2.1

Determining Average Rate of Change

average rate of change

in a relation, the change in the quantity given by the dependent variable (Δy) divided by the corresponding change in the quantity represented by the independent variable (Δx); for a function $y = f(x)$, the average rate of change in the internal $x_1 \leq x \leq x_2$ is $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

GOAL

Calculate and interpret the average rate of change on an interval of the independent variable.

LEARN ABOUT the Math

The following table represents the growth of a bacteria population over a 10 h period.

Time (h)	0	2	4	6	8	10
Number of Bacteria	850	1122	1481	1954	2577	3400

? During which 2 h interval did the bacteria population grow the fastest?

EXAMPLE 1 Reasoning about rate of change

Use the data in the table of values to determine the 2 h interval in which the bacteria population grew the fastest.

Solution A: Using a table

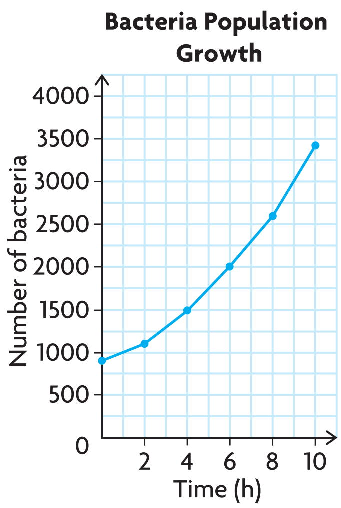
Time Interval (h)	Δb = Change in Number of Bacteria	Δt = Change in Time (h)	$\frac{\Delta b}{\Delta t}$ = Average Rate of Change (bacteria/h)
$0 \leq t \leq 2$	$1122 - 850 = 272$	$2 - 0 = 2$	$\frac{272}{2} = 136$
$2 \leq t \leq 4$	$1481 - 1122 = 359$	$4 - 2 = 2$	$\frac{359}{2} = 179.5$
$4 \leq t \leq 6$	$1954 - 1481 = 473$	$6 - 4 = 2$	$\frac{473}{2} = 236.5$
$6 \leq t \leq 8$	$2577 - 1954 = 623$	$8 - 6 = 2$	$\frac{623}{2} = 311.5$
$8 \leq t \leq 10$	$3400 - 2577 = 823$	$10 - 8 = 2$	$\frac{823}{2} = 411.5$

Calculate the average rate of change in the dependent variable (bacteria population) for each 2 h interval. Divide the change in the number of bacteria by the corresponding change in time (2 h for each interval). Identify the interval with the greatest change in population.

The greatest change in the bacteria population occurred during the last 2 h, when the population increased by an average of 412 bacteria per hour.

The average rate of change is expressed using the units of the two related quantities.

Solution B: Using points on a graph

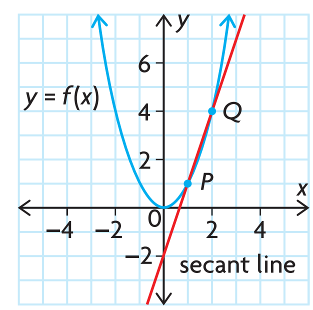


Create a scatter plot using the data in the table of values. Draw a **secant line** that passes through each pair of the endpoints for each 2 h interval. The slope of each secant line is equivalent to the average rate of change in the number of bacteria over each interval.

From the graph, it appears that the secant line with the greatest slope occurs during the last interval, from 8 to 10 h.

secant line

a line that passes through two points on the graph of a relation



Calculate the slope of each secant line to verify.

Recall that $m = \frac{y_2 - y_1}{x_2 - x_1}$, where (x_1, y_1) and (x_2, y_2) are points on the line.

In the first interval, the secant line passes through $(0, 850)$ and $(2, 1122)$.

$$m_1 = \frac{1122 - 850}{2 - 0} = \frac{272}{2}$$
$$= 136$$

Perform the same calculations for the other intervals.

$$m_2 = \frac{1481 - 1122}{4 - 2}$$
$$= 179.5$$

$$m_3 = \frac{1954 - 1481}{6 - 4}$$
$$= 236.5$$

$$m_4 = \frac{2577 - 1954}{8 - 6}$$
$$= 311.5$$

$$m_5 = \frac{3400 - 2577}{10 - 8}$$
$$= 411.5$$

The greatest change in the bacteria population occurred when the secant line is the steepest, during the last 2 h. The bacteria population increased by an average of 412 bacteria per hour in this interval.

Slope has no units, but average rate of change does.

Reflecting

A. Why is the average rate of change of the bacteria population positive on each interval, and what does this mean? How is this represented by the secant lines on the graph of the data?

B. How is calculating an average rate of change like calculating the slope of a secant line?

C. Why does rate of change have units, even though slope does not?

D. Why is the rate of change in the bacteria population not a constant?

APPLY the Math

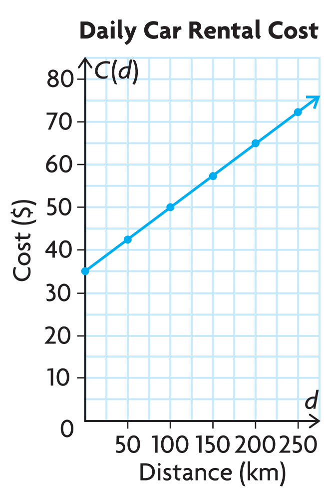
EXAMPLE 2 Reasoning about average rates of change in linear relationships

Sarah rents a car from a rental company. She is charged \$35 a day, plus a fee of \$0.15/km for the distances she drives each day. The equation $C(d) = 0.15d + 35$ can be used to calculate her daily cost to rent the car, where $C(d)$ is her daily cost in dollars and d is the daily distance she drives in kilometres.

Discuss the average rate of change of her daily costs in relation to the distance she drives.

Solution

Graph the equation $C(d) = 0.15d + 35$.



The relationship is linear, so the rate of change in the daily cost is constant. This means that the average rate of change between any two points on the graph is always constant. The secant lines that are drawn between any two pairs of points on the graph have the same slope.

Using the distance interval $0 \leq d \leq 100$,

$$\frac{\Delta C}{\Delta d} = \frac{C(100) - C(0)}{100 - 0}$$

Calculate some average rates of change in the daily cost, using different distance intervals to verify.

$$= \frac{50 - 35}{100} = \$0.15/\text{km}$$

Using the distance interval $100 \leq d \leq 250$,

$$\frac{\Delta C}{\Delta d} = \frac{C(250) - C(100)}{250 - 100}$$

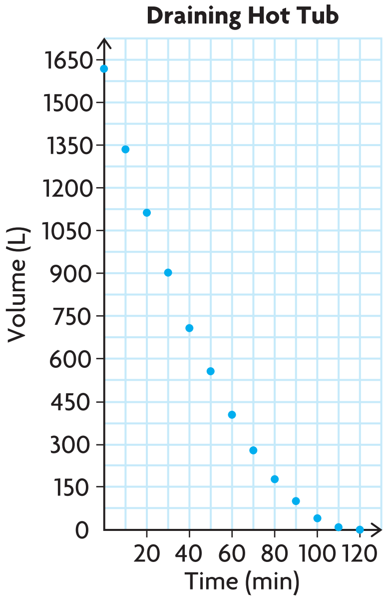
$$= \frac{72.50 - 50}{150} = \$0.15/\text{km}$$

The farther she drives each day, the more she will pay to rent the car. However, the rate at which the daily cost increases does not change. For every additional kilometre she drives, her daily cost increases by \$0.15.

EXAMPLE 3 Using a graph to determine the average rate of change

Andrew drains the water from a hot tub. The tub holds 1600 L of water. It takes 2 h for the water to drain completely. The volume V, in litres, of water remaining in the tub at various times t, in minutes, is shown in the table and graph.

Volume of Water in a Draining Hot Tub



Time (min)	Volume (L)
0	1600
10	1344
20	1111
30	900
40	711
50	544
60	400
70	278
80	178
90	100
100	44
110	10
120	0

a) Calculate the average rate of change in volume during each of the following time intervals.

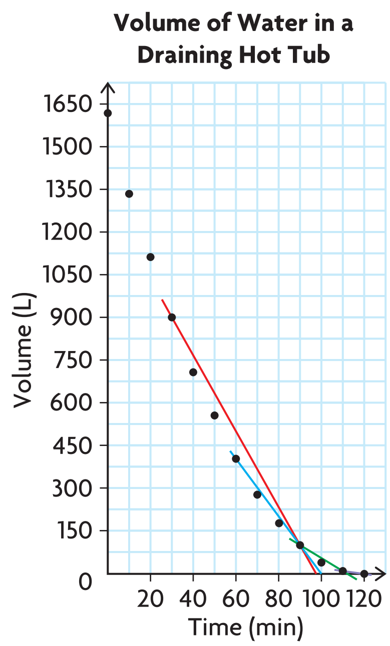
i) $30 \leq t \leq 90$ **iii)** $90 \leq t \leq 110$

ii) $60 \leq t \leq 90$ **iv)** $110 \leq t \leq 120$

b) Why is the rate of change in volume negative during each of these time intervals?

c) Does the hot tub drain at a constant rate? Explain.

Solution



Draw secant lines that pass through the endpoints for each given interval.

a) i) $$m = \frac{\Delta V}{\Delta t} = \frac{V(90) - V(30)}{90 - 30}$$
$$= \frac{100 - 900}{60}$$
$$\doteq -13.3$$

Use the points (30, 900) and (90, 100) on the graph for the red secant line to calculate the slope of the line. The average rate of change in volume that corresponds to this change in time is equivalent to the slope of the secant line.

The volume of water is decreasing, on average, at the rate of 13.3 L/min between 30 min and 90 min.

ii) $$m = \frac{\Delta V}{\Delta t} = \frac{V(90) - V(60)}{90 - 60}$$
$$= \frac{100 - 400}{30}$$
$$= -10$$

Use the points (60, 400) and (90, 100) on the graph for the blue secant line to calculate its slope.

The volume of water is decreasing, on average, at the rate of 10 L/min between 60 min and 90 min.

iii) $m = \frac{\Delta V}{\Delta t} = \frac{V(110) - V(90)}{110 - 90}$

$= \frac{10 - 100}{20}$

$= -4.5$

Use the points (90, 100) and (110, 10) on the graph for the green secant line to calculate its slope.

The volume of water is decreasing, on average, at the rate of 4.5 L/min between 90 min and 110 min.

iv) $m = \frac{\Delta V}{\Delta t} = \frac{V(120) - V(110)}{120 - 110}$

$= \frac{0 - 10}{10}$

$= -1$

Use the points (110, 10) and (120, 0) on the graph for the purple secant line to calculate its slope.

The volume of water is decreasing, on average, at the rate of 1 L/min between 110 min and 120 min.

b) The volume decreases as the time increases. So the numerator in each slope calculation is negative, while the denominator is positive. This makes the **rational number** that represents the rate of change negative.

The water is flowing out of the tub, so the volume that remains in the tub decreases with time.

c) The water is not draining at a constant rate over the 2 h period. This can be seen from the graph, because it is a non-linear relationship. The water is draining from the tub faster over time intervals at the beginning of the 2 h period. As the volume of water decreases, the pressure also decreases, causing the water to flow out of the tub more slowly.

Looking at the slope calculations, the slopes of the red and blue secant lines have a greater magnitude than the slopes of the green and purple secant lines. Also, the slopes of the secant lines between points over each 10 min interval are smaller in magnitude as time increases.

magnitude
the absolute value of a quantity

If you are given the equation of a relation or function, the average rate of change on a given interval can be calculated.

EXAMPLE 4 Using an equation to determine the average rate of change

A rock is tossed upward from a cliff that is 120 m above the water. The height of the rock above the water is modelled by $h(t) = -5t^2 + 10t + 120$, where $h(t)$ is the height in metres and t is time in seconds.

a) Calculate the average rate of change in height during each of the following time intervals.
i) $0 \leq t \leq 1$ **ii)** $1 \leq t \leq 2$ **iii)** $2 \leq t \leq 3$ **iv)** $3 \leq t \leq 4$

b) As the time increases, what do you notice about the average rate of change in height during each 1 s interval? What does this mean?

c) Describe what the average rate of change means in this situation.

Solution

a) i)
$$\frac{\Delta h}{\Delta t} = \frac{h(1) - h(0)}{1 - 0} = \frac{125 - 120}{1} = 5 \text{ m/s}$$

ii)
$$\frac{\Delta h}{\Delta t} = \frac{h(2) - h(1)}{2 - 1} = \frac{120 - 125}{1} = -5 \text{ m/s}$$

iii)
$$\frac{\Delta h}{\Delta t} = \frac{h(3) - h(2)}{3 - 2} = \frac{105 - 120}{1} = -15 \text{ m/s}$$

iv)
$$\frac{\Delta h}{\Delta t} = \frac{h(4) - h(3)}{4 - 3} = \frac{80 - 105}{1} = -25 \text{ m/s}$$

Substitute $t = 1$ and $t = 0$ into the equation to determine $h(1)$ and $h(0)$.

Calculate the change in height. Divide by the corresponding change in time to determine the average rate of change.

Repeat this process for the other three intervals.

b) The average rates of change are positive and then negative because the rock's height increases and then decreases. The average rates of change in height are also changing for each 1 s interval. After 1 s, as time increases, the rock is dropping a greater distance. The magnitude of the average rates of change are increasing. The rock is not falling at a constant rate.

Between 0 s and 1 s, the rock rises 5 m.
Between 1 s and 2 s, the rock drops 5 m.
Between 2 s and 3 s, the rock drops 15 m.
Between 3 s and 4 s, the rock drops 25 m.

c) Since the rate of change compares a change in distance over an interval of time, the rate of change represents the speed of the rock over the interval.

In this situation, as time increases, the rock picks up speed once it has passed its maximum height, because the distance it drops increases with each second.

In Summary

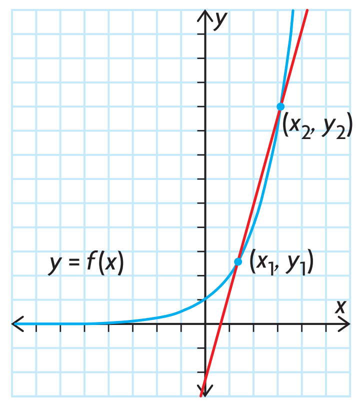
Key Ideas

- The average rate of change is the change in the quantity represented by the dependent variable (Δy) divided by the corresponding change in the quantity represented by the independent variable (Δx) over an interval. Algebraically, the average rate of change for any function $y = f(x)$ over the interval $x_1 \leq x \leq x_2$ can be determined by

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{change in } y}{\text{change in } x} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}\end{aligned}$$

- Graphically, the average rate of change for any function $y = f(x)$ over the interval $x_1 \leq x \leq x_2$ is equivalent to the slope of the secant line passing through two points (x_1, y_1) and (x_2, y_2).

$$\begin{aligned}\text{Average rate of change} = m_{\text{secant}} &= \frac{\text{change in } y}{\text{change in } x} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1}\end{aligned}$$



Need to Know

- Average rate of change is expressed using the units of the two quantities that are related to each other.
- A positive average rate of change indicates that the quantity represented by the dependent variable is increasing on the specified interval, compared with the quantity represented by the independent variable. Graphically, this is indicated by a secant line that has a positive slope (the secant line rises from left to right).
- A negative average rate of change indicates that the quantity represented by the dependent variable is decreasing on the specified interval, compared with the quantity represented by the independent variable. Graphically, this is indicated by a secant line that has a negative slope (the secant line falls from left to right).
- All linear relationships have a constant rate of change. Average rate of change calculations over different intervals of the independent variable give the same result.
- Nonlinear relationships do not have a constant rate of change. Average rate of change calculations over different intervals of the independent variable give different results.

CHECK Your Understanding

1. Calculate the average rate of change for the function $g(x) = 4x^2 - 5x + 1$ over each interval.

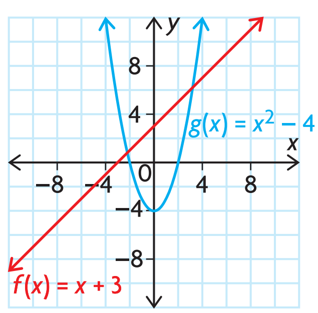
a) $2 \leq x \leq 4$
b) $2 \leq x \leq 3$
c) $2 \leq x \leq 2.5$
d) $2 \leq x \leq 2.25$
e) $2 \leq x \leq 2.1$
f) $2 \leq x \leq 2.01$

2. An emergency flare is shot into the air. Its height, in metres, above the ground at various times in its flight is given in the following table.

Time (s)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Height (m)	2.00	15.75	27.00	35.75	42.00	45.75	47.00	45.75	42.00

a) Determine the average rate of change in the height of the flare during each interval.
i) $1.0 \leq t \leq 2.0$ **ii)** $3.0 \leq t \leq 4.0$

b) Use your results from part a) to explain what is happening to the height of the flare during each interval.



3. Given the functions $f(x)$ and $g(x)$ shown on the graph, discuss how the average rates of change, $\frac{\Delta y}{\Delta x}$, differ in each relationship.

PRACTISING

4. This table shows the growth of a crowd at a rally over a 3 h period.
K

Time (h)	0.0	0.5	1.0	1.5	2.0	2.5	3.0
Number of People	0	176	245	388	402	432	415

a) Determine the average rate of change in the size of the crowd for each half hour of the rally.
b) What do these numbers represent?
c) What do positive and negative rates of change mean in this situation?

Day	**Cumulative Distance (km)**
0	0
1	203
2	396
3	561
4	739.5
5	958
6	1104

5. a) The cumulative distance travelled over several days of the 2007 Tour de France bicycle race is shown in the table to the left. Calculate the average rate of change in cumulative distance travelled between consecutive days.
b) Does the Tour de France race travel over the same distance each day? Explain.

6. What is the average rate of change in the values of the function $f(x) = 4x$ from $x = 2$ to $x = 6$? What about from $x = 2$ to $x = 26$? What do your results indicate about $f(x)$?

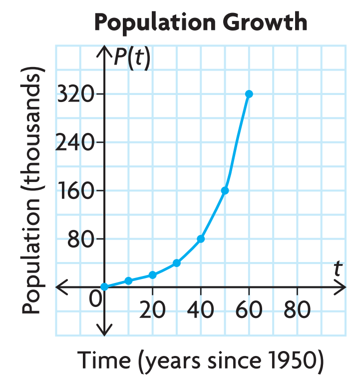
7. Shelly has a cell phone plan that costs \$39 per month and allows her 250 free anytime minutes. Any minutes she uses over the 250 free minutes cost \$0.10 per minute. The function

$$C(m) = \begin{cases} 39, \text{ if } 0 \le m \le 250 \\ 0.10(m - 250) + 39, \text{ if } m > 250 \end{cases}$$

can be used to determine her monthly cell phone bill, where $C(m)$ is her monthly cost in dollars and m is the number of minutes she talks. Discuss how the average rate of change in her monthly cost changes as the minutes she talks increases.

8. **A** The population of a city has continued to grow since 1950. The population P, in thousands, and the time t, in years, since 1950 are given in the table below and in the graph.

Time, *t* (years)	0	10	20	30	40	50	60
Population, *P* (thousands)	5	10	20	40	80	160	320



a) Calculate the average rate of change in population for the following intervals of time.

i) $0 \le t \le 20$
ii) $20 \le t \le 40$
iii) $40 \le t \le 60$
iv) $0 \le t \le 60$

b) Is the population growth constant?

c) To predict what the population will be in 2050, what assumptions must you make?

9. During the Apollo 14 mission, Alan Shepard hit a golf ball on the Moon. The function $h(t) = 18t - 0.8t^2$ models the height of the golf ball's trajectory on the Moon, where $h(t)$ is the height, in metres, of the ball above the surface of the Moon and t is the time in seconds. Determine the average rate of change in the height of the ball over the time interval $10 \le t \le 15$.

10. A company that sells sweatshirts finds that the profit can be modelled by $P(s) = -0.30s^2 + 3.5s + 11.15$, where $P(s)$ is the profit, in thousands of dollars, and s is the number of sweatshirts sold (expressed in thousands).

a) Calculate the average rate of change in profit for the following intervals.

i) $1 \le s \le 2$ **ii)** $2 \le s \le 3$ **iii)** $3 \le s \le 4$ **iv)** $4 \le s \le 5$

b) As the number of sweatshirts sold increases, what do you notice about the average rate of change in profit on each sweatshirt? What does this mean?

c) Predict if the rate of change in profit will stay positive. Explain what this means.

11. The population of a town is modelled by $P(t) = 50t^2 + 1000t + 20\,000$, where $P(t)$ is the size of the population and t is the number of years since 2000.

a) Use graphing technology to graph $P(t)$.

b) Predict if the average rate of change in the population size will be greater closer to the year 2000 or farther in the future. Explain how you made your prediction.

c) Calculate the average rate of change in the population size for each time period.

i) 2000–2010	**iii)** 2005–2015
ii) 2002–2012	**iv)** 2010–2020

d) Evaluate your earlier prediction using the data you developed when answering part c).

12. C Your classmate was absent today and phones you for help with today's lesson. Share with your classmate

a) two real-life examples of when someone might calculate an average rate of change (one positive and one negative)

b) an explanation of when an average rate of change might be useful

c) an explanation of how an average rate of change is calculated

13. T Vehicles lose value over time. A car is purchased for \$23 500, but is worth only \$8750 after eight years. What is the average annual rate of change in the value of the car, as a percent?

14. Complete the following table by providing a definition in your own words, a personal example, and a visual representation of an average rate of change.

AVERAGE RATE OF CHANGE		
Definition in your own words	Personal example	Visual representation

Extending

15. The function $F(x) = -0.005x^2 + 0.8x + 12$ models the relationship between a certain vehicle's speed and fuel economy, where $F(x)$ is the fuel economy in kilometres per litre and x is the speed of the vehicle in kilometres per hour. Determine the rate of change in fuel economy for 10 km/h intervals in speed, and use your results to determine the speed that gives the best fuel economy.