# 3.2 Characteristics of Polynomial Functions

**GOAL**

Investigate the turning points and end behaviours of polynomial functions.

**YOU WILL NEED**

- graphing calculator or graphing software

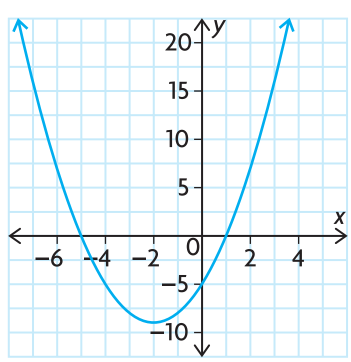
## *INVESTIGATE* the Math

Karel knows that he can describe the graph of a linear function from its equation, using the slope and the $y$-intercept. He can also describe the graph of a quadratic function from its equation, using the vertex, $y$-intercept, and the direction of opening. Now he is wondering whether he can describe the graphs of polynomial functions of higher degree, using characteristics that can be predicted from their equations.

**?** How can you predict some of the characteristics of a polynomial function from its equation?

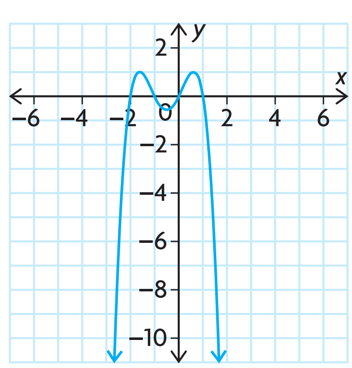
**A.** The graphs of some polynomial functions are shown below and on the following page.

**a)**



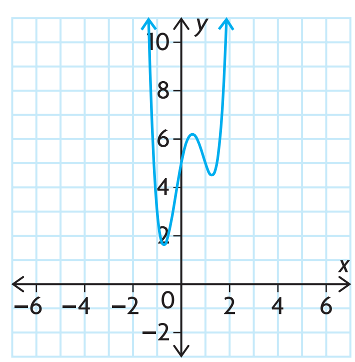

$f(x) = x^2 + 4x - 5$

**b)**



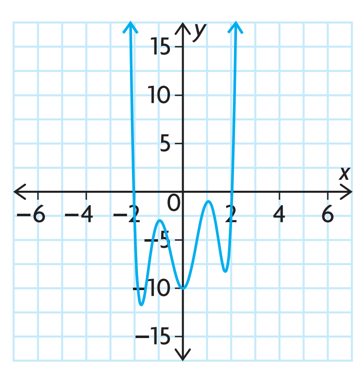

$f(x) = -x^4 - 2x^3 + x^2 + 2x$

**c)**



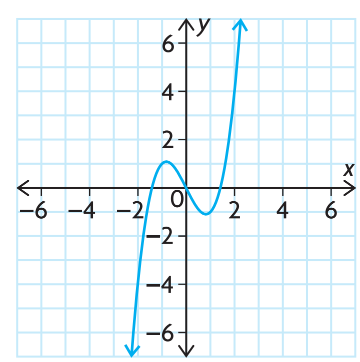

$f(x) = 3x^4 - 4x^3 - 4x^2 + 5x + 5$

**d)**



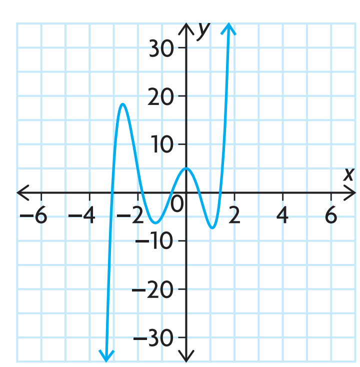

$P(x) = 2x^6 - 12x^4 + 18x^2 + x - 10$

**e)**



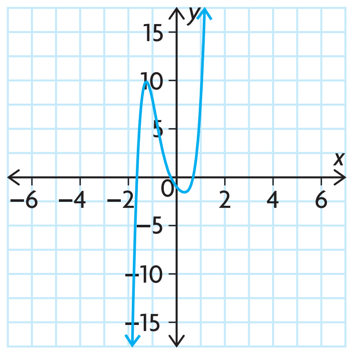

$f(x) = x^3 - 2x$

**f)**



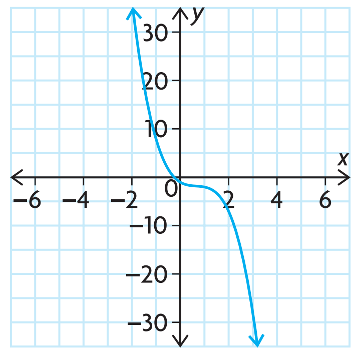

$f(x) = 2x^5 + 7x^4 - 3x^3 - 18x^2 + 5$

g)



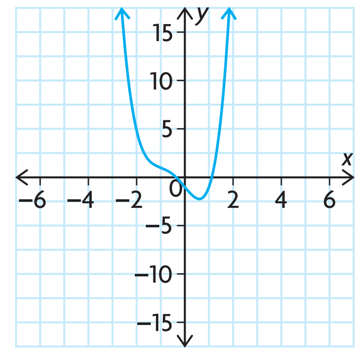
$f(x) = 5x^5 + 5x^4 - 2x^3 + 4x^2 - 3x$

h)



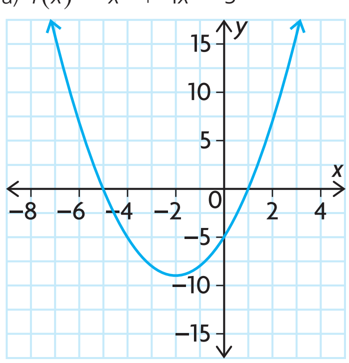
$f(x) = -2x^3 + 4x^2 - 3x - 1$

i)



$f(x) = x^4 + 2x^3 - 3x - 1$

Copy the following table, and complete it using the remaining equations and graphs given.

| Equation and Graph | Degree | Even or Odd Degree? | Leading Coefficient | End Behaviours | | Number of Turning Points |
|---|---|---|---|---|---|---|
| | | | | $x \to -\infty$ | $x \to +\infty$ | |
| a) $f(x) = x^2 + 4x - 5$  | 2 | even | $+1$ | $y \to +\infty$ | $y \to +\infty$ | 1 |

**leading coefficient**

the coefficient of the term with the highest degree in a polynomial

**B.** Describe any patterns that you see in your table.

**C.** Create two new polynomial functions of degree greater than 2, one of even degree and one of odd degree. Do these new polynomial functions support your observations in part B?

**D.** What do you think is the maximum number of turning points that a polynomial function of degree $n$ can have?

**E.** Graph the following functions using a graphing calculator. Copy each graph and its equation into the appropriate column of a table like the one shown on the next page.

**i)** $f(x) = x^4 - 2x^2 + 1$

**ii)** $f(x) = x^3 + 3x^2 - 2x - 5$

**iii)** $f(x) = \frac{1}{2}x^{10} - \frac{1}{3}x^4 + x^2$

**iv)** $f(x) = x^3 + x$

**v)** $f(x) = -2x^6 + 3x^4$

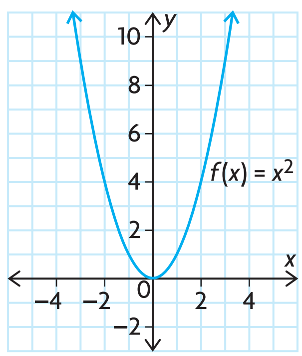
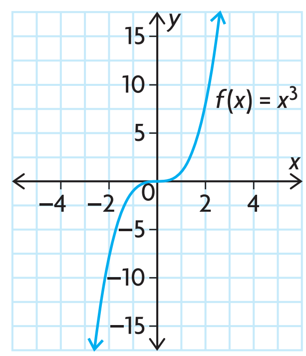
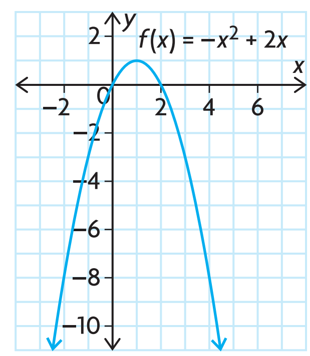
**vi)** $f(x) = x^5 - 3x$

**vii)** $f(x) = x^2 - 3x + 4$

**viii)** $f(x) = 2x^7 - 3x^3 + 2x$

**ix)** $f(x) = -3x^4 + 2x^3 - 3x + 1$

**x)** $f(x) = x^2 - x$

| Even Functions | Odd Functions | Neither Even nor Odd Functions |
|---|---|---|
| (symmetry in the *y*-axis) | (rotational symmetry around the origin) | (neither of these symmetries) |
|  |  |  |
| | | |
| | | |

**F.** Determine $f(-x)$ for each function in your table. Discuss any patterns that you see.

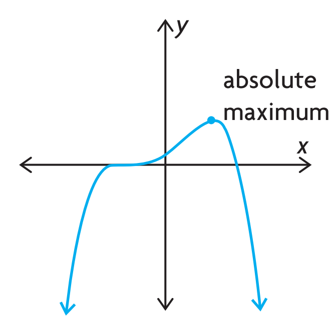
**G.** Is every function of even degree an even function? Why or why not?

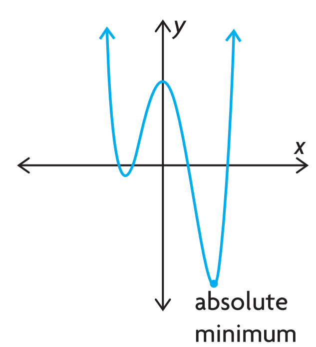
**H.** Is every function of odd degree an odd function? Why or why not?

**I.** How can you use the equation of a polynomial function to describe its end behaviours, number of turning points, and symmetry?

**absolute maximum/ absolute minimum**

the greatest/least value attained by a function for all values in its domain




## Reflecting

**J.** Why must all polynomial functions of even degree have an absolute maximum or absolute minimum?

**K.** Why must all polynomial functions of odd degree have at least one zero?

**L.** Can the graph of a polynomial function have no zeros? Explain.

**M.** Examine all the graphs you have investigated and their equations. Is it possible to predict the maximum number of zeros that a graph will have if you are given its equation? Explain.

## *APPLY* the Math

### EXAMPLE 1 | Reasoning about characteristics of a given polynomial function

Describe the end behaviours of each function, the possible number of turning points, and the possible number of zeros. Use these characteristics to sketch possible graphs of the function.

**a)** $f(x) = -3x^5 + 4x^3 - 8x^2 + 7x - 5$ **b)** $g(x) = 2x^4 + x^2 + 2$

### Solution

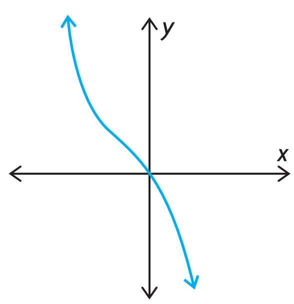
**a)** $f(x) = -3x^5 + 4x^3 - 8x^2 + 7x - 5$

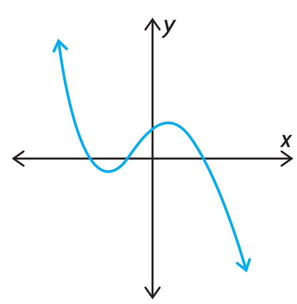
The degree is odd, so the function has opposite end behaviours. The leading coefficient is negative, so the graph must extend from the second quadrant to the fourth quadrant.

As $x \rightarrow -\infty$, $y \rightarrow +\infty$.

As $x \rightarrow +\infty$, $y \rightarrow -\infty$.

If $x$ is a very large negative number, such as $-1000$, $-3x^5$ will have a large positive value and will have a greater effect on the value of the function than the other terms. Therefore, the graph will pass through the second quadrant. For very large positive values of $x$, $-3x^5$ will have a large negative value. Therefore, the graph will extend into the fourth quadrant.

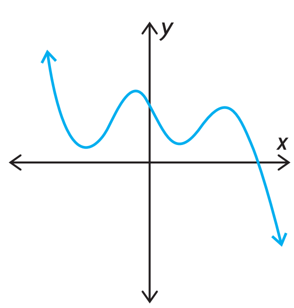


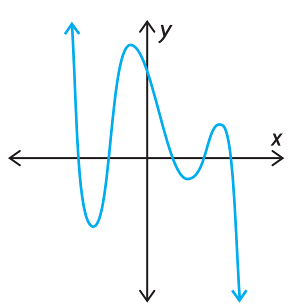



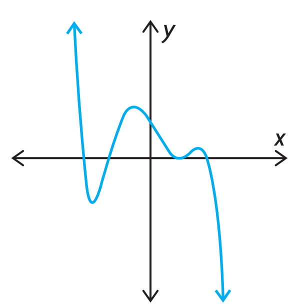

Using the end behaviours of the function, sketch possible graphs of a fifth-degree polynomial.

To pass through the second quadrant and extend into the fourth quadrant, the graph must have an even number of turning points.

$f(x)$ may have zero, two, or four turning points.




Since the function is a fifth-degree polynomial, it must have at least one zero and no more than five zeros.

$f(x)$ may have one, two, three, four, or five zeros.

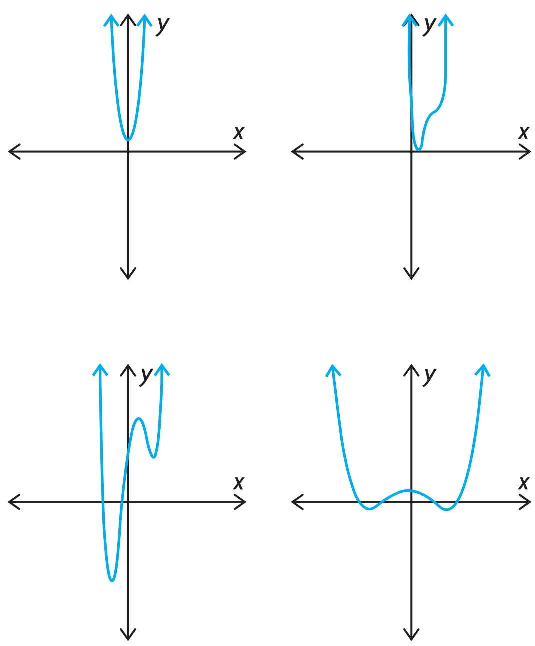
**b)** $g(x) = 2x^4 + x^2 + 2$

The degree is even, so the function has the same end behaviours. The leading coefficient is positive, so the graph must extend from the second quadrant to the first quadrant.

As $x \rightarrow -\infty$, $y \rightarrow +\infty$.

As $x \rightarrow +\infty$, $y \rightarrow +\infty$.

If $x$ is a very large negative number, $2x^4$ will have a large positive value and will have a greater effect on the value of the function than the other terms. Therefore, the graph will pass through the second quadrant. For very large positive values of $x$, $2x^4$ will have a large positive value. Therefore, the graph will extend into the first quadrant.




Using the end behaviours of the function, sketch possible graphs of a fourth-degree polynomial.

To start in the second quadrant and end in the first quadrant, the graph must have an odd number of turning points.

Since the function is a fourth-degree polynomial, it may have anywhere from zero to four $x$-intercepts.

$f(x)$ may have one or three turning points and zero, one, two, three, or four zeros.

## EXAMPLE 2 Reasoning about how given characteristics fit particular functions

What could the graph of a polynomial function that has range $\{y \in \mathbf{R} \mid y \leq 10\}$ and three turning points look like? What can you conclude about its equation?

### Solution

End behaviours of the function:

As $x \to -\infty$, $y \to -\infty$.

As $x \to +\infty$, $y \to -\infty$.

Since the range has an upper limit, both ends of the function extend downward toward $-\infty$ in the third and fourth quadrants. For this to occur, the leading coefficient in the equation must be negative.

The function has at least two zeros.

Because the function has a maximum value that is positive and both ends extend downward, the function must cross the *x*-axis at least twice.

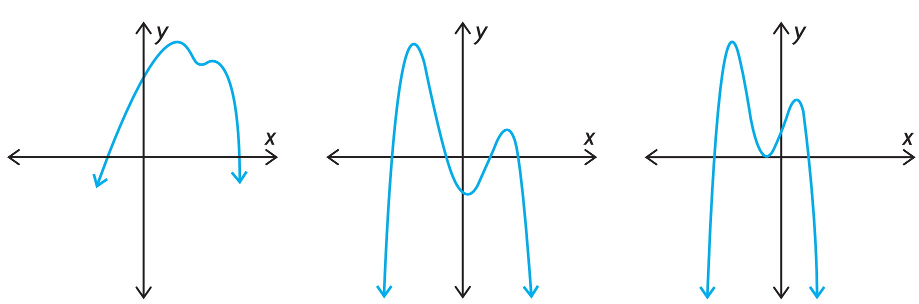
The function has an even degree.

Since the function has an absolute maximum, it must have an even degree. This is confirmed by the end behaviours, because they are the same.

The degree of the function is at least 4.

It is not possible to be sure about the degree of the function, but the degree must be at least one more than the number of turning points.

Here are some possible graphs of the function.



## In Summary

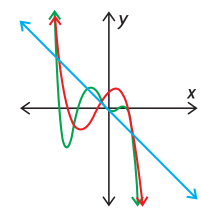
### Key Ideas

- Polynomial functions of the same degree have similar characteristics.
- The degree and the leading coefficient in the equation of a polynomial function indicate the end behaviours of the graph.
- The degree of a polynomial function provides information about the shape, turning points, and zeros of the graph.

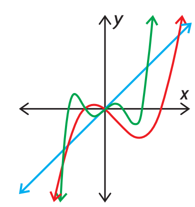
### Need to Know

**End Behaviours**

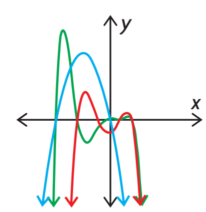
- An odd-degree polynomial function has opposite end behaviours.
  - If the leading coefficient is negative, then the function extends from the second quadrant to the fourth quadrant; that is, as $x \rightarrow -\infty$, $y \rightarrow \infty$ and as $x \rightarrow \infty$, $y \rightarrow -\infty$.



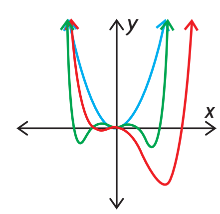

  - If the leading coefficient is positive, then the function extends from the third quadrant to the first quadrant; that is, as $x \rightarrow -\infty$, $y \rightarrow -\infty$ and as $x \rightarrow \infty$, $y \rightarrow \infty$.




- An even-degree polynomial function has the same end behaviours.
  - If the leading coefficient is negative, then the function extends from the third quadrant to the fourth quadrant; that is, as $x \rightarrow \pm\infty$, $y \rightarrow -\infty$.




  - If the leading coefficient is positive, then the function extends from the second quadrant to the first quadrant; that is, as $x \rightarrow \pm\infty$, $y \rightarrow \infty$.




**Turning Points**

- A polynomial function of degree $n$ has at most $n - 1$ turning points.

**Number of Zeros**

- A polynomial function of degree $n$ may have up to $n$ distinct zeros.
- A polynomial function of odd degree must have at least one zero.
- A polynomial function of even degree may have no zeros.

**Symmetry**

- Some polynomial functions are symmetrical in the $y$-axis. These are even functions, where $f(-x) = f(x)$.
- Some polynomial functions have rotational symmetry about the origin. These are odd functions, where $f(-x) = -f(x)$.
- Most polynomial functions have no symmetrical properties. These are functions that are neither even nor odd, with no relationship between $f(-x)$ and $f(x)$.

## *CHECK Your Understanding*

**1.** State the degree, leading coefficient, and end behaviours of each polynomial function.

**a)** $f(x) = -4x^4 + 3x^2 - 15x + 5$

**b)** $g(x) = 2x^5 - 4x^3 + 10x^2 - 13x + 8$

**c)** $p(x) = 4 - 5x + 4x^2 - 3x^3$

**d)** $h(x) = 2x(x - 5)(3x + 2)(4x - 3)$

**2.** **a)** Determine the minimum and maximum number of turning points for each function in question 1.

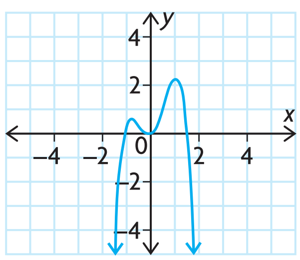
**b)** Determine the minimum and maximum number of zeros that each function in question 1 may have.

**3.** For each of the following graphs, decide if

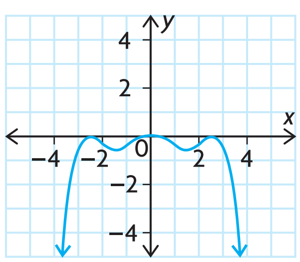
**a)** the function has an even or odd degree

**b)** the leading coefficient is positive or negative

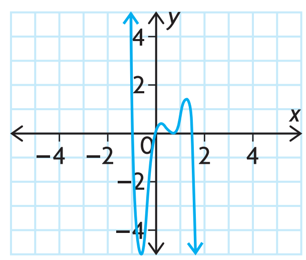
**i)**



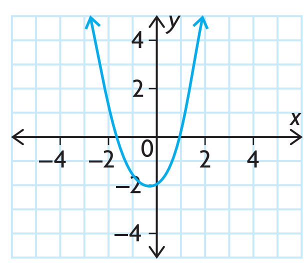
**ii)**



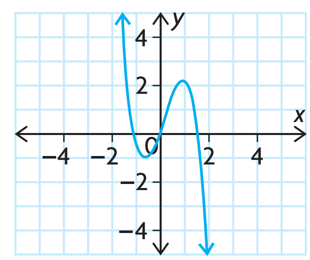
**iii)**



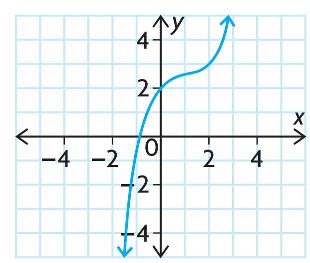
**iv)**



**v)**



**vi)**



## *PRACTISING*

**4.** K Describe the end behaviour of each polynomial function using the degree and the leading coefficient.

**a)** $f(x) = 2x^2 - 3x + 5$

**b)** $f(x) = -3x^3 + 2x^2 + 5x + 1$

**c)** $f(x) = 5x^3 - 2x^2 - 2x + 6$

**d)** $f(x) = -2x^4 + 5x^3 - 2x^2 + 3x - 1$

**e)** $f(x) = 0.5x^4 + 2x^2 - 6$

**f)** $f(x) = -3x^5 + 2x^3 - 4x$

**5.** Use end behaviours, turning points, and zeros to match each polynomial equation with the most likely graph below. Explain.

**a)** $y = 2x^3 - 4x^2 + 3x + 2$

**b)** $y = -4x^4 + 3x^2 + 4$

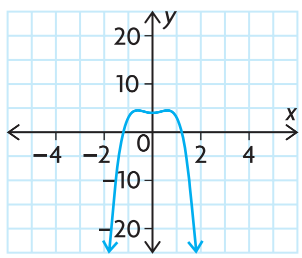
**c)** $y = x^2 + 3x - 5$

**d)** $y = x^4 - x^3 - 4x^2 + 5x$

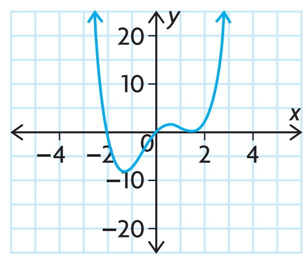
**e)** $y = -2x^5 + 3x^4 + 6x^3 - 10x^2 + 2x + 5$

**f)** $y = 3x^3 + 5x^2 - 3x + 1$

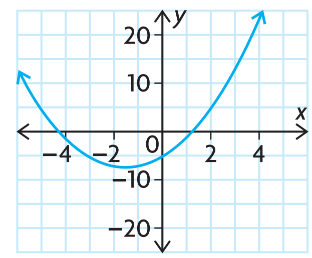
**A**



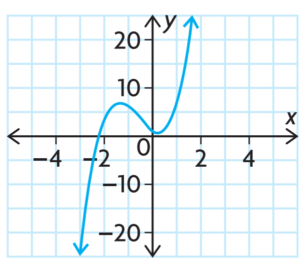

**C**



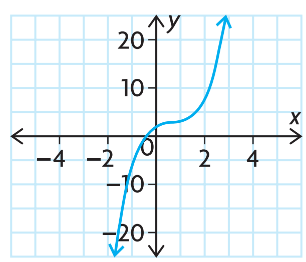

**E**



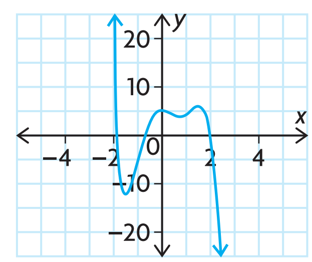

**B**




**D**




**F**




**6.** Give an example of a polynomial function that has each of the following end behaviours:

**a)** As $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to \infty$.

**b)** As $x \to \pm\infty$, $y \to \infty$.

**c)** As $x \to \pm\infty$, $y \to -\infty$.

**d)** As $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$.

**7.** Sketch the graph of a polynomial function that satisfies each set of conditions.

**a)** degree 4, positive leading coefficient, 3 zeros, 3 turning points

**b)** degree 4, negative leading coefficient, 2 zeros, 1 turning point

**c)** degree 4, positive leading coefficient, 1 zero, 3 turning points

**d)** degree 3, negative leading coefficient, 1 zero, no turning points

**e)** degree 3, positive leading coefficient, 2 zeros, 2 turning points

**f)** degree 4, two zeros, three turning points, Range $= \{y \in \mathbf{R} \mid y \leq 5\}$

**8.** C Explain why odd-degree polynomial functions can have only local maximums and minimums, but even-degree polynomial functions can have absolute maximums and minimums.

**9.** T Rei noticed that the graph of the function $f(x) = ax^b - cx$ is symmetrical with respect to the origin, and that it has some turning points. Does the graph have an odd or even number of turning points?

**10.** Sketch an example of a cubic function with a graph that intersects the $x$-axis at each number of points below.

**a)** only one point **b)** two different points **c)** three different points

**11.** Sketch an example of a quartic function with a graph that intersects the $x$-axis at each number of points below.

**a)** no points
**b)** only one point
**c)** two different points
**d)** three different points
**e)** four different points

**12.** The graph of a polynomial function has the following characteristics:

- Its domain and range are the set of all real numbers.
- There are turning points at $x = -2$, 0, and 3.

**a)** Draw the graphs of two different polynomial functions that have these three characteristics.
**b)** What additional characteristics would ensure that only one graph could be drawn?

**13.** A The mining town of Brighton was founded in 1900. Its population, $y$, in hundreds, is modelled by the equation $y = -0.1x^4 + 0.5x^3 + 0.4x^2 + 10x + 7$, where $x$ is the number of years since 1900.

**a)** What was the population of the town in 1900?
**b)** Based on the equation, describe what happened to the population of Brighton over time. Justify your answer.

**14.** $f$ is a polynomial function of degree $n$, where $n$ is a positive even integer. Decide whether each of the following statements is true or false. If the statement is false, give an example that illustrates why it is false.

**a)** $f$ is an even function.
**b)** $f$ cannot be an odd function.
**c)** $f$ will have at least one zero.
**d)** As $x \rightarrow \infty$, $y \rightarrow \infty$ and as $x \rightarrow -\infty$, $y \rightarrow \infty$.

**15.** If you needed to predict the graph or equation of a polynomial function and were only allowed to ask three questions about the function, what questions would you ask to help you the most? Why?

## Extending

**16.** **a)** Suppose that $f(x) = ax^2 + bx + c$. What must be true about the coefficients if $f$ is an even function?
**b)** Suppose that $g(x) = ax^3 + bx^2 + cx + d$. What must be true about the coefficients if $g$ is an odd function?