2.5 Solving Problems Involving Rates of Change

GOAL

Use rates of change to solve problems that involve functions.

YOU WILL NEED

- dynamic geometry software, spreadsheet, or graphing calculator

INVESTIGATE the Math

A theatre company's profit $P(x)$, in dollars, is described by the equation $P(x) = -60x^2 + 1800x + 16\,500$, where x is the cost of a ticket in dollars.

? What ticket price will give the maximum profit?

A. Calculate the average rate of change in profit for each interval of ticket prices:

$12 \le x \le 15$	$15 \le x \le 17$
$14 \le x \le 15$	$15 \le x \le 16$
$14.5 \le x \le 15$	$15 \le x \le 15.5$
$14.8 \le x \le 15$	$15 \le x \le 15.2$

B. What do all the values for the first four rate of change calculations have in common? The last four?

C. Use your results to estimate the instantaneous rate of change when $x = 15$.

D. Graph the profit function. Where does the maximum occur on your graph and what ticket price gives this maximum profit?

Reflecting

E. What is the relationship between the instantaneous rate of change in profit and the cost of a ticket at the point where the maximum profit occurs? How do you know?

F. How else could you use your graph and your knowledge of rates of change to verify that a maximum occurs at this point?

G. What would the tangent line look like at the point where a maximum occurs on your graph?

H. Explain how you could use tangents and rates of change to identify the value where a minimum occurs on a graph.

APPLY the Math

EXAMPLE 1 Selecting a strategy to identify the location of a minimum value

Leonard is riding a Ferris wheel. Leonard's elevation $h(t)$, in metres above the ground at time t in seconds, can be modelled by the function $h(t) = 5\cos(4(t - 10)°) + 6$. Shu thinks that Leonard will be closest to the ground at 55 s. Do you agree? Support your answer.

Solution

Using $t = 54$ and $t = 55$,
Average rate of change in elevation

$$= \frac{h(55) - h(54)}{55 - 54}$$

$$\doteq \frac{1 - 1.0122}{1}$$

$$= -0.0122 \text{ m/s}$$

Estimate the instantaneous rate of change in height near $t = 55$. If it is equal to zero, then a minimum could happen there.

Using $t = 56$ and $t = 55$,
Average rate of change in elevation

$$= \frac{h(56) - h(55)}{56 - 55}$$

$$\doteq \frac{1.0122 - 1}{1}$$

$$= 0.0122 \text{ m/s}$$

A minimum could occur at $t = 55$.
$h(54) = 1.0122$, $h(55) = 1$, and $h(56) = 1.0122$

Since the rate of change in height using a point to the left of $t = 55$ is negative, and using a point to the right of $t = 55$ is positive, and since both are close to 0, the instantaneous rate of change could be zero at $t = 55$.

Since the estimate of the instantaneous rate of change at $t = 55$ is zero, and since $h(55)$ is less than both $h(54)$ and $h(56)$, Leonard is closest to the ground at $t = 55$ s, just as Shu predicted.

Check that $h(55)$ is really lower than $h(54)$ and $h(56)$ to be sure that a minimum occurs at $t = 55$.

EXAMPLE 2 Selecting an algebraic strategy to identify the location of a minimum value

Show that the minimum value for the function $f(x) = x^2 + 4x - 21$ happens when $x = -2$.

Solution

Estimate the slope of the tangent to the curve when $a = -2$ by writing an equation for the slope of any secant line on the graph of $f(x)$.

$$m = \frac{f(-2 + h) - f(-2)}{h}$$

$$= \frac{(-2 + h)^2 + 4(-2 + h) - 21 - (-25)}{h}$$

$$= \frac{4 - 4h + h^2 - 8 + 4h + 4}{h}$$

$$= \frac{h^2}{h}$$

$$= h$$

To estimate the slope of the tangent at $x = -2$, use two points and the difference quotient. For one point, use $(-2, -25)$, the point where you want the tangent line to be. For the other point, use the general point on the graph of $f(x)$, for example, $(-2 + h, f(-2 + h))$.

To estimate the slope of the tangent to the curve when $x = -2$, replace h with small values.

When $h = -0.01$, $m = -0.01$.
When $h = 0.01$, $m = 0.01$.

Since the slope of the tangent is close to 0, there could be a minimum value at $x = -2$.

Take the average of these rates of change to improve your estimate of the instantaneous rate of change at $x = -2$.

$$\text{Instantaneous rate of change} = \frac{0.01 + (-0.01)}{2}$$

$$= 0$$

$$f(-1.9) = -24.99$$
$$f(-2) = -25$$
$$f(-2.1) = -24.99$$

Use two values of x that are close to -2, but on opposite sides of it, to calculate values for the function. Compare these values for the function to the value at $x = -2$.

Since the slope of the tangent is equal to zero when $x = -2$, and since the values of the function when $x = -2.1$ and $x = -1.9$ are greater than the value when $x = -2$, a minimum value occurs at $x = -2$.

EXAMPLE 3 Selecting a strategy that involves instantaneous rate of change to solve a problem

Tim has a culture of 25 bacteria that is growing at a rate of 15%/h. He observes the culture for 12 h. During this time period, when is the instantaneous rate of change the greatest?

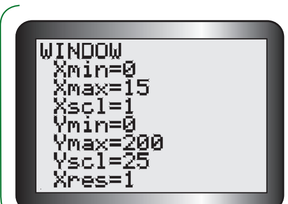
Solution

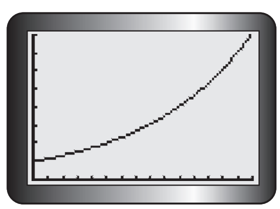
Determine an algebraic model for the situation.

$P(t) = 25(1.15)^t$

This situation involves exponential growth. The algebraic model will be in the form $y = ab^x$, where a represents the initial size of the population, 25, and b is 1 + the growth rate, which is 15%.

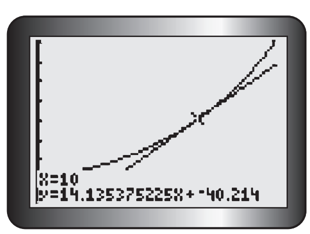
Graph the algebraic model over the given domain, $0 \leq t \leq 12$, using a suitable window setting.



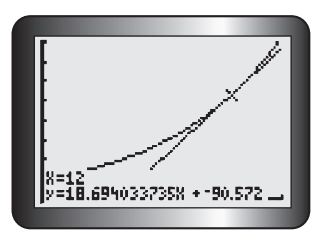


It looks like the instantaneous rate of change is the greatest near the end of the time period, because the graph is increasing faster then.

Estimate the instantaneous rate of change at $t = 10$ and $t = 12$ by drawing tangent lines at each of these points.



At $x = 10$, the slope of the tangent line is about 14.1. So here the bacteria population is increasing by about 14 bacteria per hour.



At $x = 12$, the slope of the tangent line is about 18.7. At this point, the bacteria population is increasing by about 19 bacteria per hour.

The instantaneous rate of change is the greatest at 12 h.

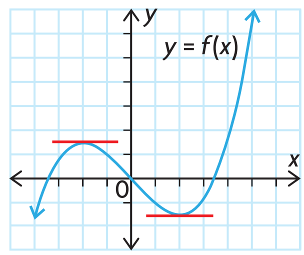
Tech Support

For help using the graphing calculator to draw tangent lines, see Technical Appendix, T-17.

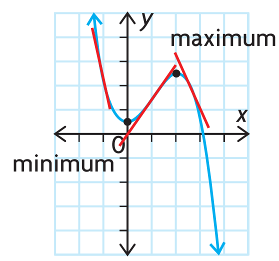
In Summary

Key Idea

- The instantaneous rate of change is zero at both a maximum point and a minimum point. As a result, the tangent lines drawn at these points will be horizontal lines.



Need to Know



- If the instantaneous rate of change is negative before the value where the rate of change is zero and positive after this value, then a minimum occurs. Graphically, the tangent lines must have a negative slope before the minimum point and a positive slope after.
- If the instantaneous rate of change is positive before the value where the rate of change is zero and negative after this value, then a maximum occurs. Graphically, the tangent lines must have a positive slope before the maximum point and a negative slope after.

CHECK Your Understanding

1. The cost of running an assembly line can be modelled by the function $C(x) = 0.3x^2 - 0.9x + 1.675$, where $C(x)$ is the cost per hour in thousands of dollars and x is the number of items produced per hour in thousands. The most economical production level occurs when 1500 items are produced. Verify this using the appropriate calculations for rate of change in cost.

2. For a person at rest, the function $P(t) = -20\cos(300°t) + 100$ models blood pressure, in millimetres of mercury (mm Hg), at time t seconds. What is the rate of change in blood pressure at 3 s?

3. If a function has a maximum value at $(a, f(a))$, what do you know about the slopes of the tangent lines at the following points?

a) points to the left of, and very close to, $(a, f(a))$

b) points to the right of, and very close to, $(a, f(a))$

4. If a function has a minimum value at $(a, f(a))$, what do you know about the slopes of the tangent lines at the following points?

a) points to the left of, and very close to, $(a, f(a))$

b) points to the right of, and very close to, $(a, f(a))$

PRACTISING

5. For each function, the point given is the maximum or minimum. **K** Use the difference quotient to verify that the slope of the tangent at this point is zero.
   a) $f(x) = 0.5x^2 + 6x + 7.5$; $(-6, -10.5)$
   b) $f(x) = -6x^2 + 6x + 9$; $(0.5, 10.5)$
   c) $f(x) = 5 \sin (x)$; $(90°, 5)$
   d) $f(x) = -4.5 \cos (2x)$; $(0°, -4.5)$

6. Use an algebraic strategy to verify that the point given for each function is either a maximum or a minimum.
   a) $f(x) = x^2 - 4x + 5$; $(2, 1)$
   b) $f(x) = -x^2 - 12x + 5.75$; $(-6, 41.75)$
   c) $f(x) = x^2 - 9x$; $(4.5, -20.25)$
   d) $f(x) = 3 \cos (x)$; $(0°, 3)$
   e) $f(x) = x^3 - 3x$; $(-1, 2)$
   f) $f(x) = -x^3 + 12x - 1$; $(2, 15)$

7. **A** A pilot who is flying at an altitude of 10 000 feet is forced to eject from his airplane. The path that his ejection seat takes is modelled by the equation $h(t) = -16t^2 + 90t + 10\,000$, where $h(t)$ is his altitude in feet and t is the time since his ejection in seconds. Estimate at what time, t, the pilot is at a maximum altitude. Explain how the maximum altitude is related to the slope of the tangent line at certain points.

8. a) Graph each function using a graphing calculator. Then find the minimum or maximum point for the function.
      i) $f(x) = x^2 + 10x - 15$
      ii) $f(x) = -3x^2 + 45x + 16$
      iii) $f(x) = 4x^2 - 26x - 3$
      iv) $f(x) = -0.5x^2 + 6x + 0.45$
   b) Draw tangent lines on either side of the points you found in part a).
   c) Explain how the tangent lines you drew confirm the existence of the minimum or maximum points you found in part a).

9. **T** a) Find the maximum *and* minimum values for each exponential growth or decay equation on the given interval.
      i) $y = 100(0.85)^t$, for $0 \le t \le 5$
      ii) $y = 35(1.15)^x$, for $0 \le x \le 10$
   b) Examine your answers for part a). Use your answers to hypothesize about where the maximum value will occur in a given range of values, $a \le x \le b$. Explain and support your hypothesis thoroughly.

10. The height of a diver above the water is modelled by the function $h(t) = -5t^2 + 5t + 10$, where t represents the time in seconds and $h(t)$ represents the height in metres. Use the appropriate calculations for the rate of change in height to show that the diver reaches her maximum height at $t = 0.5$ s.

11. The top of a flagpole sways back and forth in high winds. The function $f(t) = 8 \sin (180°t)$ represents the displacement, in centimetres, that the flagpole sways from vertical, where t is the time in seconds. The flagpole is vertical when $f(t) = 0$. It is 8 cm to the right of its resting place when $f(t) = 8$, and 8 cm to the left of its resting place when $f(t) = -8$. If the flagpole is observed for 2 s, it appears to be farthest to the left when $t = 1.5$ s. Is this observation correct? Justify your answer using the appropriate calculations for the rate of change in displacement.

12. C The weekly revenue for battery sales at Discount H hardware store can be modelled by the function $R(x) = -x^2 + 10x + 30\,000$, where revenue, R, and the cost of a package of batteries, x, are in dollars. The maximum revenue occurs when a package of batteries costs \$5. Write detailed instructions, using the appropriate calculations for the rate of change in revenue, to verify that the maximum revenue occurs when a package of batteries costs \$5. Exchange instructions with a partner. Follow your partner's instructions to verify when the maximum revenue occurs.

Extending

13. Explain how to determine the value of x that gives a maximum for a transformed sine function in the form $y = a \sin (k(x - d)) + c$, if the maximum for $y = \sin x$ occurs at $(90°, 1)$.

14. The speedometer in a car shows the vehicle's instantaneous velocity, or rate of change in position, at any moment. Every 5 s, Myra records the speedometer reading in a vehicle driven by a friend. She then plots these values. When Myra begins considering rates of change shown on her graph, what quantity is she looking at? Explain what different scenarios on Myra's graph mean, such as, her graph is increasing, but the rate of change between points on her graph is decreasing.

15. Estimate the instantaneous rate of change for $f(x) = x^2$ at $x = -2, -1, 2,$ and 3. Does there appear to be a rule for determining the instantaneous rate of change for the function at given values of x? If so, state the rule. Repeat for $f(x) = x^3$.