2.3 Exploring Instantaneous Rates of Change Using Graphs

GOAL

Estimate instantaneous rates of change using slopes of lines.

YOU WILL NEED

- graphing calculator or graphing software

EXPLORE the Math

In the previous lesson, you used numerical and algebraic techniques to estimate instantaneous rates of change. Graphically, you have seen that the average rate of change is equivalent to the slope of a secant line that passes through two points on the graph of a function.

? How can you use the slopes of secant lines to estimate the instantaneous rate of change?

A. Enter the function $f(x) = x^2$ into the equation editor of your graphing calculator, graph it, and draw a sketch of the graph.

B. On your sketch, draw a secant line that passes through the points $(1, f(1))$ and $(3, f(3))$.

C. Calculate the slope of the secant line. Copy the table and record the slope. Calculate and record the slopes of other secant lines using the points listed.

Points	Slope of Secant
$(1, f(1))$ and $(3, f(3))$	
$(1, f(1))$ and $(2, f(2))$	
$(1, f(1))$ and $(1.5, f(1.5))$	
$(1, f(1))$ and $(1.1, f(1.1))$	
$(1, f(1))$ and $(1.01, f(1.01))$	

D. Create a formula for calculating the slope of any secant line that passes through $(1, f(1))$ and the general point $(x, f(x))$.

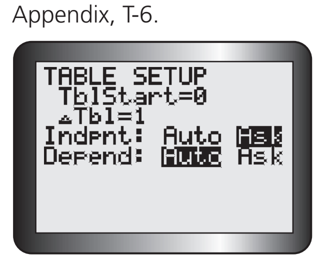
E. Enter this formula into Y1 of the equation editor.

F. Set the TBLSET feature of your graphing calculator by scrolling down and across so that the cursor is over Ask in the Indpnt: row of the screen as shown.

G. Confirm that the first slope you calculated in part C, for the secant line that passes through the points $(1, f(1))$ and $(3, f(3))$, is correct by entering X = 3 in the TABLE on your graphing calculator. (If there are already x-values in the table, delete them by moving the cursor over each value and pressing DEL.)

Tech Support

For help using the TBLSET and TABLE features of a graphing calculator, see Technical Appendix, T-6.



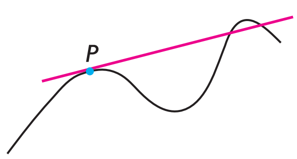
H. On your sketch, draw another secant line that passes through the points $(1, f(1))$ and $(2, f(2))$. Calculate its slope by entering $X = 2$ in the TABLE on your graphing calculator, and compare this to the slope in the table you created in part F.

I. Draw and calculate three other secant lines, always using $(1, f(1))$ as a fixed point and moving the other points closer to $(1, f(1))$ each time. You can do this by using the points given in the table in part C.

J. Examine your sketch and your table of secant slopes. Describe what happens to each secant line in your sketch, and compare this with the values of the slopes in your table as the points get closer and closer to the fixed point $(1, f(1))$.

tangent line

a line that touches the graph at only one point, *P*, within a small interval of a relation; it could, but does not have to, cross the graph at another point outside this interval. The tangent line goes in the same direction as the relation at point *P* (called the point of tangency).



K. Estimate the slope of the tangent line to the curve $f(x) = x^2$ at the point $(1, f(1))$ by examining the trend in the secant slopes you calculated.

L. Repeat parts B to K using the points in the table below.

Points	Slope of Secant
$(1, f(1))$ and $(-1, f(-1))$	
$(1, f(1))$ and $(0, f(0))$	
$(1, f(1))$ and $(0.5, f(0.5))$	
$(1, f(1))$ and $(0.9, f(0.9))$	
$(1, f(1))$ and $(0.99, f(0.99))$	

Tech Support

For help using the graphing calculator to draw tangent lines, see Technical Appendix, T-17.

M. Verify your estimates by drawing the tangent line to the graph of $f(x) = x^2$ at $x = 1$ using your graphing calculator.

N. Repeat parts B to L with two other functions of your choice. Use two different types of functions, such as an exponential function, a **sinusoidal function**, or a different quadratic function.

Reflecting

O. What happens to the slopes of the secant lines as the points move closer to the fixed point?

P. How do the slopes of the secant lines relate to the slope of the tangent line when $x = 1$? Explain.

Q. How is estimating the slope of a tangent like estimating the instantaneous rate of change?

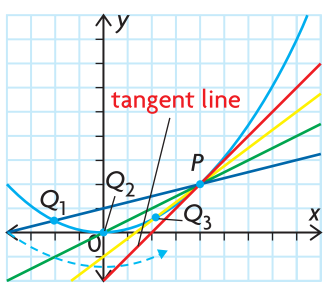
In Summary

Key Ideas

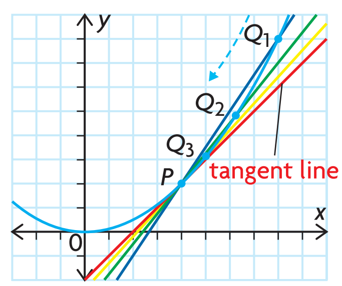
- The slope of a secant line is equivalent to the average rate of change over the interval defined by the *x*-coordinates of the two points that are used to define the secant line.
- The slope of a tangent at a point on a graph is equivalent to the instantaneous rate of change of a function at this point.

Need to Know

- The slope of a tangent cannot be calculated directly using the slope formula, because the coordinates of only one point are known. The slope can be estimated, however, by calculating the slopes of a series of secant lines that go through the fixed point of tangency *P* and points that get closer and closer to this fixed point, Q_1, Q_2, and Q_3.



As *Q* approaches *P* from the left, the slope of *QP* increases and approaches the slope of the tangent line.



As *Q* approaches *P* from the right, the slope of *QP* decreases and approaches the slope of the tangent line.

FURTHER Your Understanding

1. Graph each of the following functions using a graphing calculator, and then sketch the graph. On your sketch, draw a series of secant lines that you could use to estimate the slope of the tangent when $x = 2$. Calculate and record the slopes of these secant lines. Use the slopes to estimate the slope of the tangent line when $x = 2$.

a) $f(x) = 3x^2 - 5x + 1$

b) $f(x) = 3^x + 1$

c) $f(x) = \sqrt{x + 2}$

d) $f(x) = 2x - 7$

2. Verify your estimates for each function in question 1 by drawing the tangent line when $x = 2$ on your graphing calculator.

3. a) For each of the following sets of functions, estimate the slopes of the tangents at the given values of x.
   b) What do all the slopes in each set of functions have in common?

Set A
$f(x) = -x^2 + 6x - 4$ when $x = 3$
$g(x) = \sin x$ when $x = 90°$
$h(x) = x^2 + 4x + 11$ when $x = -2$
$j(x) = 5$ when $x = 1$

Set B
$f(x) = 3x^2 + 2x - 1$ when $x = 2$
$g(x) = 2^x + 3$ when $x = 1$
$h(x) = 5x + 4$ when $x = 3$
$j(x) = \sin x$ when $x = 60°$

Set C
$f(x) = 3x^2 + 2x - 1$ when $x = -1$
$g(x) = -2^x + 3$ when $x = 0$
$h(x) = -3x + 5$ when $x = 2$
$j(x) = \sin x$ when $x = 120°$

4. The following table gives the temperature of an oven as it heats up.

Time (min)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Temperature (°F)	70	125	170	210	250	280	310	335	360	380	400	415	430	440	445

   a) Graph the data.
   b) Draw a **curve of best fit** and the tangent line at $x = 5$.
   c) Determine the slope of the tangent line using the y-intercept of the tangent line and the point of tangency $(5, 280)$.
   d) Estimate the instantaneous rate of change in temperature at exactly 5 min using a centred interval from the table of values.
   e) Compare your answers to parts c) and d).

5. In the first two sections of this chapter, you calculated the slopes of successive secant lines to estimate the slope of a tangent line, and you calculated the average rate of change to estimate the instantaneous rate of change. How are these two calculations similar and different?

6. a) On graph paper, sketch the graph of $f(x) = x^2$.
   b) Draw the secant line that passes through $(1, 2)$ and $(2, 4)$.
   c) Estimate the location of the point of tangency on the graph of $f(x)$ whose tangent line has the same slope as the secant line you drew in part b).