK *Your Understanding*

1. Which graphical characteristic is the least helpful for differentiating among the parent functions? Why?

2. Which graphical characteristic is the most helpful for differentiating among the parent functions? Why?

3. One of the seven parent functions examined in this lesson is transformed to yield a graph with these characteristics:
- $D = \{x \in \mathbf{R}\}$
- $R = \{y \in \mathbf{R} \mid y > 2\}$
- As $x \to -\infty$, $y \to 2$.

What is the equation of the transformed function?

PRACTISING

4. K For each pair of functions, give a characteristic that the two functions have in common and a characteristic that distinguishes between them.

a) $f(x) = \frac{1}{x}$ and $g(x) = x$

b) $f(x) = \sin x$ and $g(x) = x$

c) $f(x) = x$ and $g(x) = x^2$

d) $f(x) = 2^x$ and $g(x) = |x|$

5. For each function, determine $f(-x)$ and $-f(-x)$ and compare it with $f(x)$. Use this to decide whether each function is even, odd, or neither.

a) $f(x) = x^2 - 4$

b) $f(x) = \sin x + x$

c) $f(x) = \frac{1}{x} - x$

d) $f(x) = 2x^3 + x$

e) $f(x) = 2x^2 - x$

f) $f(x) = |2x + 3|$

6. Determine a possible parent function that could serve as a model for each of the following situations, and explain your choice.
A

a) The number of marks away from the class average that a student's test score is

b) The height of a person above the ground during several rotations of a Ferris wheel

c) The population of Earth throughout time

d) The amount of total money saved if you put aside exactly one dollar every day

7. Identify a parent function whose graph has the given characteristics.

a) The domain is not all real numbers, and $f(0) = 0$.

b) The graph has an infinite number of zeros.

c) The graph is even and has no sharp corners.

d) As x gets negatively large, so does y. As x gets positively large, so does y.

8. Each of the following situations involves a parent function whose graph has been translated. Draw a possible graph that fits the situation.

a) The domain is $\{x \in \mathbf{R}\}$, the interval of increase is $(-\infty, \infty)$, and the range is $\{f(x) \in \mathbf{R} \mid f(x) > -3\}$.

b) The range is $\{g(x) \in \mathbf{R} \mid 2 \leq g(x) \leq 4\}$.

c) The domain is $\{x \in \mathbf{R} \mid x \neq 5\}$, and the range is $\{h(x) \in \mathbf{R} \mid h(x) \neq -3\}$.

9. Sketch a possible graph of a function that has the following characteristics:

- $f(0) = -1.5$
- $f(1) = 2$
- There is a vertical asymptote at $x = -1$.
- As x gets positively large, y gets positively large.
- As x gets negatively large, y approaches zero.

10. **a)** $f(x)$ is a quadratic function. The graph of $f(x)$ decreases on the interval $(-\infty, -2)$ and increases on the interval $(2, \infty)$. It has a y-intercept at $(0, 4)$. What is a possible equation for $f(x)$?
T

b) Is there only one quadratic function, $f(x)$, that has the characteristics given in part a)?

c) If $f(x)$ is an absolute value function that has the characteristics given in part a), is there only one such function? Explain.

11. $f(x) = x^2$ and $g(x) = |x|$ are similar functions. How might you describe the difference between the two graphs to a classmate, so that your classmate can tell them apart?

12. Copy and complete the following table. In your table, highlight the graphical characteristics that are unique to each function and could be used to distinguish it easily from other parent functions.

Parent Function	$f(x) = x$	$g(x) = x^2$	$h(x) = \frac{1}{x}$	$k(x) = \lvert x \rvert$	$m(x) = \sqrt{x}$	$p(x) = 2^x$	$r(x) = \sin x$
Sketch							
Domain							
Range							
Intervals of Increase							
Intervals of Decrease							
Location of Discontinuities and Asymptotes							
Zeros							
y-Intercepts							
Symmetry							
End Behaviours							

13. C Linear, quadratic, reciprocal, absolute value, square root, exponential, and sine functions are examples of different types of functions, with different properties and characteristics. Why do you think it is useful to name these different types of functions?

Extending

14. Consider the parent function $f(x) = x^3$. Graph $f(x)$, and compare and contrast this function with the parent functions you have learned about in this lesson.

15. Explain why it is not necessary to have $h(x) = \cos(x)$ defined as a parent function.

16. Suppose that $g(x) = |x|$ is translated around the coordinate plane. How many zeros can its graph have? Discuss all possibilities, and give an example of each.