2.4 Using Rates of Change to Create a Graphical Model

YOU WILL NEED

- graphing calculator or graphing software

GOAL

Represent verbal descriptions of rates of change using graphs.

LEARN ABOUT the Math

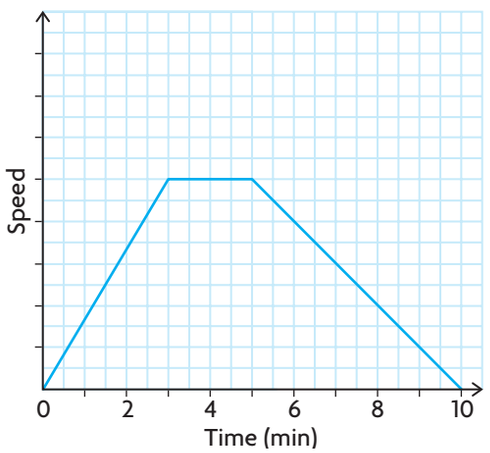
Today Steve walked to his part-time job. As he started walking, he sped up for 3 min. Then he walked at a constant pace for another 2 min. When he realized that he would be early for work, he slowed down. His walk ended and he came to a complete stop once he reached his destination 10 min after he started.

? What would the speed versus time graph of Steve's walk to work look like?

EXAMPLE 1 Representing the situation with a graph

Create a speed versus time graph for Steve's walk to work.

Solution A: Assuming that he changed speed at a constant rate

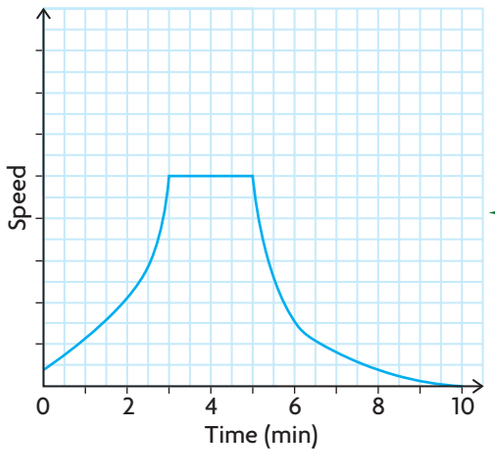


Because Steve was speeding up, his speed increased as time increased. His speed increased at a constant rate, so the graph should be a straight line with a positive slope that begins at (0, 0) and ends at $x = 3$.

Between 3 min and 5 min, Steve walked at the same rate, so his speed did not change. The graph should be a horizontal line that connects to the first line.

After 5 min, Steve slowed down at a constant rate, decreasing his speed as time increased, so you might draw a straight line with a negative slope that begins at $x = 5$ and ends at $x = 10$.

Solution B: Assuming that he walked at a variable speed



Because Steve was speeding up, his speed increased as time increased. His speed increased at a variable rate, so you might draw an increasing curve that starts at (0, 0) and ends at $x = 3$.

Between 3 min and 5 min, Steve walked at the same rate, so his speed did not change. The graph should be a horizontal line that connects to the first line.

After 5 min, Steve slowed down at a variable rate, decreasing his speed as time increased, so you might draw a decreasing curve that begins at $x = 5$ and ends at $x = 10$.

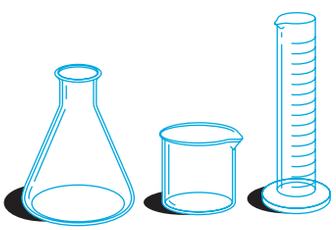
Reflecting

A. Which details in the given description were most important for determining the shape of the graph?

B. Are these the only two graphs that could represent Steve's walk to work? Explain.

APPLY the Math

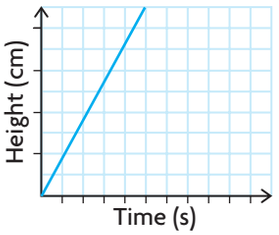
EXAMPLE 2 Representing the situation with a graph



A flask, a beaker, and a graduated cylinder are being filled with water. The rate at which the water flows from the tap is the same when filling all three containers. Draw possible water level versus time graphs for the three containers.

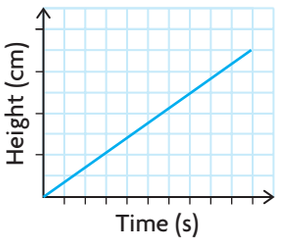
Solution

Graduated Cylinder



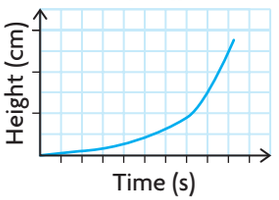
Since the containers are being filled with water, the height of the water in the containers increases as time increases. All the graphs should be increasing curves. Both the graduated cylinder and the beaker have a constant diameter so the water level increases at a constant rate. The water level will rise the fastest in the container with the smallest diameter.

Beaker



The water level in the graduated cylinder increases faster than the water level in the beaker, so the slope of the line for the graduated cylinder must be greater than the slope of the line for the beaker.

Flask



The diameter of the flask varies, so the water level will increase at different rates. As the water level rises, the diameter of each cross-section gets smaller, causing the water level to increase more rapidly. So the graph must be nonlinear.

EXAMPLE 3 Using a graph to determine the rate of change

A cyclist is observed moving at a speed of 10 m/s. She begins to slow down at a constant rate and, 4 s later, is at a speed of 5 m/s. She continues to slow down at a different constant rate and finally comes to a stop 6 s later.

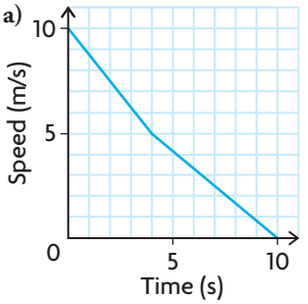
a) Sketch a graph of speed versus time.

b) What is the average rate of change of the cyclist's speed in the first 4 s?

c) Estimate the instantaneous rate of change in speed at 3 s.

Solution

a)



The cyclist begins at 10 m/s and slows down at a constant rate. Place a point at (0, 10) and another at (4, 5), and connect the points with a straight line.

Because the cyclist stops at 10 s, her speed is 0 m/s. Place another point at (10, 0), and connect it to the previous point with a straight line.

b) Average rate of change $= \dfrac{5 - 10}{4 - 0}$

$= -1.25$

The cyclist's speed is decreasing at the rate of 1.25 m/s^2.

To determine the average rate of change of the cyclist's speed in the first 4 s, calculate the slope of the secant line between (0, 10) and (4, 5).

c) The equation of the line for the first part of the graph is $y = -\frac{5}{4}x + 10$, since the slope is $-\frac{5}{4}$ and the y-intercept is 10.

When $x = 3.1$, $y = 6.125$.

When $x = 2.9$, $y = 6.375$.

Substitute $x = 3.1$ and $x = 2.9$ into the equation of the line. These values are close to $x = 3$, but on opposite sides of it. Determine the corresponding y-values.

Average rate of change $= \dfrac{\Delta y}{\Delta x}$

$= \dfrac{6.125 - 6.375}{3.1 - 2.9}$

$= -1.25$ m/s^2

Calculate the average rate of change between these points, and estimate the instantaneous rate of change based on your answer.

The instantaneous rate of change is the same as the average rate of change calculated in part b). This should not be surprising, since the tangent line at $x = 3$ has the same slope as the secant line on the interval $0 \leq x \leq 4$.

Recall that the rate of change of a linear function is constant, so the rate of change at 3 s will be same as the rate of change at any time between 0 s and 4 s.

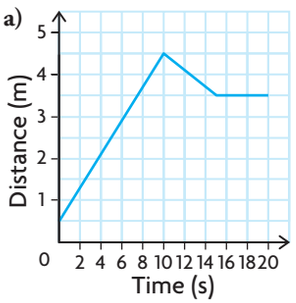
EXAMPLE 4 Using reasoning to represent and analyze a situation

Adam and his friend are testing a motion sensor. Adam stands 0.5 m in front of the sensor and then walks 4 m away from it at a constant rate for 10 s. Next, Adam walks 1 m toward the sensor for 5 s and then waits there for another 5 s.

a) Draw a distance versus time graph for Adam's motion sensor walk.
b) What is the average rate of change in his distance in the first 10 s?
c) What are the instantaneous rates of change at $t = 1$ s and $t = 7$ s?
d) What is the average rate of change in the next 5 s?
e) What are the instantaneous rates of change at $t = 12$ s and $t = 14$ s?
f) What is the instantaneous rate of change at $t = 18$ s?
g) Draw a speed versus time graph for Adam's motion sensor walk.

Solution

a)



The graph begins with a straight line since the rate at which Adam walks is constant. The graph has a positive slope since he walks away from the sensor, and his distance from the sensor increases as time increases.

Adam starts 0.5 m from the sensor. Use (0, 0.5) as the distance intercept.

Adam walks 4 m away from the sensor at a constant rate for 10 s, so use the point (10, 4.5).

Adam then walks toward the sensor. The line has a negative slope, because his distance from the sensor decreases as time increases. The line is not very steep because he is walking slowly.

The graph ends with a horizontal line that has a slope of 0 because Adam is not moving. The slope indicates that his distance from the sensor does not change.

b) Average rate of change
$= \text{slope of secant}$

$$= \frac{4.5 - 0.5}{10 - 0}$$

$$= 0.4$$

Adam's distance from the sensor is increasing, on average, by 0.4 m/s.

Calculate the slope of the secant line between the points (0, 0.5) and (10, 4.5).

c) Instantaneous rate of change
$= \text{slope of tangent}$
$= 0.4$
Adam's distance from the sensor is increasing. He is moving away from the sensor at a rate of 0.4 m/s.

Estimate the slopes of the tangent lines at 1 s and 7 s. Both of the tangent lines have the same slope as the secant line since the graph is linear on the interval $0 \leq t \leq 10$.

d) Average rate of change
$= \text{slope of secant}$
$= \dfrac{3.5 - 4.5}{15 - 10}$
$= -0.2$
Adam's distance from the sensor is decreasing. He is moving toward the sensor at a rate of 0.2 m/s.

Calculate the slope of the secant line between the points (10, 4.5) and (15, 3.5).

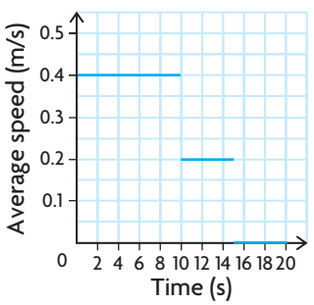
e) Instantaneous rate of change
$= \text{slope of tangent}$
$= -0.2$
Adam's distance from the sensor is decreasing by 0.2 m/s at 12 s and 14 s.

Estimate the slope of the tangent lines at 12 s and 14 s. As in part c), both of these tangent lines have the same slope as the secant line since the graph is linear on the interval $10 < t \leq 15$.

f) Instantaneous rate of change
$= \text{slope of tangent}$
$= 0$
Adam's distance from the sensor is not changing at 18 s.

Estimate the slope of the tangent line at 18 s. Again, the tangent line has the same slope as the secant line since the graph is linear on the interval $15 < t \leq 20$. Since the line is horizontal, its slope is 0.

g)



There are three different speeds at which Adam walks, over three different intervals of time. Using the previous calculations,
Speed $= 0.4$ m/s when $0 \leq t \leq 10$
Speed $= 0.2$ m/s when $10 < t \leq 15$
Speed $= 0$ m/s when $15 < t \leq 20$
Note that speed is a non-negative quantity.
Speed $= \left|\dfrac{\Delta d}{\Delta t}\right|$

In Summary

Key Ideas

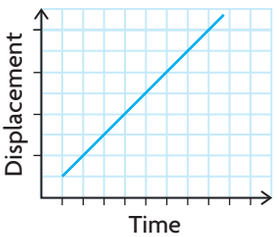
- In a problem that involves movement, a possible graph shows **displacement** (distance, height, or depth) versus time. Distance, height, or depth is the dependent variable, and time is the independent variable. The rate of change in these relationships is speed:

$$\text{Speed} = \left|\frac{\text{change in displacement}}{\text{change in time}}\right|, \quad S = \left|\frac{\Delta d}{\Delta t}\right|$$

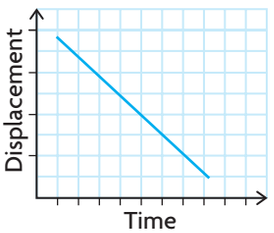
- On a displacement (distance, height, or depth) versus time graph, the magnitude of the slope of a secant line represents the average speed on the corresponding interval. The magnitude of the slope of a tangent line represents the instantaneous speed at a specific point. As a result, observing how the slopes of tangent lines change at different points on a graph gives you insight into how the speed changes over time.

Need to Know

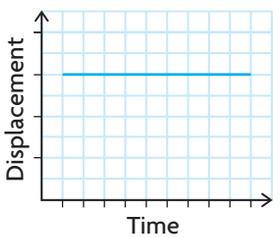
- When the rate of change of displacement (or speed) is constant:



An increasing line indicates that displacement increases as time increases.



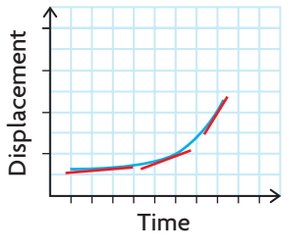
A decreasing line indicates that displacement decreases as time increases.



A horizontal line indicates that there is no change in displacement as time increases.

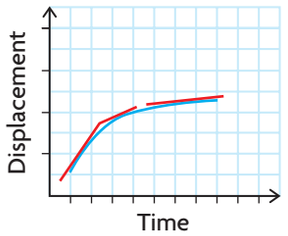
(*continued*)

- When the rate of change of displacement (or speed) is variable, an increasing curve indicates that displacement increases as time increases.



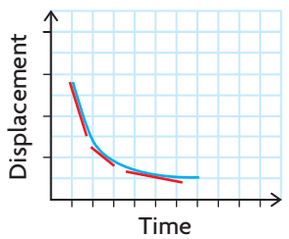
The speed is increasing as the time increases.

or



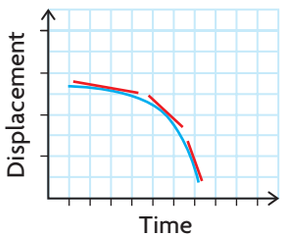
The speed is decreasing as the time increases.

- A decreasing curve indicates that displacement decreases as time increases.



The speed is decreasing as the time increases.

or

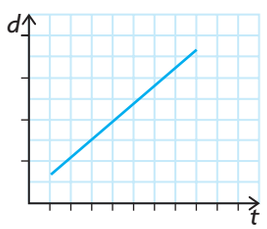


The speed is increasing as the time increases.

CHECK Your Understanding

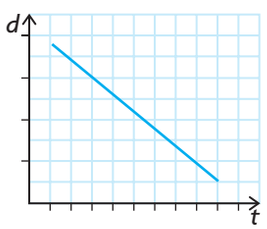
1. The following graphs show distance versus time. Match each graph with the description given below.

a)



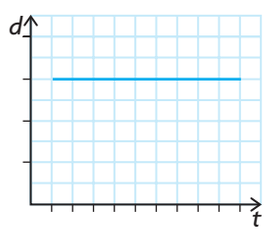
A Distance is decreasing over time.

b)



B There is no change in distance over time.

c)



C Distance is increasing over time.

2. Which of the graphs in question 1 show that the speed is constant? Explain.

3. Jan stands 5 m away from a motion sensor and then walks 4 m toward it at a constant rate for 5 s. Then she walks 2 m away from the location where she changed direction at a variable rate for the next 3 s. She stops and waits at this location for 2 s. Draw a distance versus time graph to show Jan's motion sensor walk.

PRACTISING

4. Rachel climbed Mt. Fuji while in Japan. There are 10 levels to the mountain. She was able to drive to Level 5, where she began her climb.

- She walked at a constant rate for 40 min from Level 5 to Level 6.
- She slowed slightly but then continued at a constant rate for a total of 90 min from Level 6 to Level 7.
- She decided to eat and rest there, which took approximately 2 h.
- From Level 7 to Level 8, a 40 min trip, she travelled at a constant rate.
- Continuing on to Level 9, a 45 min trip, she decreased slightly to a new constant rate.
- During most of the 1 h she took to reach Level 10, the top of Mt. Fuji, she maintained a constant rate. As she neared the top, however, she accelerated.

a) Using the information given and the following table, draw an elevation versus time graph to describe Rachel's climb.

Level	5	6	7	8	9	10
Elevation (m)	2100	2400	2700	3100	3400	3740

b) Calculate Rachel's average speed over each part of her climb.

c) Draw a speed versus time graph to describe Rachel's climb.

5. K The containers shown are being filled with water at a constant rate. Draw a graph of the water level versus time for each container.

a) a 2 L plastic pop bottle **b)** a vase

6. John is riding a bicycle at a constant cruising speed along a flat road. He slows down as he climbs a hill. At the top of the hill, he speeds up, back to his constant cruising speed on a flat road. He then accelerates down the hill. He comes to another hill and coasts to a stop as he starts to climb.

a) Sketch a possible graph to show John's speed versus time, and another graph to show his distance travelled versus time.

b) Sketch a possible graph of John's elevation (in relation to his starting point) versus time.

7. A
A swimming pool is 50 m long. Kommy swims from one end of the pool to the other end in 45 s. He rests for 10 s and then takes 55 s to swim back to his starting point.

a) Use the information given to find the average speed for Kommy's first length of the pool.

b) What is the average speed for Kommy's second length of the pool?

c) If you were to graph Kommy's distance versus time for his first and second lengths of the pool, how would the two graphs compare? How is this related to Kommy's speed?

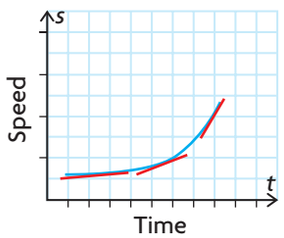
d) Draw a distance versus time graph for Kommy's swim.

e) What is Kommy's speed at time $t = 50$?

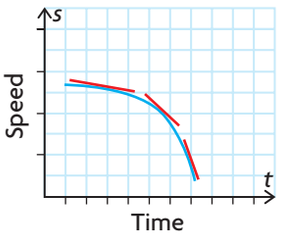
f) Draw a speed versus time graph for Kommy's swim.

8. The following graphs show speed versus time. Match each graph with the description given below.

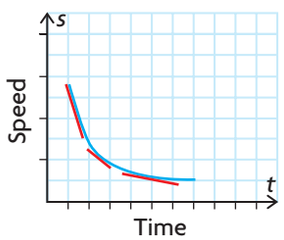
a)



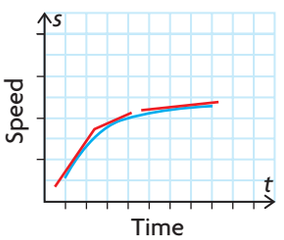
c)



b)



d)



A The rate at which the speed increases is increasing as time increases.

B The rate at which the speed increases is decreasing as time increases.

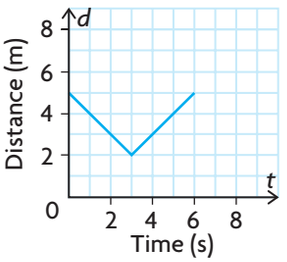
C The rate at which the speed decreases is decreasing as time increases.

D The rate at which the speed decreases is increasing as time increases.

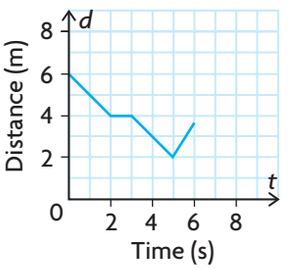
9. T
A jockey is warming up a horse. Whenever the jockey has the horse accelerate or decelerate, she does so at a nonconstant rate—at first slowly and then more quickly. The jockey begins by having the horse trot around the track at a constant rate. She then increases the rate to a canter and allows the horse to canter at a constant rate for several laps. Next, she slowly begins to decrease the speed of the horse to a trot and then to a walk. To finish, the jockey walks the horse around the track once. Draw a speed versus time graph to represent this situation.

10. C **a)** Describe how you would walk toward or away from a motion sensor detector to give each distance versus time graph shown below.

i)



ii)



b) For each part of each graph, determine the speed at which you must walk.

11. A cross-country runner is training for a marathon. His training program requires him to run at different speeds for different lengths of time. His program also requires him to accelerate and decelerate at a constant rate. Today he begins by jogging for 10 min at a rate of 5 miles per hour. He then spends 1 min accelerating to a rate of 10 miles per hour. He stays at this rate for 5 min. He then decelerates for 1 min to a rate of 7 miles per hour. He stays at this rate for 30 min. Finally, to cool down, he decelerates for 2 min to a rate of 3 miles per hour. He stays at this rate for a final 10 min and then stops.

a) Make a speed versus time graph to represent this situation.

b) What is the instantaneous rate of change in the runner's speed at 10.5 min?

c) Calculate the runner's average rate at which he changed speeds from minute 11 to minute 49.

d) Explain why your answer for part c) does not accurately represent the runner's training schedule from minute 11 to minute 49.

12. Create a scenario that could be used to create either a distance versus time graph or a speed versus time graph. Exchange your scenario with a partner and create the corresponding graph.

Extending

13. Two women are running on the same track. One has just finished her workout and is decelerating—at first slowly and then more quickly as she comes to a complete stop. The other woman is just starting her workout and is accelerating—at first quickly and then more slowly as she reaches her target speed. Use one graph to illustrate the rates of both women.

14. A graph displays changes in rate of speed versus time. The graph has straight lines from point to point. If the graph had been drawn to display changes in distance versus time, how would it be different?