3.1 Exploring Polynomial Functions

YOU WILL NEED

- graphing calculator or graphing software

GOAL

Identify polynomial functions.

EXPLORE the Math

Beth knows that linear functions result in graphs of straight lines, while quadratic functions result in parabolas. She wonders what happens when the **degree** of a function is larger than 2. Beth searched for polynomials on the Internet and found the following table.

These are polynomial expressions.	These are not polynomial expressions.
$3x^2 - 5x + 3$	$\sqrt{x} + 5x^3$
$-4x + 5x^7 - 3x^4 + 2$	$\frac{1}{2x + 5}$
$\frac{2}{5}x^3 - 3x^5 + 4$	$6x^3 + 5x^2 - 3x + 2 + 4x^{-1}$
$\sqrt{4}x^3 - \frac{\sqrt{5}}{3}x^2 + 2x - \frac{1}{4}$	$\frac{3x^2 + 5x - 1}{2x^2 + x - 3}$
$3x - 5$	$4^x + 5$
-7	$\sin(x - 30)$
$-4x$	$x^2y + 3x - 4y^{-2}$
$(2x - 3)(x + 1)^2$	$3x^3 + 4x^{2.5}$

Communication *Tip*

A polynomial expression in one variable is usually written with the powers arranged from highest to lowest degrees, as in $5x^3 - 7x^2 + 4x + 3$. The phrase "polynomial expression" is often shortened to just "polynomial."

? What makes an expression a *polynomial* expression, and what do functions that involve polynomial expressions look like graphically and algebraically?

A. Look carefully at the expressions in the two columns of the table. What do all of the polynomial expressions have in common?

B. The expressions in the right column are not polynomials. How are they different from the polynomial expressions in the left column?

C. In your own words, define a polynomial expression.

D. The simplest polynomial functions are functions that contain a single term. Use a graphing calculator to graph each of the following polynomial functions. Then copy and complete the table.

Polynomial Function	Type	Sketch of Graph	Description of Graph	Domain and Range	Existence of Asymptotes?
$f(x) = x$	linear				
$f(x) = x^2$	quadratic				
$f(x) = x^3$	cubic				
$f(x) = x^4$	quartic				
$f(x) = x^5$	quintic				

E. Which polynomial functions in part D have similar graphical characteristics? How are the equations of these functions related?

F. For each function in part D, create a table of values for $-3 \leq x \leq 3$. Calculate the finite differences until they are constant. What do you notice?

G. Create equations for four different polynomial functions that are neither linear nor quadratic. Make sure that each function has a different degree and contains at least three terms. Graph each function on a graphing calculator, make a detailed sketch, and create a table of finite differences.

H. Create equations for four non-polynomial functions. Make detailed sketches of their graphs, and create a table of finite differences.

I. Compare and contrast the graphs, the equations, and the finite difference tables for the polynomial and non-polynomial functions you created. Explain how you can tell whether or not a function is a polynomial by looking at

i) its graph
ii) its equation
iii) its finite difference table

Reflecting

J. Explain how you can tell whether a polynomial equation is a function and not just a relation.

K. Why are the equations of the form $y = mx + b$ and $y = ax^2 + bx + c$ examples of polynomial functions?

polynomial functions

a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers and n is a whole number; the equation of a polynomial function is defined by a polynomial expression, as in $f(x) = 5x^3 + 6x^2 - 3x + 7$

Communication *Tip*

Polynomial functions are named according to their degree. Polynomial functions of degree 1, 2, 3, 4, and 5 are commonly called linear, **quadratic**, **cubic**, **quartic**, and **quintic** functions, respectively.

L. As the degree of a polynomial function increases, describe what happens to
   i) the graph of the function
   ii) the finite differences

M. Would you change your definition in part C now, after having completed part G? Explain.

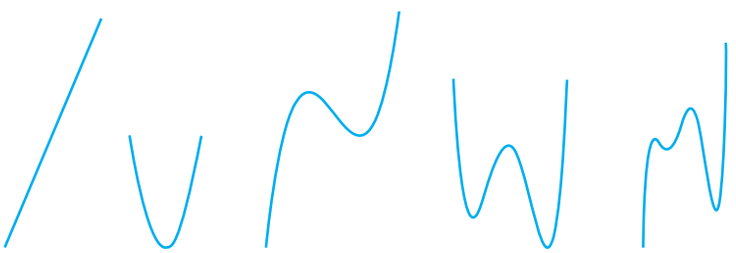
In Summary

Key Idea

- A polynomial in one variable is an expression of the form $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers and n is a whole number. The expression contains only one variable, with the powers arranged in descending order. For example, $2x + 5$, $3x^2 + 2x - 1$, and $5x^4 + 3x^3 - 6x^2 + 5x - 8$.

Need to Know

- In any polynomial expression, the exponents on the variable must be whole numbers.
- A polynomial function is any function that contains a polynomial expression in one variable. The degree of the function is the highest exponent in the expression. For example, $f(x) = 6x^3 - 3x^2 + 4x - 9$ has a degree of 3.
- The nth finite differences of a polynomial function of degree n are constant.
- The domain of a polynomial function is the set of real numbers, $\{x \in \mathbf{R}\}$.
- The range of a polynomial function may be all real numbers, or it may have a lower bound or an upper bound (but not both).
- The graphs of polynomial functions do not have horizontal or vertical asymptotes.
- The graphs of polynomial functions of degree zero are horizontal lines. The shape of other graphs depends on the degree of the function. Five typical shapes are shown for various degrees:

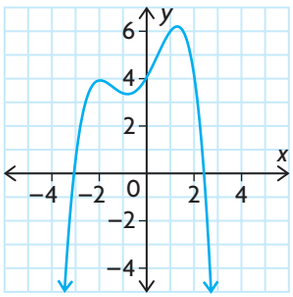


Linear ($n = 1$)	Quadratic ($n = 2$)	Cubic ($n = 3$)	Quartic ($n = 4$)	Quintic ($n = 5$)

FURTHER Your Understanding

1. Determine which graphs represent polynomial functions. Explain how you know.

a)



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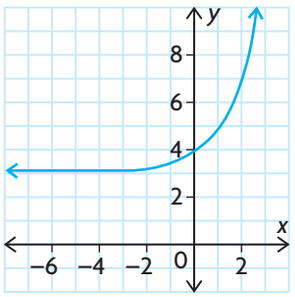
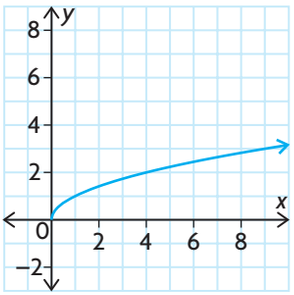


e)



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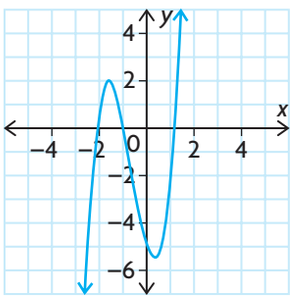

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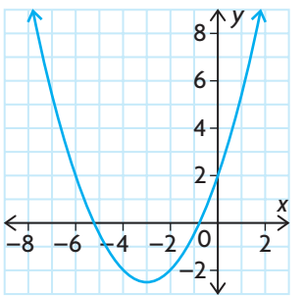

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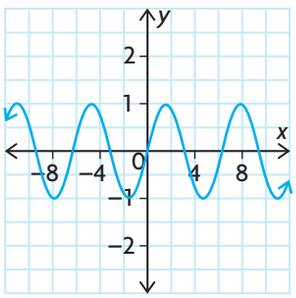

2. Determine whether each function is a polynomial function or another type of function. Justify your decision.

a) $f(x) = 2x^3 + x^2 - 5$

b) $f(x) = x^2 + 3x - 2$

c) $y = 2x - 7$

d) $y = \sqrt{x + 1}$

e) $y = \dfrac{x^2 - 4x + 1}{x + 2}$

f) $f(x) = x(x - 1)^2$

3. Use finite differences to determine the type of polynomial function that could model each relationship.

a) Michelle earns $200 per week, plus 5% of sales.

Sales	0	500	1000	1500	2000
Earnings ($)	200	225	250	275	300

b) A model rocket is launched from the roof of a school.

Time (s)	0	1	2	3	4
Height above Ground (m)	10	25	30	25	10

c) The volume of a box varies at different widths.

Width (cm)	1	2	3	4	5
Volume (cm^3)	200	225	250	275	300

d) The input for a function gives a certain output.

Input	0	1	2	3	4	5	6
Output	200	204	232	308	456	700	1064

4. Graph the function $y = 2^x$ on the domain $0 \leq x \leq 3$.

a) Explain why a person who sees only the graph you created (not the equation) might think that the graph represents a polynomial function.

b) Explain why this function is not a polynomial function.

5. Draw a graph of a polynomial function that satisfies all of the following characteristics:

- $f(-3) = 16, f(3) = 0$, and $f(-1) = 0$
- The y-intercept is 2.
- $f(x) \geq 0$ when $x < 3$.
- $f(x) \leq 0$ when $x > 3$.
- The domain is the set of real numbers.

6. Explain why there are many different graphs that fit different combinations of the characteristics in question 5. Draw two graphs that are different from each other, and explain how they satisfy some, but not all, of the characteristics in question 5.

7. Create equations for a linear, a quadratic, a cubic, and a quartic polynomial function that all share the same y-intercept of 5.

8. Complete the following chart to summarize your understanding of polynomials.