2.2 Estimating Instantaneous Rates of Change from Tables of Values and Equations

GOAL

Estimate and interpret the rate of change at a particular value of the independent variable.

YOU WILL NEED

- graphing calculator or graphing software

INVESTIGATE the Math

A small pebble was dropped into a 3.0 m tall cylindrical tube filled with thick glycerine. A motion detector recorded the time and the total distance that the pebble fell after its release. The table below shows some of the measurements between 6.0 s and 7.0 s after the initial drop.

Time, t (s)	6.0	6.2	6.4	6.6	6.8	7.0
Distance, $d(t)$ (cm)	208.39	221.76	235.41	249.31	263.46	277.84

? How can you estimate the rate of change in the distance that the pebble fell at exactly $t = 6.4$ s?

A. Calculate the average rate of change in the distance that the pebble fell during each of the following time intervals.

i) $6.0 \leq t \leq 6.4$
ii) $6.2 \leq t \leq 6.4$
iii) $6.4 \leq t \leq 7.0$
iv) $6.4 \leq t \leq 6.8$
v) $6.4 \leq t \leq 6.6$

B. Use your results for part A to estimate the **instantaneous rate of change** in the distance that the pebble fell at exactly $t = 6.4$ s. Explain how you determined your estimate.

instantaneous rate of change

the exact rate of change of a function $y = f(x)$ at a specific value of the independent variable $x = a$; estimated using average rates of change for small intervals of the independent variable very close to the value $x = a$

C. Calculate the average rate of change in the distance that the pebble fell during the time interval $6.2 \leq t \leq 6.6$. How does your calculation compare with your estimate?

Reflecting

D. Why do you think each of the intervals you used to calculate the average rate of change in part A included 6.4 as one of its endpoints?

E. Why did it make sense to examine the average rates of change using time intervals on both sides of $t = 6.4$ s? Which of these intervals provided the best estimate for the instantaneous rate of change at $t = 6.4$ s?

F. Even though 6.4 is not an endpoint of the interval used in the average rate of change calculation in part C, explain why this calculation gave a reasonable estimate for the instantaneous rate of change at $t = 6.4$ s.

G. Using the table of values given, is it possible to get as accurate an estimate of the instantaneous rate of change for $t = 7.0$ s as you did for $t = 6.4$ s? Explain.

APPLY the Math

EXAMPLE 1 Selecting a strategy to estimate instantaneous rate of change using an equation

The population of a small town appears to be growing exponentially. Town planners think that the equation $P(t) = 35\,000\,(1.05)^t$, where $P(t)$ is the number of people in the town and t is the number of years after 2000, models the size of the population. Estimate the instantaneous rate of change in the population in 2015.

Solution A: Selecting a strategy using intervals

preceding interval
an interval of the independent variable of the form $a - h \leq x \leq a$, where h is a small positive value; used to determine an average rate of change

Using a **preceding interval** in which $14 \leq t \leq 15$,

$$\frac{\Delta P}{\Delta t} = \frac{P(15) - P(14)}{15 - 14}$$
$$\doteq \frac{72\,762 - 69\,298}{15 - 14}$$
$$= \frac{3464}{1}$$
$$= 3464 \text{ people/year}$$

Calculate average rates of change using some dates that precede the year 2015. Since 2015 is 15 years after 2000, use $t = 15$ to represent the year 2015.

Use $14 \leq t \leq 15$ and $14.5 \leq t \leq 15$ as preceding intervals (intervals on the left side of 15) to calculate the average rates of change in the population.

Using a preceding interval in which $14.5 \leq t \leq 15$,

$$\frac{\Delta P}{\Delta t} = \frac{P(15) - P(14.5)}{15 - 14.5}$$
$$\doteq \frac{72\,762 - 71\,009}{15 - 14.5}$$
$$= 3506 \text{ people/year}$$

following interval
an interval of the independent variable of the form $a \leq x \leq a + h$, where h is a small positive value; used to determine an average rate of change

Using a **following interval** in which $15 \leq t \leq 16$,

$$\frac{\Delta P}{\Delta t} = \frac{P(16) - P(15)}{16 - 15}$$
$$\doteq \frac{76\,401 - 72\,762}{16 - 15}$$
$$= 3639 \text{ people/year}$$

Calculate average rates of change using some dates that follow the year 2015. Use $15 \leq t \leq 16$ and $15 \leq t \leq 15.5$ as following intervals (intervals on the right side of 15) to calculate the average rates of change in the population.

Using a following interval in which $15 \leq t \leq 15.5$,

$$\frac{\Delta P}{\Delta t} = \frac{P(15.5) - P(15)}{15.5 - 15}$$
$$\doteq \frac{74\,559 - 72\,762}{15.5 - 15}$$
$$= 3594 \text{ people/year}$$

As the size of the preceding interval decreases, the average rate of change increases.

As the size of the following interval decreases, the average rate of change decreases.

Examine the average rates of change in population on both sides of $t = 15$ to find a trend.

The instantaneous rate of change in the population is somewhere between the values above.

$$\text{Estimate} = \frac{3506 + 3594}{2}$$
$$= 3550 \text{ people/year}$$

Make an estimate using the average of the two calculations for smaller intervals on either side of $t = 15$.

Solution B: Selecting a different interval strategy

Calculate some average rates of change using intervals that have the year 2015 as their midpoint.

Using a centred interval in which $14 \leq t \leq 16$,

$$\frac{\Delta P}{\Delta t} = \frac{P(16) - P(14)}{16 - 14}$$
$$\doteq \frac{76\,401 - 69\,298}{16 - 14}$$
$$\doteq 3552 \text{ people/year}$$

Using a centred interval in which $14.5 \leq t \leq 15.5$,

$$\frac{\Delta P}{\Delta t} = \frac{P(15.5) - P(14.5)}{15.5 - 14.5}$$
$$\doteq \frac{74\,559 - 71\,009}{15.5 - 14.5}$$
$$= 3550 \text{ people/year}$$

Use $14 \leq t \leq 16$ and $14.5 \leq t \leq 15.5$ as centred intervals (intervals with 15 as their midpoint) to calculate the average rates of change in the population. Examine the corresponding rates of change to find a trend. Using centred intervals allows you to move in gradually to the value that you are interested in. Sometimes this is called the *squeeze technique*.

The instantaneous rate of change in the population is about 3550 people/year.

The average rates of change are very similar. Make an estimate using the smallest centred interval.

centred interval

an interval of the independent variable of the form $a - h \leq x \leq a + h$, where h is a small positive value; used to determine an average rate of change

EXAMPLE 2 Selecting a strategy to estimate the instantaneous rate of change

The volume of a cubic crystal, grown in a laboratory, can be modelled by $V(x) = x^3$, where $V(x)$ is the volume measured in cubic centimetres and x is the side length in centimetres. Estimate the instantaneous rate of change in the crystal's volume with respect to its side length when the side length is 5 cm.

Solution A: Squeezing the centred intervals

Look at the average rates of change near $x = 5$ using a series of centred intervals that get progressively smaller. By using intervals that get systematically smaller and smaller, you can make a more accurate estimate for the instantaneous rate of change than if you were to use intervals that are all the same size.

Using $4.5 \le x \le 5.5$,

$$\frac{\Delta V}{\Delta x} = \frac{166.375 - 91.125}{5.5 - 4.5}$$
$$= 75.25\ \text{cm}^3/\text{cm}$$

Using $4.9 \le x \le 5.1$,

In this case, use $4.5 \le x \le 5.5$, then $4.9 \le x \le 5.1$, and finally $4.99 \le x \le 5.01$.

$$\frac{\Delta V}{\Delta x} = \frac{132.651 - 117.649}{5.1 - 4.9}$$
$$= 75.01\ \text{cm}^3/\text{cm}$$

Using $4.99 \le x \le 5.01$,

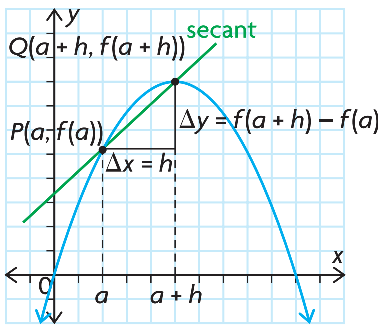
$$\frac{\Delta V}{\Delta x} = \frac{125.751\,501 - 124.251\,499}{5.01 - 4.99}$$
$$= 75.0001\ \text{cm}^3/\text{cm}$$

As the interval gets smaller, the average rate of change in the volume of the cube appears to be getting closer to 75 cm^3/cm. So it seems that the instantaneous rate of change in volume should be 75 cm^3/cm.

When the side length of the cube is exactly 5 cm, the volume of the cube is increasing at the rate of 75 cm^3/cm.

difference quotient

if $P(a, f(a))$ and $Q(a + h, f(a + h))$ are two points on the graph of $y = f(x)$, then the instantaneous rate of change of y with respect to x at P can be estimated using $\frac{\Delta y}{\Delta x} = \frac{f(a + h) - f(a)}{h}$, where h is a very small number. This expression is called the difference quotient.



Solution B: Using an algebraic approach and a general point

Write the difference quotient for the average rate of change in volume as the side length changes between 5 and any value: $(5 + h)$.

Use two points. Let one point be $(5, 5^3)$ or $(5, 125)$ because you are investigating the rate of change for $V(x) = x^3$ when $x = 5$. Let the other point be $(5 + h, (5 + h)^3)$, where h is a very small number, such as 0.01 or -0.01.

$$\frac{\Delta V}{\Delta x} = \frac{(5 + h)^3 - 125}{5 + h - 5}$$
$$= \frac{(5 + h)^3 - 125}{h}$$

Let $h = -0.01$.

$$\frac{\Delta V}{\Delta x} = \frac{(5 + (-0.01))^3 - 125}{h}$$

$$= \frac{124.251\,499 - 125}{-0.01}$$

$$= 74.8501 \text{ cm}^3/\text{cm}$$

The value $h = -0.01$ corresponds to a very small preceding interval, where $4.99 \le x \le 5$. This gives an estimate of the instantaneous rate of change when the side length changes from 4.99 cm to 5 cm.

Let $h = 0.01$.

$$\frac{\Delta V}{\Delta x} = \frac{(5 + 0.01)^3 - 125}{0.01}$$

$$= \frac{125.751\,501 - 125}{0.01}$$

$$= 75.1501 \text{ cm}^3/\text{cm}$$

The value $h = 0.01$ corresponds to a very small following interval, where $5 \le x \le 5.01$. This gives an estimate of the instantaneous rate of change when the side length changes from 5 cm to 5.01 cm.

The instantaneous rate of change in the volume of the cube is somewhere between the two values calculated.

$$\text{Estimate} = \frac{74.8501 + 75.1501}{2}$$

$$\doteq 75.0001 \text{ cm}^3/\text{cm}$$

Determine an estimate using the average of the two calculations on either side of $x = 5$.

EXAMPLE 3 Selecting a strategy to estimate an instantaneous rate of change

The following table shows the temperature of an oven as it heats from room temperature to 400°F.

Time (min)	0	1	2	3	4	5	6	7	8	9	10
Temperature (°F)	70	125	170	210	250	280	310	335	360	380	400

a) Estimate the instantaneous rate of change in temperature at exactly 5 min using the given data.

b) Estimate the instantaneous rate of change in temperature at exactly 5 min using a quadratic model.

Solution

a) Using the interval $2 \le t \le 8$,

$$\frac{\Delta T}{\Delta t} = \frac{360 - 170}{8 - 2}$$

$$\doteq 31.67\text{°F/min}$$

Choose some centred intervals around 5 min. Examine the average rates of change as the intervals of time get smaller, and find a trend.

Tech Support

For help using a graphing calculator to create scatter plots and determine an algebraic model using quadratic regression, see Technical Appendix, T-11.

Using the interval $3 \leq t \leq 7$,

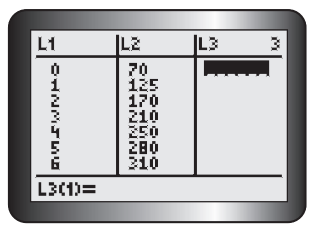
$$\frac{\Delta T}{\Delta t} = \frac{335 - 210}{7 - 3}$$
$$= 31.25°\text{F/min}$$

Using the interval $4 \leq t \leq 6$,

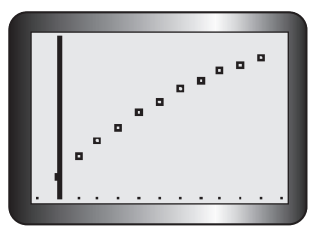
$$\frac{\Delta T}{\Delta t} = \frac{310 - 250}{6 - 4}$$
$$= 30°\text{F/min}$$

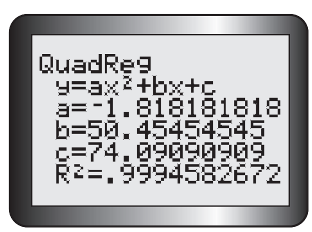
As the centred intervals around 5 min get smaller, it appears that the average rates of change in the temperature of the oven get closer to about 30°F/min.

b)



Enter the data into the lists of the graphing calculator, and create a scatter plot.





Using quadratic regression, determine an equation to represent the data. Rounding to two decimal places, the model is $f(x) = -1.82x^2 + 50.45x + 74.09$, where $f(x)$ is oven temperature and x is time.

Next, calculate the average rate of change in oven temperature using a very small centred interval near $x = 5$. For example, use $4.99 \leq x \leq 5.01$.

Interval	$\Delta f(x)$	Δx	$\frac{\Delta f(x)}{\Delta x}$
$4.99 \leq x \leq 5.01$	$f(5.01) - f(4.99)$ $\doteq 281.16 - 280.52$ $= 0.64$	$5.01 - 4.99$ $= 0.02$	$0.64/0.02$ $= 32°\text{F/min}$

The instantaneous rate of change in temperature at 5 min is about 32°F/min.

In Summary

Key Idea

- The instantaneous rate of change of the dependent variable is the rate at which the dependent variable changes at a specific value of the independent variable, $x = a$.

Need to Know

- The instantaneous rate of change of the dependent variable, in a table of values or an equation of the relationship, can be estimated using the following methods:
  - Using a series of preceding $(a - h \leq x \leq a)$ and following $(a \leq x \leq a + h)$ intervals: Calculate the average rate of change by keeping one endpoint of each interval fixed. (This is $x = a$, the location where the instantaneous rate of change occurs.) Move the other endpoint of the interval closer and closer to the fixed point from either side by making h smaller and smaller. Based on the trend for the average rates of change, make an estimate for the instantaneous rate of change at the specific value.
  - Using a series of centred intervals $(a - h \leq x \leq a + h)$: Calculate the average rate of change by picking endpoints for each interval on either side of $x = a$, where the instantaneous rate of change occurs. Choose these endpoints so that the value where the instantaneous rate of change occurs is the midpoint of the interval. Continue to calculate the average rate of change by moving both endpoints closer and closer to where the instantaneous rate of change occurs. Based on the trend, make an estimate for the instantaneous rate of change.
  - Using the difference quotient and a general point: Calculate the average rate of change using the location where the instantaneous rate of change occurs $(a, f(a))$ and a general point $(a + h, f(a + h))$, i.e., $\frac{f(a + h) - f(a)}{h}$. Choose values for h that are very small, such as ± 0.01 or ± 0.001. The smaller the value used for h, the better the estimate will be.
- The best estimate for the instantaneous rate of change occurs when the interval used to calculate the average rate of change is made as small as possible.

CHECK Your Understanding

1. **a)** Copy and complete the tables, if $f(x) = 5x^2 - 7$.

Preceding Interval	$\Delta f(x)$	Δx	Average Rate of Change, $\frac{\Delta f(x)}{\Delta x}$
$1 \leq x \leq 2$	$13 - (-2) = 15$	$2 - 1 = 1$	
$1.5 \leq x \leq 2$	8.75	0.5	
$1.9 \leq x \leq 2$			
$1.99 \leq x \leq 2$			

Following Interval	$\Delta f(x)$	Δx	Average Rate of Change, $\frac{\Delta f(x)}{\Delta x}$
$2 \le x \le 3$	$38 - 13 = 25$	$3 - 2 = 1$	
$2 \le x \le 2.5$	11.25	0.5	
$2 \le x \le 2.1$			
$2 \le x \le 2.01$			

b) Based on the trend in the average rates of change, estimate the instantaneous rate of change when $x = 2$.

2. A soccer ball is kicked into the air. The following table of values shows the height of the ball above the ground at various times during its flight.

Time (s)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Height (m)	0.5	11.78	20.6	26.98	30.9	32.38	31.4	27.98	22.1	13.78	3.0

a) Estimate the instantaneous rate of change in the height of the ball at exactly $t = 2.0$ s using the preceding and following interval method.

b) Estimate the instantaneous rate of change in the height of the ball at exactly $t = 2.0$ s using the centred interval method.

c) Which estimation method do you prefer? Explain.

3. A population of raccoons moves into a wooded area. At t months, the number of raccoons, $P(t)$, can be modelled using the equation $P(t) = 100 + 30t + 4t^2$.

a) Determine the population of raccoons at 2.5 months.

b) Determine the average rate of change in the raccoon population over the interval from 0 months to 2.5 months.

c) Estimate the rate of change in the raccoon population at exactly 2.5 months.

d) Explain why your answers for parts a), b), and c) are different.

PRACTISING

4. For the function $f(x) = 6x^2 - 4$, estimate the instantaneous rate of change for the given values of x.

a) $x = -2$ **b)** $x = 0$ **c)** $x = 4$ **d)** $x = 8$

5. An object is sent through the air. Its height is modelled by the function $h(x) = -5x^2 + 3x + 65$, where $h(x)$ is the height of the object in metres and x is the time in seconds. Estimate the instantaneous rate of change in the object's height at 3 s.

6. A family purchased a home for \$125 000. Appreciation of the home's value, in dollars, can be modelled by the equation $H(t) = 125\ 000(1.06)^t$, where $H(t)$ is the value of the home and t is the number of years that the family owns the home. Estimate the instantaneous rate of change in the home's value at the start of the eighth year of owning the home.

7. **K** The population of a town, in thousands, is described by the function $P(t) = -1.5t^2 + 36t + 6$, where t is the number of years after 2000.

a) What is the average rate of change in the population between the years 2000 and 2024?

b) Does your answer to part a) make sense? Does it mean that there was no change in the population from 2000 to 2024?

c) Explain your answer to part b) by finding the average rate of change in the population from 2000 to 2012 and from 2012 to 2024.

d) For what value of t is the instantaneous rate of change in the population 0?

8. Jacelyn purchased a new car for \$18 999. The yearly depreciation of the value of the car can be modelled by the equation $V(t) = 18\ 999(0.93)^t$, where $V(t)$ is the value of the car and t is the number of years that Jacelyn owns the car. Estimate the instantaneous rate of change in the value of the car when the car is 5 years old. What does this mean?

9. **A** A diver is on the 10 m platform, preparing to perform a dive. The diver's height above the water, in metres, at time t can be modelled using the equation $h(t) = 10 + 2t - 4.9t^2$.

a) Determine when the diver will enter the water.

b) Estimate the rate at which the diver's height above the water is changing as the diver enters the water.

10. **T** To make a snow person, snow is being rolled into the shape of a sphere. The volume of a sphere is given by the function $V(r) = \frac{4}{3}\pi r^3$, where r is the radius in centimetres. Use two different methods to estimate the instantaneous rate of change in the volume of the snowball with respect to the radius when $r = 5$ cm.

11. David plans to drive to see his grandparents during his winter break. How can he determine his average speed for a part of his journey along the way? Be as specific as possible. Describe the steps he must take and the information he must know.

12. The following table shows the temperature of an oven as it cools.

Time (min)	0	1	2	3	4	5	6
Temperature (°F)	400	390	375	350	330	305	270

a) Use the data in the table to estimate the instantaneous rate of change in the temperature of the oven at exactly 4 min.

b) Use a graphing calculator to determine a quadratic model. Use your quadratic model to estimate the instantaneous rate of change in the temperature of the oven at exactly 4 min.

c) Discuss why your answers for parts a) and b) are different.

d) Which is the better estimate? Explain.

13. C In a table like the one below, list all the methods that can be used to estimate the instantaneous rate of change. What are the advantages and disadvantages of each method?

Method of Estimating Instantaneous Rate of Change	**Advantage**	**Disadvantage**

Extending

14. Concentric circles form when a stone is dropped into a pool of water.

a) What is the average rate of change in the area of one circle with respect to the radius as the radius grows from 0 cm to 100 cm?

b) How fast is the area changing with respect to the radius when the radius is 120 cm?

15. A crystal in the shape of a cube is growing in a laboratory. Estimate the rate at which the surface area is changing with respect to the side length when the side length of the crystal is 3 cm.

16. A spherical balloon is being inflated. Estimate the rate at which its surface area is changing with respect to the radius when the radius measures 20 cm.