1.5 Inverse Relations

YOU WILL NEED

- graph paper
- graphing calculator

GOAL

Determine the equation of an inverse relation and the conditions for an inverse relation to be a function.

LEARN ABOUT the Math

The owners of a candy company are creating a spherical container to hold their small chocolates. They are trying to decide what size to make the sphere and how much volume the sphere will hold, based on its radius.

The volume of a sphere is given by the relationship $V = \frac{4}{3}\pi r^3$.

? How can you use this relationship to find the radius of any sphere for a given volume?

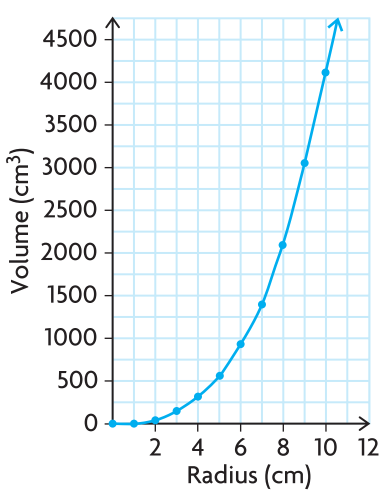
EXAMPLE 1 Representing the inverse using a table of values and a graph

Use a table of values and a graphical model to represent the relationship between the radius of a sphere and any given volume.

Solution

$$V = \frac{4}{3}\pi r^3$$

Radius (cm)	Volume (cm^3)
0.0	0.0
1.0	4.2
2.0	33.5
3.0	113.1
4.0	268.1
5.0	523.6
6.0	904.8
7.0	1436.8
8.0	2144.7
9.0	3053.6
10.0	4188.8

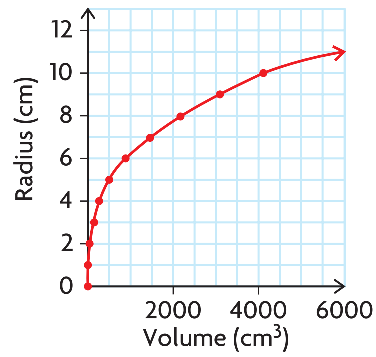


Radius is the independent variable, and volume is the dependent variable.

Create a table of values, and calculate the volume for a specific radius.

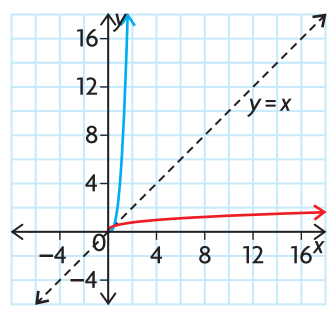
Draw a scatter plot of volume in terms of radius. Draw a smooth curve through the points since the function is continuous.

Volume (cm^3)	Radius (cm)
0.0	0
4.2	1
33.5	2
113.1	3
268.1	4
523.6	5
904.8	6
1436.8	7
2144.7	8
3053.6	9
4188.8	10



To graph radius in terms of volume, switch the variables in the table, making radius the dependent variable and volume the independent variable.

The red curve shows volume as the independent variable and radius as the dependent variable.



If we ignore units and plot both relations on the same graph, the red curve is a reflection of the blue curve in the line $y = x$. This is reasonable, given that the x-values and y-values were switched on the graph. The red curve is the inverse relation, and it is also a function.

The inverse was found by switching the independent and dependent variables in the table of values. The independent and dependent variables can also be switched in the equation of the relation to determine the equation of the inverse relation.

EXAMPLE 2 Representing the inverse using an equation

Recall that the volume of a sphere is given by the relationship $V = \frac{4}{3}\pi r^3$.

Determine the equation of the inverse.

Solution

$$V = \frac{4}{3}\pi r^3$$

To express V in terms of r, rearrange the formula using inverse operations.

$$3 \times V = 3 \times \left(\frac{4}{3}\pi r^3\right)$$

Multiply both sides by 3 to eliminate the fraction.

$$3V = 4\pi r^3$$

$$\frac{3V}{4\pi} = \frac{4\pi r^3}{4\pi}$$

$$\sqrt[3]{\frac{3V}{4\pi}} = \sqrt[3]{r^3}$$

Divide both sides by 4π (the coefficient of r^3) to isolate r^3. Take the cube root of both sides to isolate r.

$$\sqrt[3]{\frac{3V}{4\pi}} = r$$

The radius is now expressed as a function of volume and can be determined for different values of V.

Reflecting

A. Compare the domain and range of this function and its inverse.

B. Will an **inverse of a function** always be a function? Explain.

C. Why is it reasonable to switch the V and the r in Example 2 to determine the inverse relation?

APPLY the Math

EXAMPLE 3 Using an algebraic strategy to determine the inverse relation

Given $f(x) = x^2$.

a) Find the inverse relation.
b) Compare the domain and range of the function and its inverse.
c) Determine if the inverse relation is also a function.

Solution

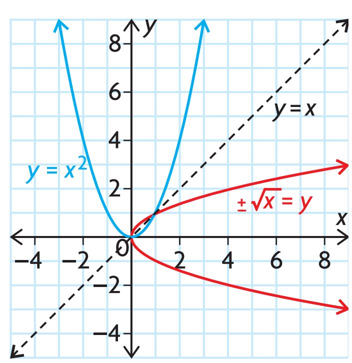
a) $y = x^2$ — Rewrite the function using x and y.

$x = y^2$ — Interchange x and y in the relation.

$\pm\sqrt{x} = \sqrt{y^2}$ — Solve for y by taking the square root of both sides.

$\pm\sqrt{x} = y$

b)



The graph of the inverse relation is a reflection of the original relation in the line $y = x$.

Only non-negative values of x work in the square root function. The square root of a negative number is undefined. Since $\pm$ in the inverse indicates that the output, y, will include both positive and negative values, the range will include all the real numbers.

The domain of $y = x^2$ is $\{x \in \mathbf{R}\}$. The range is $\{x \in \mathbf{R} \mid y \geq 0\}$.

The domain of the inverse relation is $\{x \in \mathbf{R} \mid x \geq 0\}$. The range is $\{y \in \mathbf{R}\}$.

c) The inverse relation is not a function, but it can be split in the middle into the two functions, $y = \sqrt{x}$ and $y = -\sqrt{x}$.

Based on the equation of the inverse relation, each input of x will have two outputs for y, one positive and one negative. The only exception is $x = 0$.

Communication *Tip*

The domain of the square root function is $\{x \in \mathbf{R} \mid x \geq 0\}$; we say the values of x are non-negative. The range of the exponential function $y = 2^x$ is $\{y \in \mathbf{R} \mid y > 0\}$; we say the values of y are positive. The distinction is because zero is neither negative nor positive.

The inverse relation is useful to solve problems, particularly when you are given a value of the dependent variable and need to determine the value of the corresponding independent variable.

EXAMPLE 4 Selecting a strategy that involves the inverse relation to solve a problem

Archaeologists use models for the relationship between height and footprint length to determine the height of a person based on the lengths of the bones they discover. The relationship between height, $h(x)$, in centimetres and footprint length, x, in centimetres is given by $h(x) = 1.1x + 143.6$. Use this relationship to predict the footprint length for a person who is 170 cm tall.

Solution

$h(x) = 1.1x + 143.6$

Let $y = h(x)$.

$y = 1.1x + 143.6$

To predict the footprint length, rewrite the relationship with footprint length as the dependent variable and $h(x)$ as the independent variable.

$x = 1.1y + 143.6$

Interchange x and y.

$x - 143.6 = 1.1y$

$\frac{x - 143.6}{1.1} = y = h^{-1}(x)$

Solve for y.

$h^{-1}(170) = \frac{170 - 143.6}{1.1}$

$= 24$ cm

Evaluate $h^{-1}(170)$.

A person who is 170 cm tall may have a footprint length of 24 cm.

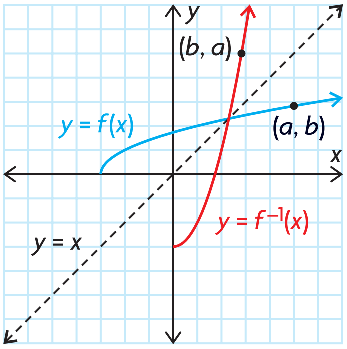
Communication *Tip*

When an inverse relation is also a function, the notation $f^{-1}(x)$ can be used to define the inverse function.

In Summary

Key Ideas

- The inverse function of $f(x)$ is denoted by $f^{-1}(x)$. Function notation can only be used when the inverse is a function.
- The graph of the inverse function is a reflection in the line $y = x$.



(continued)

Need to Know

- Not all inverse relations are functions. The domain and/or range of the original function may need to be restricted to ensure that the inverse of a function is also a function.
- To find the inverse algebraically, write the function equation using y instead of $f(x)$. Interchange x and y. Solve for y.
- If (a, b) represents a point on the graph of $f(x)$, then (b, a) represents a point on the graph of the corresponding f^{-1}.
- Given a table of values or a graph of a function, the independent and dependent variables can be interchanged to get a table of values or a graph of the inverse relation.
- The domain of a function is the range of its inverse. The range of a function is the domain of its inverse.

CHECK *Your Understanding*

1. Each of the following ordered pairs is a point on a function. State the corresponding point on the inverse relation.

a) $(2, 5)$ **c)** $(4, -8)$ **e)** $g(-3) = 0$

b) $(-5, -6)$ **d)** $f(1) = 2$ **f)** $h(0) = 7$

2. Given the domain and range of a function, state the domain and range of the inverse relation.

a) $D = \{x \in \mathbf{R}\}, R = \{y \in \mathbf{R}\}$

b) $D = \{x \in \mathbf{R} \mid x \geq 2\}, R = \{y \in \mathbf{R}\}$

c) $D = \{x \in \mathbf{R} \mid x \geq -5\}, R = \{y \in \mathbf{R} \mid y < 2\}$

d) $D = \{x \in \mathbf{R} \mid x < -2\}, R = \{y \in \mathbf{R} \mid -5 < y < 10\}$

3. Match the inverse relations to their corresponding functions.

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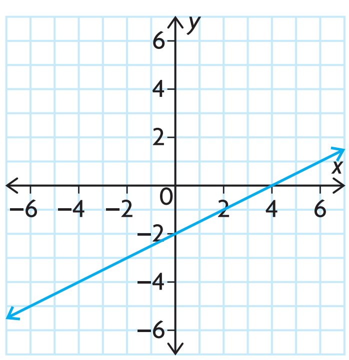

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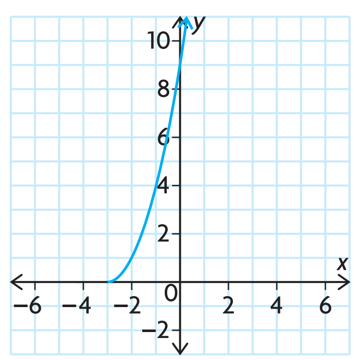

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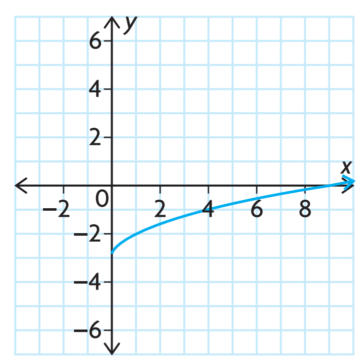

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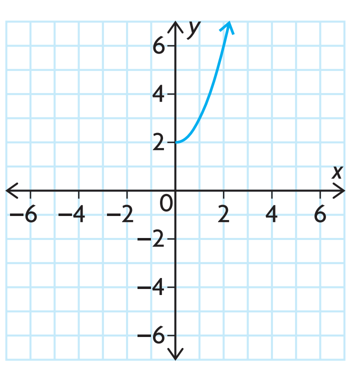

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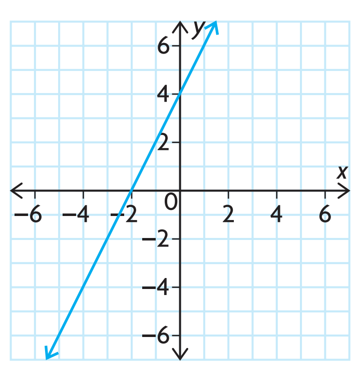

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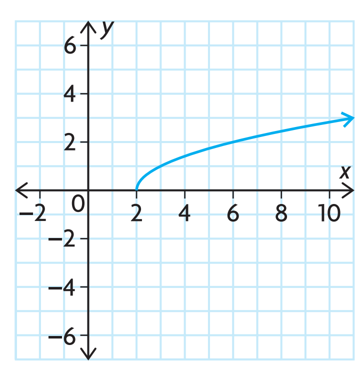

PRACTISING

4. Consider the function $f(x) = 2x^3 + 1$.

K

a) Find the ordered pair $(4, f(4))$ on the function.
b) Find the ordered pair on the inverse relation that corresponds to the ordered pair from part a).
c) Find the domain and range of f.
d) Find the domain and the range of the inverse relation of f.
e) Is the inverse relation a function? Explain.

5. Repeat question 4 for the function $g(x) = x^4 - 8$.

6. Graph each function and its inverse relation on the same set of axes. Determine whether the inverse relation is a function.
a) $f(x) = x^2 + 1$
b) $g(x) = \sin x$, where $-360° \leq x \leq 360°$
c) $h(x) = -x$
d) $m(x) = |x| + 1$

7. **a)** The equation $F = \frac{9}{5}C + 32$ can be used to convert a known Celsius temperature, C, to the equivalent Fahrenheit temperature, F. Find the inverse of this relation, and describe what it can be used for.

A

b) Use the equation given in part a) to convert 20 °C to its equivalent Fahrenheit temperature. Use the inverse relation to convert this Fahrenheit temperature back to its equivalent Celsius temperature.

8. **a)** The formula $A = \pi r^2$ is convenient for calculating the area of a circle when the radius is known. Find the inverse of the relation, and describe what it can be used for.
b) Use the equation given in part a) to calculate the area of a circle with a radius of 5 cm. (Express the area as an exact value in terms of π.) Use the inverse relation to calculate the radius of the circle with the area you calculated.

9. If $f(x) = kx^3 - 1$ and $f^{-1}(15) = 2$, find k.

T

10. Given the function $h(x) = 2x + 7$, find
a) $h(3)$
b) $h(9)$
c) $\dfrac{h(9) - h(3)}{9 - 3}$
d) $h^{-1}(3)$
e) $h^{-1}(9)$
f) $\dfrac{h^{-1}(9) - h^{-1}(3)}{9 - 3}$

11. Suppose that the variable a represents a particular student and $f(a)$ represents the student's overall average in all their subjects. Is the inverse relation of f a function? Explain.

12. Determine the inverse of each function.

a) $f(x) = 3x + 4$

b) $h(x) = -x$

c) $g(x) = x^3 - 1$

d) $m(x) = -2(x + 5)$

13. A function g is defined by $g(x) = 4(x - 3)^2 + 1$.

a) Determine an equation for the inverse of $g(x)$.

b) Solve for y in the equation for the inverse of $g(x)$.

c) Graph $g(x)$ and its inverse using graphing technology.

d) At what points do the graphs of $g(x)$ and its inverse intersect?

e) State **restrictions** on the domain or range of g so that its inverse is a function.

f) Suppose that the domain of $g(x)$ is $\{x \in \mathbf{R} \mid 2 \leq x \leq 5\}$. Is the inverse a function? Justify your answer.

14. A student writes, "The inverse of $y = -\sqrt{x + 2}$ is $y = x^2 - 2$." Explain why this statement is not true.

15. Do you have to restrict either the domain or the range of the function $y = \sqrt{x + 2}$ to make its inverse a function? Explain.

16. C John and Katie are discussing inverse relationships. John says, "A function is a rule, and the inverse is the rule performed in reverse order with opposite operations. For example, suppose that you cube a number, divide by 4, and add 2. The inverse is found by subtracting 2, multiplying by 4, and taking the cube root." Is John correct? Justify your answer algebraically, numerically, and graphically.

Extending

17. $f(x) = x$ is an interesting function because it is its own inverse. Can you find three more functions that have the same property? Can you convince yourself that there are an infinite number of functions that satisfy this property?

18. The inverse relation of a function is also a function if the original function passes the horizontal line test (in other words, if any horizontal line hits the function in at most one location). Explain why this is true.