1.6 Piecewise Functions

YOU WILL NEED

- graph paper
- graphing calculator

GOAL

Understand, interpret, and graph situations that are described by piecewise functions.

LEARN ABOUT the Math

A city parking lot uses the following rules to calculate parking fees:

- A flat rate of $5.00 for any amount of time up to and including the first hour
- A flat rate of $12.50 for any amount of time over 1 h and up to and including 2 h
- A flat rate of $13 plus $3 per hour for each hour after 2 h

? How can you describe the function for parking fees in terms of the number of hours parked?

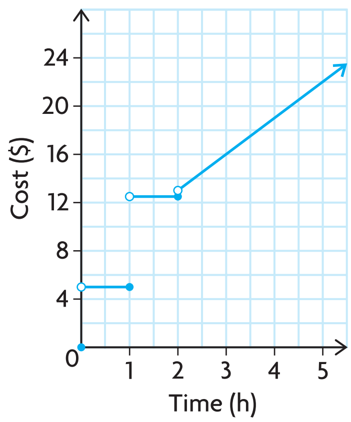
EXAMPLE 1 Representing the problem using a graphical model

Use a graphical model to represent the function for parking fees.

Solution

Time (h)	Parking Fee ($)
0	0
0.25	5.00
0.50	5.00
1.00	5.00
1.25	12.50
1.50	12.50
2.00	12.50
2.50	14.50
3.00	16.00
4.00	19.00

Create a table of values.



The domain of this **piecewise function** is $x \geq 0$.

The function is linear over the domain, but it is discontinuous at $x = 0$, 1, and 2.

Plot the points in the table of values. Use a solid dot to include a value in an interval. Use an open dot to exclude a value from an interval.

There is a solid dot at (0, 0) and an open dot at (0, 5) because the parking fee at 0 h is \$0.00.

There is a closed dot at (1, 5) and an open dot at (1, 12.50) because the parking fee at 1 h is \$5.00.

There is a closed dot at (2, 12.50) and an open dot at (2, 13) because the parking fee at 2 h is \$12.50.

The last part of the graph continues in a straight line since the rate of change is constant after 2 h.

piecewise function
a function defined by using two or more rules on two or more intervals; as a result, the graph is made up of two or more pieces of similar or different functions

Each part of a piecewise function can be described using a specific equation for the interval of the domain.

EXAMPLE 2 Representing the problem using an algebraic model

Use an algebraic model to represent the function for parking fees.

Solution

$y_1 = 0$ if $x = 0$

$y_2 = 5$ if $0 < x \leq 1$

$y_3 = 12.50$ if $1 < x \leq 2$

$y_4 = 3x + 13$ if $x > 2$

Write the relation for each rule.

$$f(x) = \begin{cases} 0, \text{ if } x = 0 \\ 5, \text{ if } 0 < x \leq 1 \\ 12.50, \text{ if } 1 < x \leq 2 \\ 3x + 13, \text{ if } x > 2 \end{cases}$$

Combine the relations into a piecewise function.

The domain of the function is $x \geq 0$.

The function is discontinuous at $x = 0$, 1, and 2 because there is a break in the function at each of these points.

Reflecting

A. How do you sketch the graph of a piecewise function?

B. How do you create the algebraic representation of a piecewise function?

C. How do you determine from a graph or from the algebraic representation of a piecewise function if there are any discontinuities?

APPLY the Math

EXAMPLE 3 Representing a piecewise function using a graph

Graph the following piecewise function.

$$f(x) = \begin{cases} x^2, \text{ if } x < 2 \\ 2x + 3, \text{ if } x \geq 2 \end{cases}$$

Solution

Create a table of values.

$f(x) = x^2$

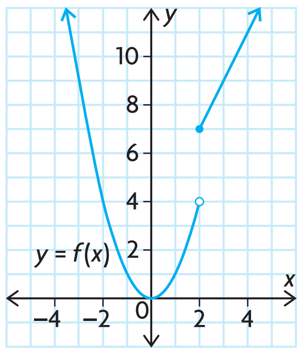
x	f(x)
−2	4
−1	1
0	0
1	1
2	4

$f(x) = 2x + 3$

x	f(x)
2	7
3	9
4	11
5	13
6	15

From the equations given, the graph consists of part of a parabola that opens up and a line that rises from left to right.

Both tables include $x = 2$ since this is where the description of the function changes.



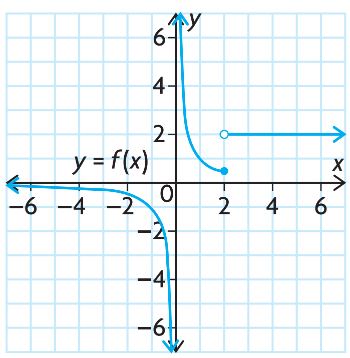
$f(x)$ is discontinuous at $x = 2$.

Plot the points, and draw the graph.

A solid dot is placed at (2, 7) since $x = 2$ is included with $f(x) = 2x + 3$. An open dot is placed at (2, 4) since $x = 2$ is excluded from $f(x) = x^2$.

EXAMPLE 4 Representing a piecewise function using an algebraic model

Determine the algebraic representation of the following piecewise function.



Solution

$$f(x) = \begin{cases} \frac{1}{x}, \text{ if } x \leq 2 \\ 2, \text{ if } x > 2 \end{cases}$$

The graph is made up of two pieces. One piece is part of the reciprocal function defined by $y = \frac{1}{x}$ when $x \leq 2$. The other piece is a horizontal line defined by $y = 2$ when $x > 2$. The solid dot indicates that point $\left(2, \frac{1}{2}\right)$ belongs with the reciprocal function.

EXAMPLE 5 Reasoning about the continuity of a piecewise function

Is this function continuous at the points where it is pieced together? Explain.

$$g(x) = \begin{cases} x + 1, \text{ if } x \leq 0 \\ 2x + 1, \text{ if } 0 < x < 3 \\ 4 - x^2, \text{ if } x \geq 3 \end{cases}$$

Solution

The function is continuous at the points where it is pieced together if the functions being joined have the same y-values at these points.

The graph is made up of three pieces. One piece is part of an increasing line defined by $y = x + 1$ when $x \leq 0$. The second piece is an increasing line defined by $y = 2x + 1$ when $0 < x < 3$. The third piece is part of a parabola that opens down, defined by $y = 4 - x^2$ when $x \geq 3$.

Calculate the values of the function at $x = 0$ using the relevant equations:

$y = x + 1$	$y = 2x + 1$
$y = 0 + 1$	$y = 2(0) + 1$
$y = 1$	$y = 1$

The two y-values are the same, so the two linear pieces join each other at $x = 0$.

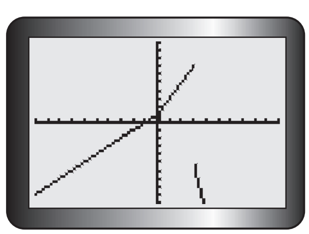
Calculate the values of the function at $x = 3$ using the relevant equations:

$y = 2x + 1$	$y = 4 - x^2$
$y = 2(3) + 1$	$y = 4 - 3^2$
$y = 7$	$y = -5$

The two y-values are different, so the second linear piece does not join with the parabola at $x = 3$.

The function is discontinuous, since there is a break in the graph at $x = 3$.

Verify by graphing.



Tech Support

For help using a graphing calculator to graph a piecewise function, see Technical Appendix, T-16.

In Summary

Key Ideas

- Some functions are represented by two or more "pieces." These functions are called piecewise functions.
- Each piece of a piecewise function is defined for a specific interval in the domain of the function.

Need to Know

- To graph a piecewise function, graph each piece of the function over the given interval.
- A piecewise function can be either continuous or not. If all the pieces of the function join together at the endpoints of the given intervals, then the function is continuous. Otherwise, it is discontinuous at these values of the domain.

CHECK Your Understanding

1. Graph each piecewise function.

a) $f(x) = \begin{cases} 2, \text{ if } x < 1 \\ 3x, \text{ if } x \geq 1 \end{cases}$

b) $f(x) = \begin{cases} -2x, \text{ if } x < 0 \\ x + 4, \text{ if } x \geq 0 \end{cases}$

c) $f(x) = \begin{cases} |x|, \text{ if } x \leq -2 \\ -x^2, \text{ if } x > -2 \end{cases}$

d) $f(x) = \begin{cases} |x + 2|, \text{ if } x \leq -1 \\ -x^2 + 2, \text{ if } x > -1 \end{cases}$

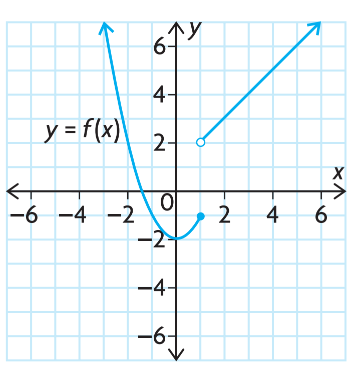
e) $f(x) = \begin{cases} \sqrt{x}, \text{ if } x < 4 \\ 2^x, \text{ if } x \geq 4 \end{cases}$

f) $f(x) = \begin{cases} \frac{1}{x}, \text{ if } x < 1 \\ -x, \text{ if } x \geq 1 \end{cases}$

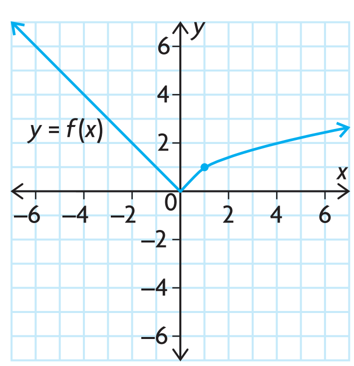
2. State whether each function in question 1 is continuous or not. If not, state where it is discontinuous.

3. Write the algebraic representation of each piecewise function, using function notation.

a)



b)



4. State the domain of each piecewise function in question 3, and comment on the continuity of the function.

PRACTISING

5. **K** Graph the following piecewise functions. Determine whether each function is continuous or not, and state the domain and range of the function.

a) $f(x) = \begin{cases} 2, \text{ if } x < -1 \\ 3, \text{ if } x \geq -1 \end{cases}$

b) $f(x) = \begin{cases} -x, \text{ if } x \leq 0 \\ x, \text{ if } x > 0 \end{cases}$

c) $f(x) = \begin{cases} x^2 + 1, \text{ if } x < 2 \\ 2x + 1, \text{ if } x \geq 2 \end{cases}$

d) $f(x) = \begin{cases} 1, \text{ if } x < -1 \\ x + 2, \text{ if } -1 \leq x \leq 3 \\ 5, \text{ if } x > 3 \end{cases}$

6. **A** Graham's long-distance telephone plan includes the first 500 min per month in the \$15.00 monthly charge. For each minute after 500 min, Graham is charged \$0.02. Write a function that describes Graham's total long-distance charge in terms of the number of long distance minutes he uses in a month.

7. Many income tax systems are calculated using a tiered method. Under a certain tax law, the first \$100 000 of earnings are subject to a 35% tax; earnings greater than \$100 000 and up to \$500 000 are subject to a 45% tax. Any earnings greater than \$500 000 are taxed at 55%. Write a piecewise function that models this situation.

8. **T** Find the value of k that makes the following function continuous. Graph the function.

$$f(x) = \begin{cases} x^2 - k, \text{ if } x < -1 \\ 2x - 1, \text{ if } x \geq -1 \end{cases}$$

9. The fish population, in thousands, in a lake at any time, x, in years is modelled by the following function:

$$f(x) = \begin{cases} 2^x, \text{ if } 0 \leq x \leq 6 \\ 4x + 8, \text{ if } x > 6 \end{cases}$$

This function describes a sudden change in the population at time $x = 6$, due to a chemical spill.

a) Sketch the graph of the piecewise function.
b) Describe the continuity of the function.
c) How many fish were killed by the chemical spill?
d) At what time did the population recover to the level it was before the chemical spill?
e) Describe other events relating to fish populations in a lake that might result in piecewise functions.

10. Create a flow chart that describes how to plot a piecewise function with two pieces. In your flow chart, include how to determine where the function is continuous.

11. **C** An absolute value function can be written as a piecewise function that involves two linear functions. Write the function $f(x) = |x + 3|$ as a piecewise function, and graph your piecewise function to check it.

12. The demand for a new CD is described by

$$D(p) = \begin{cases} \frac{1}{p^2}, \text{ if } 0 < p \leq 15 \\ 0, \text{ if } p > 15 \end{cases}$$

where D is the demand for the CD at price p, in dollars. Determine where the demand function is discontinuous and continuous.

Extending

13. Consider a function, $f(x)$, that takes an element of its domain and rounds it down to the nearest 10. Thus, $f(15.6) = 10$, while $f(21.7) = 20$ and $f(30) = 30$. Draw the graph, and write the piecewise function. You may limit the domain to $x \in [0, 50)$. Why do you think graphs like this one are often referred to as *step functions*?

14. Explain why there is no value of k that will make the following function continuous.

$$f(x) = \begin{cases} 5x, \text{ if } x < -1 \\ x + k, \text{ if } -1 \leq x \leq 3 \\ 2x^2, \text{ if } x > 3 \end{cases}$$

15. The *greatest integer function* is a step function that is written as $f(x) = [x]$, where $f(x)$ is the greatest integer less than or equal to x. In other words, the greatest integer function rounds any number down to the nearest integer. For example, the greatest integer less than or equal to the number [5.3] is 5, while the greatest integer less than or equal to the number $[-5.3]$ is -6. Sketch the graph of $f(x) = [x]$.

16. **a)** Create your own piecewise function using three different transformed parent functions.
b) Graph the function you created in part a).
c) Is the function you created continuous or not? Explain.
d) If the function you created is not continuous, change the interval or adjust the transformations used as required to change it to a continuous function.