1.4 Sketching Graphs of Functions

GOAL

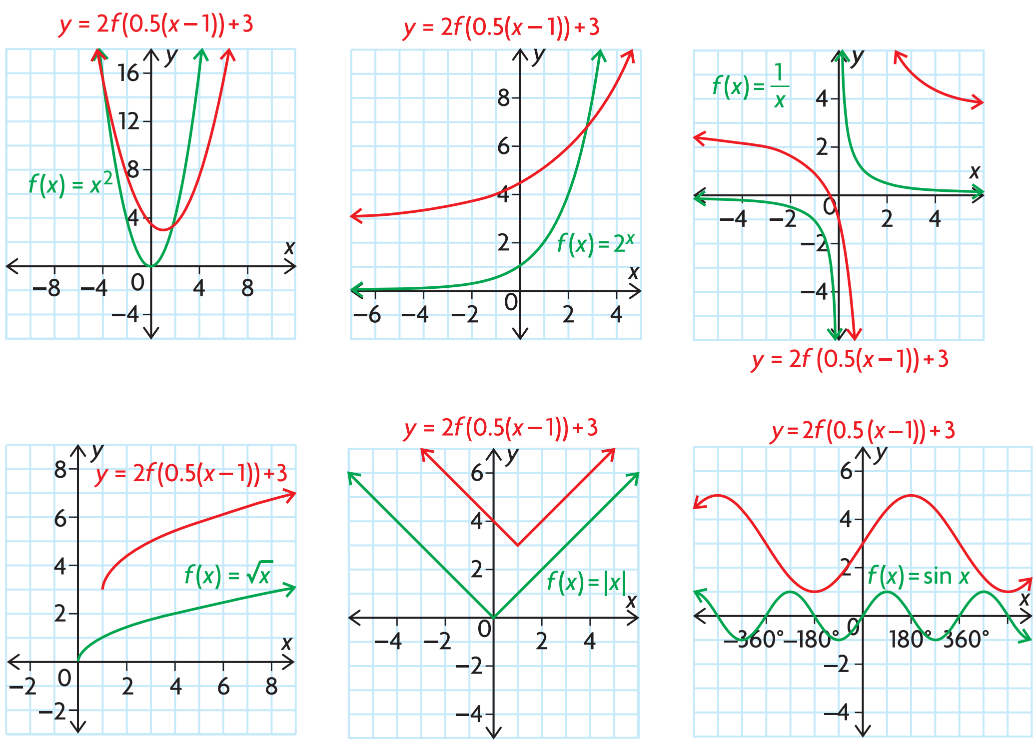
Apply transformations to parent functions, and use the most efficient methods to sketch the graphs of the functions.

YOU WILL NEED

- graph paper
- graphing calculator

INVESTIGATE the Math

The same transformations have been applied to six different parent functions, as shown below.



? How do the transformations defined by $y = 2f(0.5(x - 1)) + 3$ affect the characteristics of each parent function?

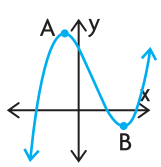
A. Identify the parent function for each graph.

B. Copy and complete the following table for each parent function.

Parent Fuction	$y = x^2$	$y = \frac{1}{x}$	$y = \lvert x \rvert$	$y = 2^x$	$y = \sqrt{x}$	$y = \sin x$
Domain						
Range						
Intervals of Increase						
Intervals of Decrease						
Turning Points						

turning point

a point on a curve where the function changes from increasing to decreasing, or vice versa; for example, A and B are turning points on the following curve



C. Identify the transformations (in the correct order) that were performed on each parent function to arrive at the transformed function.

D. State the transformation(s) that affected each of the following characteristics for each of the parent functions in the table above.

i) domain
ii) range
iii) intervals of increase/decrease
iv) turning points
v) the equation(s) of any vertical asymptotes
vi) the equation(s) of any horizontal asymptotes

E. What transformations to the graph of $y = f(x)$ result in the graph of $y = -\frac{1}{2}f(x + 2) - 1$?

Reflecting

F. For which parent functions are the domain, range, intervals of increase/decrease, and turning points affected when their graphs are transformed?

G. Describe the most efficient order that can be used to graph a transformed function when performing multiple transformations.

H. The most general equation of a transformed function is $y = af(k(x - d)) + c$, where a, k, c, and d are real numbers. Describe the transformations that would be performed on the parent function $y = f(x)$ in terms of the parameters a, k, c, and d.

APPLY the Math

EXAMPLE 1 Connecting transformations to the equation of a function

State the function that would result from vertically compressing $y = f(x)$ by a factor of $\frac{1}{2}$ and then translating the graph 5 units to the right.

Solution

$y = \frac{1}{2}f(x)$ ← This is the function that has a vertical compression by a factor of $\frac{1}{2}$.

$y = \frac{1}{2}f(x - 5)$ ← This is the function has also has a translation 5 units to the right.

EXAMPLE 2 Connecting transformations to the characteristics of a function

Use transformations to help you describe the characteristics of the transformed function $y = 3\sqrt{x} - 2$.

Solution

In the general function $y = af(k(x - d)) + c$, the parameters k and d affect the x-coordinates of each point on the parent function, and the parameters a and c affect the y-coordinates. Each point (x, y) on the parent function is mapped onto $\left(\frac{x}{k} + d, ay + c\right)$ on the transformed function. ← The parameters k and a are related to stretches/compressions and reflections, while the parameters d and c are related to translations. Since division and multiplication must be performed before addition, all stretches/compression and reflections must be applied before any translations, due to the order of operations.

The equation $y = 3\sqrt{x} - 2$ indicates that two transformations have been applied to the parent function $y = \sqrt{x}$: ← In this equation, $a = 3$ and $c = -2$.

1. a vertical stretch by a factor of 3
2. a vertical translation 2 units down

$(x, y) \rightarrow (x, 3y)$

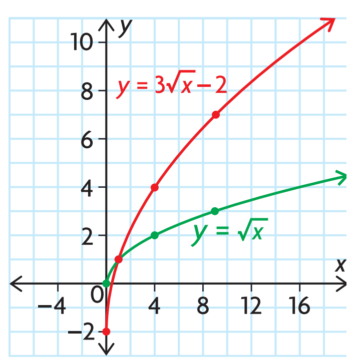
Parent Function $y = \sqrt{x}$	Stretched Function $y = 3\sqrt{x}$
(0, 0)	(0, 3(0)) = (0, 0)
(1, 1)	(1, 3(1)) = (1, 3)
(4, 2)	(4, 3(2)) = (4, 6)
(9, 3)	(9, 3(3)) = (9, 9)

Vertically stretching the graph by a factor of 3 occurs when all the *y*-coordinates on the graph of the parent function are multiplied by 3.

$(x, 3y) \rightarrow (x, 3y - 2)$

Stretched Function $y = 3\sqrt{x}$	Final Transformed Function $y = 3\sqrt{x} - 2$
(0, 0)	(0, 0 − 2) = (0, −2)
(1, 3)	(1, 3 − 2) = (1, 1)
(4, 6)	(4, 6 − 2) = (4, 4)
(9, 9)	(9, 9 − 2) = (9, 7)

Translating the graph 2 units down occurs when 2 is subtracted from all the *y*-coordinates on the graph of the stretched function.



Plot the key points of $y = \sqrt{x}$ and the new points of the transformed function.

Since the domain of both the parent function and transformed function is the same, the interval of increase is also the same: $[0, \infty)$. The difference occurs in the range. The *y*-values of the transformed function increase faster than the *y*-values of the parent function.

These two transformations act on the *y* values only; there is no change to the *x* values. The domain is unchanged; it is $\{x \in \mathbf{R} \mid x \geq 0\}$. The range changes from $\{y \in \mathbf{R} \mid y \geq 0\}$ to $\{y \in \mathbf{R} \mid y \geq -2\}$.

EXAMPLE 3 Reasoning about the characteristics of a transformed function

Graph the function $f(x) = \cos(x)$ and the transformed function $y = 2f(3x)$, where $0° \le x \le 360°$. State the impact of the transformations on the domain, range, intervals of increase/decrease, and turning points of the transformed function.

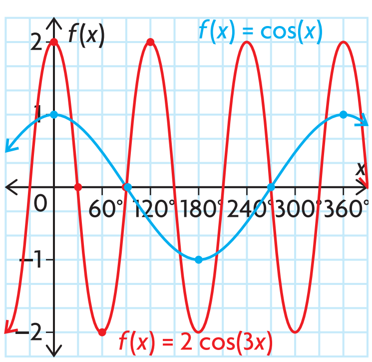
Solution

$(x, y) \rightarrow \left(\frac{1}{3}x, 2y\right)$

Parent Function $y = \cos(x)$	Final Transformed Function $y = 2\cos(3x)$
(0°, 1)	$\left(\frac{1}{3}(0°), 2(1)\right) = (0°, 2)$
(90°, 0)	$\left(\frac{1}{3}(90°), 2(0)\right) = (30°, 0)$
(180°, −1)	$\left(\frac{1}{3}(180°), 2(-1)\right) = (60°, -2)$
(270°, 0)	$\left(\frac{1}{3}(270°), 2(0)\right) = (90°, 0)$
(360°, 1)	$\left(\frac{1}{3}(360°), 2(1)\right) = (120°, 2)$

Apply a horizontal compression by a factor of $\frac{1}{3}$ and a vertical stretch by a factor of 2.

On the graph of $f(x) = \cos(x)$, multiply the x-coordinates by $\frac{1}{3}$ and the y-coordinates by 2.



Plot the key points of the parent function and the transformed points.

Within the specified domain,

- the transformed function decreases on the intervals (0°, 60°), (120°, 180°), and (240°, 300°) and increases on the intervals (60°, 120°), (180°, 240°), and (300°, 360°)
- the transformed function has the following turning points: (60°, −2), (120°, 2), (180°, −2), (240°, 2), and (300°, −2)

The domain consists of all real numbers; this is not changed by the horizontal compression and translation.
Domain $= \{x \in \mathbf{R}\}$.
The vertical stretch has changed the range from $\{y \in \mathbf{R} \mid -1 \le y \le 1\}$ to $\{y \in \mathbf{R} \mid -2 \le y \le 2\}$.

EXAMPLE 4 Reasoning about the order of transformations

Describe the order in which you would apply the transformations defined by $y = -2f(3(x + 1)) - 4$ to $f(x) = \sqrt{x}$. Then state the impact of the transformations on the domain, range, intervals of increase/decrease, and end behaviours of the transformed function.

Solution

$$(x, y) \rightarrow \left(\frac{1}{3}x, -2y\right)$$

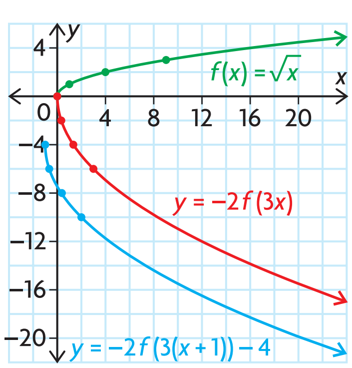
Parent Function $y = \sqrt{x}$	Stretched/Compressed Function $y = -2\sqrt{3x}$
(0, 0)	$\left(\frac{1}{3}(0), -2(0)\right) = (0, 0)$
(1, 1)	$\left(\frac{1}{3}(1), -2(1)\right) = \left(\frac{1}{3}, -2\right)$
(4, 2)	$\left(\frac{1}{3}(4), -2(2)\right) = \left(\frac{4}{3}, -4\right)$
(9, 3)	$\left(\frac{1}{3}(9), -2(3)\right) = (3, -6)$

Since multiplication must be done before addition, apply a horizontal compression by a factor of $\frac{1}{3}$, a vertical stretch by a factor of 2, and a reflection in the x-axis. To do this, multiply the x-coordinates of points on the parent function by $\frac{1}{3}$ and the y-coordinates by -2.

$$\left(\frac{1}{3}x, -2y\right) \rightarrow \left(\frac{1}{3}x - 1, -2y - 4\right)$$

Stretched/Compressed Function $y = -2\sqrt{3x}$	Final Transformed Function $y = -2\sqrt{3(x + 1)} - 4$
(0, 0)	$(0 - 1, 0 - 4) = (-1, -4)$
$\left(\frac{1}{3}, -2\right)$	$\left(\frac{1}{3} - 1, -2 - 4\right) = \left(-\frac{2}{3}, -6\right)$
$\left(\frac{4}{3}, -4\right)$	$\left(\frac{4}{3} - 1, -4 - 4\right) = \left(\frac{1}{3}, -8\right)$
(3, −6)	$(3 - 1, -6 - 4) = (2, -10)$

Apply all translations next. Translate the graph of $f(x) = -2f(3x)$ 1 unit to the left and 4 units down. To do this, subtract 1 from the x-coordinates and 4 from the y-coordinates of points on the previous function.



The transformed function is now a decreasing function on the interval $[-1, \infty)$.
The transformed function has the following end behaviours:

As $x \rightarrow -1$, $y \rightarrow -4$ and as $x \rightarrow \infty$, $y \rightarrow -\infty$.

Plot the points of the final transformed function. The horizontal translation changed the domain from $\{x \in \mathbf{R} \mid x \geq 0\}$ to $\{x \in \mathbf{R} \mid x \geq -1\}$.

The reflection in the x-axis and the vertical translation changed the range from $\{y \in \mathbf{R} \mid y \geq 0\}$ to $\{y \in \mathbf{R} \mid y \leq -4\}$.

In Summary

Key Ideas

- Transformations on a function $y = af(k(x - d)) + c$ must be performed in a particular order: horizontal and vertical stretches/compressions (including any reflections) must be performed before translations. All points on the graph of the parent function $y = f(x)$ are changed as follows: $(x, y) \rightarrow \left(\frac{x}{k} + d, ay + c\right)$
- When using transformations to graph, you can apply a and k together, and then c and d together, to get the desired graph in the fewest number of steps.

Need to Know

- The value of a determines whether there is a vertical stretch or compression, or a reflection in the x-axis:
  - When $|a| > 1$, the graph of $y = f(x)$ is stretched vertically by the factor $|a|$.
  - When $0 < |a| < 1$, the graph is compressed vertically by the factor $|a|$.
  - When $a < 0$, the graph is also reflected in the x-axis.
- The value of k determines whether there is a horizontal stretch or compression, or a reflection in the y-axis:
  - When $|k| > 1$, the graph is compressed horizontally by the factor $\frac{1}{|k|}$.
  - When $0 < |k| < 1$, the graph is stretched horizontally by the factor $\frac{1}{|k|}$.
  - When $k < 0$, the graph is also reflected in the y-axis.
- The value of d determines whether there is a horizontal translation:
  - For $d > 0$, the graph is translated to the right.
  - For $d < 0$, the graph is translated to the left.
- The value of c determines whether there is a vertical translation:
  - For $c > 0$, the graph is translated up.
  - For $c < 0$, the graph is translated down.

CHECK Your Understanding

1. State the transformations defined by each equation in the order they would be applied to $y = f(x)$.

a) $y = f(x) - 1$

b) $y = f(2(x - 1))$

c) $y = -f(x - 3) + 2$

d) $y = -2f(4x)$

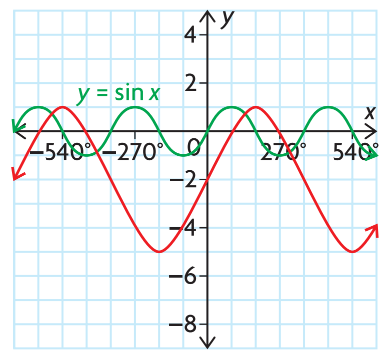
e) $y = -f(-(x + 2)) - 3$

f) $y = \frac{1}{2}f\left(\frac{1}{4}(x - 5)\right) + 6$

2. Identify the appropriate values for a, k, c, and d in $y = af(k(x - d)) + c$ to describe each set of transformations below.

a) horizontal stretch by a factor of 2, vertical translation 3 units up, reflection in the x-axis

b)



3. The point $(2, 3)$ is on the graph of $y = f(x)$. Determine the corresponding coordinates of this point on the graph of $y = -2(f(2(x + 5)) - 4$.

PRACTISING

4. The ordered pairs $(2, 3)$, $(4, 7)$, $(-2, 5)$, and $(-4, 6)$ belong to a function f. List the ordered pairs that belong to each of the following:

a) $y = 2f(x)$
b) $y = f(x - 3)$
c) $y = f(x) + 2$
d) $y = f(x + 1) - 3$
e) $y = f(-x)$
f) $y = f(2x) - 1$

5. K For each of the following equations, state the parent function and the transformation that was applied. Graph the transformed function.

a) $y = (x + 1)^2$
b) $y = 2|x|$
c) $y = \sin(3x) + 1$
d) $y = \frac{1}{x} + 3$
e) $y = 2^{0.5x}$
f) $y = \sqrt{2(x - 6)}$

6. State the domain and range of each function in question 5.

7. **a)** Graph the parent function $y = 2^x$ and the transformed function defined by $y = -2f(3(x - 1)) + 4$.

b) State the impact of the transformations on the domain and range, intervals of increase/decrease, and end behaviours.

c) State the equation of the transformed function.

8. The graph of $y = \sqrt{x}$ is stretched vertically by a factor of 3, reflected in the x-axis, and shifted 5 units to the right. Determine the equation that results from these transformations, and graph it.

9. The point $(1, 8)$ is on the graph of $y = f(x)$. Find the corresponding coordinates of this point on each of the following graphs.

a) $y = 3f(x - 2)$
b) $y = f(2(x + 1)) - 4$
c) $y = -2f(-x) - 7$
d) $y = -f(4(x + 1))$
e) $y = -f(-x)$
f) $y = 0.5f(0.5(x + 3)) + 3$

10. **A** Given $f(x) = \sqrt{x}$, find the domain and range for each of the following:

a) $g(x) = f(x - 2)$
b) $h(x) = 2f(x - 1) + 4$
c) $k(x) = f(-x) + 1$
d) $j(x) = 3f(2(x - 5)) - 3$

11. Greg thinks that the graphs of $y = 5x^2 - 3$ and $y = 5(x^2 - 3)$ are the same. Explain why he is incorrect.

12. Given $f(x) = x^3 - 3x^2$, $g(x) = f(x - 1)$, and $h(x) = -f(x)$, graph each function and compare $g(x)$ and $h(x)$ with $f(x)$.

13. **T** Consider the parent function $y = x^2$.

a) Describe the transformation that produced the equation $y = 4x^2$.
b) Describe the transformation that produced the equation $y = (2x)^2$.
c) Show algebraically that the two transformations produce the same equation and graph.

14. **C** Use a flow chart to show the sequence and types of transformations required to transform the graph of $y = f(x)$ into the graph of $y = af(k(x - d)) + c$.

Extending

15. The point $(3, 6)$ is on the graph of $y = 2f(x + 1) - 4$. Find the original point on the graph of $y = f(x)$.

16. **a)** Describe the transformations that produce $y = f(3(x + 2))$.

b) The graph of $y = f(3x + 6)$ is produced by shifting 6 units to the left and then compressing the graph by a factor of $\frac{1}{3}$. Why does this produce the same result as the transformations you described in part a)?

c) Using $f(x) = x^2$ as the parent function, graph the transformations described in parts a) and b) to show that they result in the same transformed function.