3.3 Characteristics of Polynomial Functions in Factored Form

GOAL

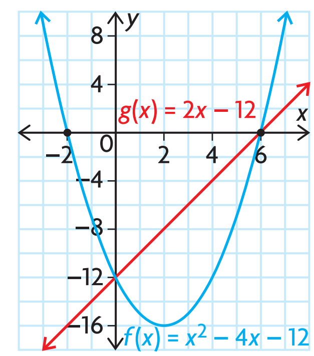
Determine the equation of a polynomial function that describes a particular graph or situation, and vice versa.

YOU WILL NEED

- graphing calculator or graphing software

INVESTIGATE the Math

The graphs of the functions $f(x) = x^2 - 4x - 12$ and $g(x) = 2x - 12$ are shown.



? What is the relationship between the real **roots** of a polynomial equation and the x-intercepts of the corresponding polynomial function?

A. Solve the equations $f(x) = 0$ and $g(x) = 0$ using the given functions. Compare your solutions with the graphs of the functions. What do you notice?

B. Create a cubic function from the family of polynomial functions of the form $h(x) = a(x - p)(x - q)(x - r)$.

C. Graph $y = h(x)$ on a graphing calculator. Describe the shape of the graph near each zero, and compare the shape to the order of each factor in the equation of the function.

D. Solve $h(x) = 0$, and compare your solutions with the zeros of the graph of the corresponding function. What do you notice?

E. Repeat parts B through D using a quartic function.

F. Repeat parts C and D using $m(x) = (x - 2)^2(x + 3)$. How would you describe the shape of the graph near the zero with the repeated factor?

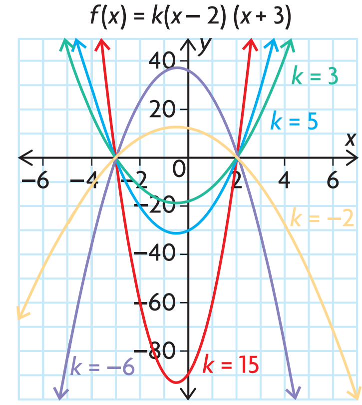
G. Repeat parts C and D using $n(x) = (x - 2)^3(x + 3)$. How would you describe the shape of the graph near the zero with the repeated factor?

H. What relationship exists between the x-intercepts of the graph of a polynomial function and the roots of the corresponding equation?

family of polynomial functions

a set of polynomial functions whose equations have the same degree and whose graphs have common characteristics; for example, one type of quadratic family has the same zeros or x-intercepts

$f(x) = k(x - 2)(x + 3)$



order

the exponent to which each factor in an algebraic expression is raised; for example, in $f(x) = (x - 3)^2(x - 1)$, the order of $(x - 3)$ is 2 and the order of $(x - 1)$ is 1

Reflecting

I. How does a squared factor in the equation of a polynomial function affect the shape of the graph near its corresponding zero?

J. How does a cubed factor in a polynomial function affect the shape of the graph near its corresponding zero?

K. Why does the relationship you described in part H make sense?

APPLY the Math

EXAMPLE 1 Using reasoning to draw a graph from the equation of a polynomial function

Sketch a possible graph of the function $f(x) = -(x + 2)(x - 1)(x - 3)^2$.

Solution

Let $x = 0$.

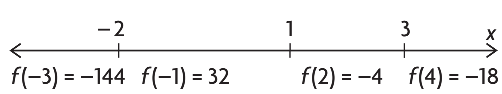
$$\begin{aligned} f(x) &= -(0 + 2)(0 - 1)(0 - 3)^2 \\ &= -(2)(-1)(-3)^2 \\ &= 18 \end{aligned}$$

Calculate the y-intercept.

$$0 = -(x + 2)(x - 1)(x - 3)^2$$
$$x = -2, x = 1, \text{ or } x = 3$$

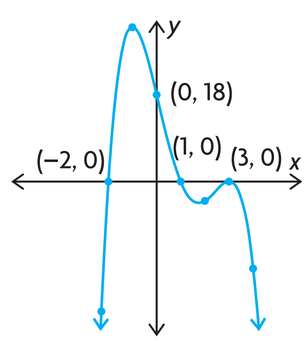
Determine the x-intercepts by letting $f(x) = 0$. Use the factors to solve the resulting equation for x.

Use values of x that fall between the x-intercepts as test values to determine the location of the function above or below the x-axis.



Since the function lies below the x-axis on both sides of $x = 3$, the graph must just touch the x-axis and not cross over at this point. The order of 2 on the factor $(x - 3)^2$ confirms the parabolic shape near $x = 3$.

Determine the end behaviours of the function.



$f(x) = -(x + 2)(x - 1)(x - 3)^2$

Because the degree is even and the leading coefficient is negative, the graph extends from third quadrant to the fourth quadrant; that is, as $x \to \pm\infty, y \to -\infty$.

This is a possible graph of $f(x)$ estimating the locations of the turning points.

EXAMPLE 2 Using reasoning to determine the equation of a function from given information

Write the equation of a cubic function that has zeros at -2, 3, and $\frac{2}{5}$. The function also has a y-intercept of 6.

Solution

$$f(x) = a(x + 2)(x - 3)(5x - 2)$$

Use the zeros of the function to create factors for the correct family of polynomials. Since this function has three zeros and it is cubic, the order of each factor must be 1.

$$6 = a(0 + 2)(0 - 3)(5(0) - 2)$$
$$6 = a(2)(-3)(-2)$$
$$6 = 12a$$
$$a = \frac{1}{2}$$

Use the y-intercept to calculate the value of a.

Substitute $x = 0$ and $y = 6$ into the equation, and solve for a.

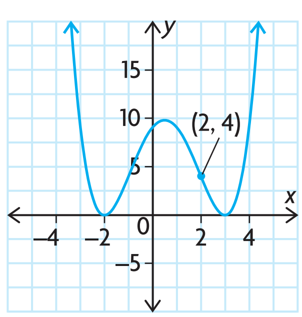
$$f(x) = \frac{1}{2}(x + 2)(x - 3)(5x - 2)$$

Write the equation in factored form.

EXAMPLE 3 Representing the graph of a polynomial function with its equation

a) Write the equation of the function shown below.

b) State the domain and range of the function.



Solution

a) $y = a(x + 2)^2(x - 3)^2$

Write the equation of the correct family of polynomials using factors created from the zeros.

Because the function must have positive values on both sides of the x-intercepts, the factors are squared. The parabolic shape of the graph near the zeros $x = -2$ and $x = 3$ confirms the order of 2 on the factors $(x + 2)^2$ and $(x - 3)^2$.

Let $x = 2$ and $y = 4$.

Substitute the coordinates of the point marked on the graph into the equation.

$$4 = a(2 + 2)^2(2 - 3)^2$$
$$4 = a(4)^2(-1)^2$$
$$4 = 16a$$
$$a = \frac{1}{4}$$

Solve to determine the value of a.

$$y = \frac{1}{4}(x + 2)^2(x - 3)^2$$

Write the equation in factored form.

b) Domain $= \{x \in \mathbf{R}\}$

Range $= \{y \in \mathbf{R} \mid y \geq 0\}$

All polynomial functions have their domain over the entire set of real numbers.

The graph has an absolute minimum value of 0 when $x = -2$ and $x = 3$. All other values of the function are greater than this.

EXAMPLE 4 Representing the equation of a polynomial function with its graph

Sketch the graph of $f(x) = x^4 + 2x^3$.

Solution

$$f(x) = x^4 + 2x^3$$
$$= x^3(x + 2)$$

Write the equation in factored form by dividing out the common factor of x^3.

The zeros are $x = 0$ and $x = -2$.

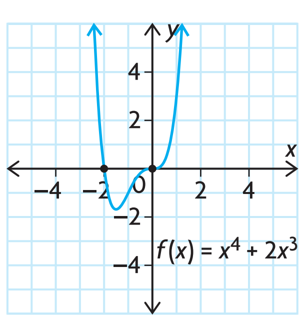
Determine the zeros, the order of the factors, and the shape of the graph near the zeros. The graph has a cubic shape at $x = 0$, since the factor x^3 has an order of 3. The graph has a linear shape near $x = -2$ since the factor $(x + 2)$ has an order of 1.

The y-intercept is $f(0) = 0^4 + 2(0)^3 = 0$.

Determine the y-intercept by letting $x = 0$.

End behaviours:
As $x \rightarrow \pm\infty$, $y \rightarrow \infty$.

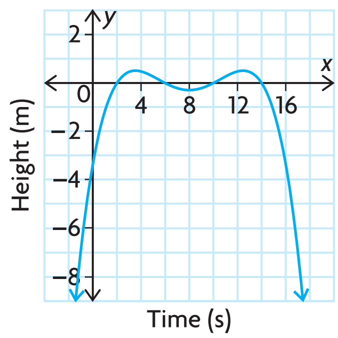
Determine the end behaviours. The function has an even degree, so the end behaviours are the same. The leading coefficient is positive, so the graph extends from the second quadrant to the first quadrant.



Use these characteristics to sketch a possible graph.

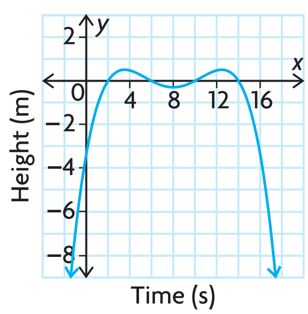
EXAMPLE 5 Representing a contextual situation with an equation of a polynomial function

While playing in the surf, a dolphin jumped twice into the air before diving deep below the surface of the water. The path of the dolphin is shown on the following graph.



Write the equation of the polynomial function that represents the height of the dolphin relative to the surface of the water.

Solution



The zeros of the function are $x = 2$, 6, 10, and 14. These are the times when the dolphin breaks the surface of the water. Use the zeros to create the factors of a family of polynomial functions. Since the shape of the graph near each zero is linear, the order of each corresponding factor must be 1.

$$f(x) = a(x - 2)(x - 6)(x - 10)(x - 14)$$

Let $f(3.5) = 0.5$.
The maximum height of the dolphin's leap was about 0.5 m when x was about 3.5 s.

Use the graph to estimate the maximum height of the dolphin's leap.

$$0.5 = a(3.5 - 2)(3.5 - 6)(3.5 - 10)(3.5 - 14)$$
$$0.5 = a(1.5)(-2.5)(-6.5)(-10.5)$$
$$0.5 = -255.9375a$$
$$a \doteq -0.002$$

Solve the equation to determine the value of a.

$$f(x) = -0.002(x - 2)(x - 6)(x - 10)(x - 14)$$

Write the equation in factored form.

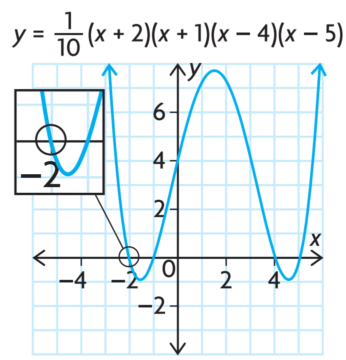
In Summary

Key Idea

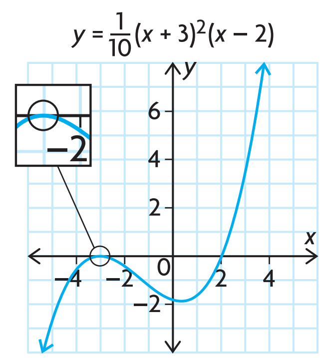
- The zeros of the polynomial function $y = f(x)$ are the same as the roots of the related polynomial equation, $f(x) = 0$.

Need to Know

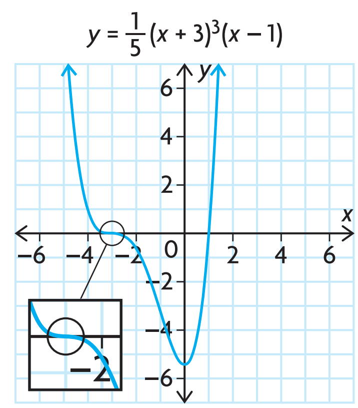
- To determine the equation of a polynomial function in factored form, follow these steps:
  - Substitute the zeros $(x_1, x_2, \ldots, x_n)$ into the general equation of the appropriate family of polynomial functions of the form $y = a(x - x_1)(x - x_2)\ldots(x - x_n)$.
  - Substitute the coordinates of an additional point for x and y, and solve for a to determine the equation.
- If any of the factors of a polynomial function are linear, then the corresponding x-intercept is a point where the curve passes through the x-axis. The graph has a linear shape near this x-intercept.



- If any of the factors of a polynomial function are squared, then the corresponding x-intercepts are turning points of the curve and the x-axis is tangent to the curve at these points. The graph has a parabolic shape near these x-intercepts.



- If any of the factors of a polynomial function are cubed, then the corresponding x-intercepts are points where the x-axis is tangent to the curve and also passes through the x-axis. The graph has a cubic shape near these x-intercepts.



CHECK Your Understanding

1. Match each equation with the most suitable graph. Explain your reasoning.

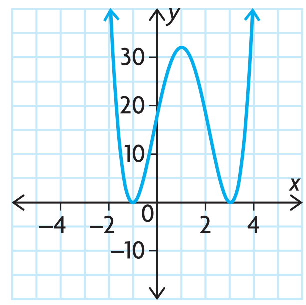
a) $f(x) = 2(x + 1)^2(x - 3)$

b) $f(x) = 2(x + 1)^2(x - 3)^2$

c) $f(x) = -2(x + 1)(x - 3)^2$

d) $f(x) = x(x + 1)(x - 3)(x - 5)$

A



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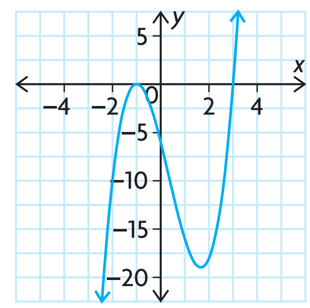

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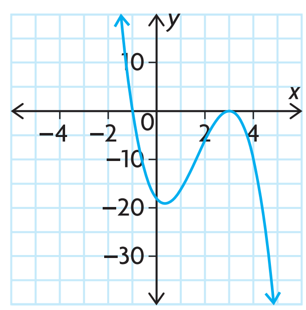

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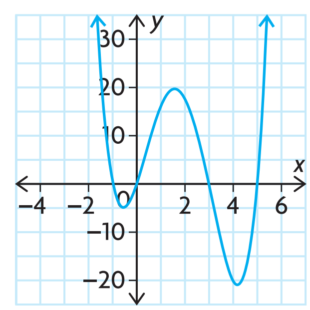


2. Sketch a possible graph of each function.

a) $f(x) = -(x - 4)(x - 1)(x + 5)$

b) $g(x) = x^2(x - 6)^3$

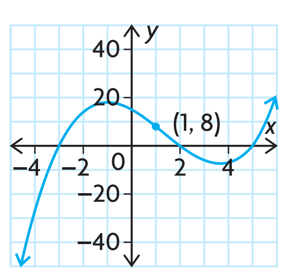
3. Each member of a family of quadratic functions has zeros at $x = -1$ and $x = 4$.

a) Write the equation of the family, and then state two functions that belong to the family.

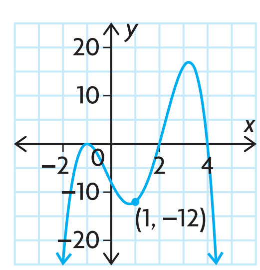
b) Determine the equation of the member of the family that passes through the point $(5, 9)$. Graph the function.

4. Write the equation of each function.

a)



b)



PRACTISING

5. Organize the following functions into families.

A	$y = 2(x - 3)(x + 5)$	**G**	$y = \frac{1}{2}(x - 3)(x + 5)$
B	$y = -1.8(x - 3)^2(x + 5)$	**H**	$y = -5(x + 8)(x)(x + 6)$
C	$y = -x(x + 6)(x + 8)$	**I**	$y = (x - 3)(x + 5)$
D	$y = 2(x + 5)(x + 3)^2$	**J**	$y = \frac{3}{5}(x + 5)(x + 3)^2$
E	$y = (x - 3)^2(x + 5)$	**K**	$y = \frac{x(x + 6)(x + 8)}{4}$
F	$y = x(x + 6)(x + 8)$	**L**	$y = 2(x + 5)(x^2 + 6x + 9)$

6. Sketch the graph of each function.

a) $y = x(x - 4)(x - 1)$
b) $y = -(x - 1)(x + 2)(x - 3)$
c) $y = x(x - 3)^2$
d) $y = (x + 1)^3$
e) $y = x(2x + 1)(x - 3)(x - 5)$
f) $y = x^2(3x - 2)^2$

7. a) Sketch an example of a cubic function with the given zeros. Then write the equation of the function.

i) $-3, 0, 2$ **iii)** $-1, 4$ (order 2)

ii) -2 (order 3) **iv)** $3, -\frac{1}{2}$ (order 2)

b) Are all the characteristics of the graphs unique? Explain.

8. Sketch an example of a quartic function with the given zeros, and write the equation of the function. Then write the equations of two other functions that belong to the same family.

a) $-5, -3, 2, 4$ **c)** $-2, \frac{3}{4}, 5$ (order 2)

b) -2 (order 2), 3 (order 2) **d)** 6 (order 4)

9. Sketch the graph of each function.

a) $y = 3x^3 - 48x$ **c)** $y = x^3 - 9x^2 + 27x - 27$

b) $y = x^4 + 4x^3 + 4x^2$ **d)** $y = -x^4 - 15x^3 - 75x^2 - 125x$

10. Sketch the graph of a polynomial function that satisfies each set of conditions.

a) degree 4, positive leading coefficient, 3 zeros, 3 turning points
b) degree 4, negative leading coefficient, 2 zeros, 1 turning point
c) degree 4, positive leading coefficient, 2 zeros, 3 turning points
d) degree 3, negative leading coefficient, 1 zero, no turning points

Year	Profit or Loss (in thousands of dollars)
1990	−216
1991	−88
1992	0
1993	54
1994	80
1995	84
1996	72
1997	50
1998	24
1999	0
2000	−16
2001	−18
2002	0
2003	44
2004	120

11. **A** A company's profits and losses during a 15-year period are shown in the table.

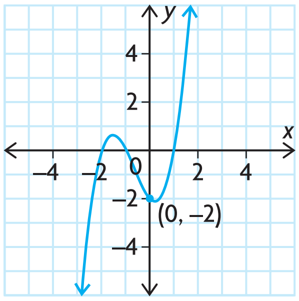
a) Sketch a graph of the data, using years since 1990 as the values of the independent variable.

b) If x represents the number of years since 1990 (with 1990 being year 0), write the polynomial equation that models the data.

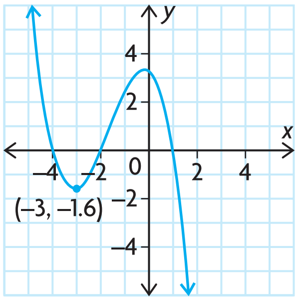
c) Is this trend likely to continue? What restrictions should be placed on the domain of the function so that it is realistic?

12. **K** Determine the equation of the polynomial function from each graph.

a)



b)



13. **a)** Determine the quadratic function that has zeros at -3 and -5, if $f(7) = -720$.

b) Determine the cubic function that has zeros at -2, 3, and 4, if $f(5) = 28$.

14. **T** The function $f(x) = kx^3 - 8x^2 - x + 3k + 1$ has a zero when $x = 2$. Determine the value of k. Graph $f(x)$, and determine all the zeros. Then rewrite $f(x)$ in factored form.

15. **C** Describe what you know about the graphs of each family of polynomials, in as much detail as possible.

a) $y = a(x - 2)^2(x - 4)^2$ **b)** $y = a(x + 4)(x - 3)^2$

Extending

16. Square corners cut from a 30 cm by 20 cm piece of cardboard create a box when the 4 remaining tabs are folded upwards. The volume of the box is $V(x) = x(30 - 2x)(20 - 2x)$, where x represents the height.

a) Calculate the volume of a box with a height of 2 cm.

b) Calculate the dimensions of a box with a volume of 1000 cm^3.

c) Solve $V(x) > 0$, and discuss the meaning of your solution in the context of the question.

d) State the restrictions in the context of the question.